Extended family of the electrovac two-soliton solutions for the Einstein-Maxwell equations

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The complex Ernst potentials for a family of two-soliton solutions of the Einstein-Maxwell equations in which 6 arbitrary complex constants correspond to 12 arbitrary relativistic multipole moments are constructed. Two new asymptotically flat members of this family are pointed out representing the exterior fields of binary systems of identical Kerr-Newman masses and of Kerr magnetized masses.

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I. INTRODUCTION

The construction of exact axisymmetric solutions of the Einstein-Maxwell equations possessing the prescribed physical properties means the obtaining of solutions in which the arbitrary parameters would correspond to the arbitrary relativistic multipole moments determining the physical structure of space-time. That this objective is not always achieved by the known solution-generating techniques may be illustrated, e.g., by Aleksejev's electrovac solution [1], where 12 arbitrary parameters do not represent 12 arbitrary multipole moments, so that the solution is only valid for the description of the exterior field of two superextreme Kerr-Newman sources and not of charged stationary black holes. An analogous solution was obtained by Guo and Ernst [2] by application of two successive Cosgrove transformations [3] to Minkowski space, in which the multipole moments are subjected to the same restrictions as in Ref. [1].

On the other hand, the problem of the physical interpretation of the parameters does not appear if the starting point in the construction of an electrovac solution is the axis behavior of its Ernst complex potentials $\mathcal{E}$ and Φ [4], since the arbitrary parameters introduced in this way guarantee the arbitrariness of the respective relativistic multipole moments, thus allowing *a priori* justification of the search for that particular solution. Among the different pseudopotential approaches [1,5–7] developed up to now on the basis of the Kinnersley-Chitre transformations [8] for the Einstein-Maxwell equations it appears that Sibgatullin's integral method has one "inborn" advantage with respect to other techniques, since it admits to know from the very beginning which physical situations will be describing electrovac solutions to be constructed with its help because it is exactly the choice of axis data for a solution, i.e., the introduction of the parameters representing certain arbitrary multipole moments, that constitutes the first necessary step in this method. In a series of papers [9,10] Sibgatullin's method was used to obtain the first known physically meaningful asymptotically flat solutions representing the exterior field of a magnetized spinning mass, and a common feature of the solutions from Ref. [10] is that all these are characterized by the axis data of the form

$$\mathcal{E}(\rho=0,z)=\frac{z^2+a_1z+a_2}{z^2+b_1z+b_2},\qquad \Phi(\rho=0,z)=\frac{c_1z+c_2}{z^2+b_1z+b_2}, \tag{1.1}$$

ρ and z being the Weyl-Papapetrou cylindrical coordinates, and a_k, b_k, c_k, $k=1,2$, being six arbitrary complex parameters, a particular choice of which leads to particular solutions from Ref. [10].

Since some other known solutions (among which are those from Refs. [1,2]) are defined by axis data of the form (1.1) with some restricted values of the parameters, it is the aim of our paper to give the general expressions of the potentials $\mathcal{E}(\rho,z)$ and $\Phi(\rho,z)$ satisfying the Ernst electrovac equations [4] and reducing at the symmetry axis to expressions (1.1). As two new particular examples obtainable from the general formulas, which may exhibit physical interest, we give the electrovac solutions for the exterior fields of the simplest binary systems of the identical Kerr-Newman masses and of identical Kerr masses endowed with magnetic dipole moment.

II. THE COMPLEX POTENTIALS OF THE EXTENDED TWO-SOLITON SOLUTION

The construction of the complex Ernst potentials corresponding to a given axis data is a straightforward procedure in Sibgatullin's method, the details of which can be found in Refs. [7,9]. In what follows, we restrict ourselves by only giving a compact determinant form of the electrovac solution resulting after the application of this method to the particular axis data (1.1). The solution was found to be

$$\mathcal{E} = \frac{E_+}{E_-}, \quad \Phi = \frac{F}{E_-} ,$$

$$E_\pm = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ \pm 1 & \frac{r_1}{\alpha_1-\beta_1} & \frac{r_2}{\alpha_2-\beta_1} & \frac{r_3}{\alpha_3-\beta_1} & \frac{r_4}{\alpha_4-\beta_1} \\ \pm 1 & \frac{r_1}{\alpha_1-\beta_2} & \frac{r_2}{\alpha_2-\beta_2} & \frac{r_3}{\alpha_3-\beta_2} & \frac{r_4}{\alpha_4-\beta_2} \\ 0 & H_1(\alpha_1) & H_1(\alpha_2) & H_1(\alpha_3) & H_1(\alpha_4) \\ 0 & H_2(\alpha_1) & H_2(\alpha_2) & H_2(\alpha_3) & H_1(\alpha_4) \end{vmatrix} ,$$

$$F = \begin{vmatrix} 0 & f(\alpha_1) & f(\alpha_2) & f(\alpha_3) & f(\alpha_4) \\ -1 & \frac{r_1}{\alpha_1-\beta_1} & \frac{r_2}{\alpha_2-\beta_1} & \frac{r_3}{\alpha_3-\beta_1} & \frac{r_4}{\alpha_4-\beta_1} \\ -1 & \frac{r_1}{\alpha_1-\beta_2} & \frac{r_2}{\alpha_2-\beta_2} & \frac{r_3}{\alpha_3-\beta_2} & \frac{r_4}{\alpha_4-\beta_2} \\ 0 & H_1(\alpha_1) & H_1(\alpha_2) & H_1(\alpha_3) & H_1(\alpha_4) \\ 0 & H_2(\alpha_1) & H_2(\alpha_2) & H_2(\alpha_3) & H_1(\alpha_4) \end{vmatrix} , \quad (2.1)$$

where the dependence on the coordinates (ρ, z) is determined by the functions

$$r_n = \sqrt{\rho^2 + (z - \alpha_n)^2}, \quad n = 1, 2, 3, 4 , \quad (2.2)$$

whereas the remaining constant coefficients are introduced in the following way.

The constants α_n are the roots of the fourth-order algebraic equation

$$\begin{aligned} S(\xi) = \xi^4 &+ \mathrm{Re}(a_1 + b_1)\xi^3 \\ &+ [c_1\bar{c}_1 + \mathrm{Re}(a_1\bar{b}_1 + a_2 + b_2)]\xi^2 \\ &+ \mathrm{Re}(a_1\bar{b}_2 + a_2\bar{b}_1 + 2c_1\bar{c}_2)\xi \\ &+ \mathrm{Re}(a_2\bar{b}_2) + c_2\bar{c}_2 = 0 , \end{aligned} \quad (2.3)$$

where an overbar denotes complex conjugation, and Re ( ) stands for a real part of the respective complex expression. Parameters β_1 and β_2 are the two poles of the common denominator of $\mathcal{E}(\rho = 0, z)$ and $\Phi(\rho = 0, z)$ in (1.1), thus being the roots of the quadratic equation

$$D(\xi) = \xi^2 + b_1\xi + b_2 = 0 . \quad (2.4)$$

The $f(\alpha_n)$ are given by

$$f(\alpha_n) = \frac{c_1\alpha_n + c_2}{a_n^2 + b_1\alpha_n + b_2}, \quad n = 1, 2, 3, 4 \quad (2.5)$$

and $H_l(\alpha_n)$ are defined by

$$H_l(\alpha_n) = \frac{\bar{e}_l + 2\bar{f}_l f(\alpha_n)}{\alpha_n - \bar{\beta}_l}, \quad l = 1, 2 \quad (2.6)$$

where

$$e_l = (-1)^{l+1}\frac{\beta_l^2 + a_1\beta_l + a_2}{\beta_1 - \beta_2} ,$$

$$f_l = (-1)^{l+1}\frac{c_1\beta_l + c_2}{\beta_1 - \beta_2} , \quad l = 1, 2 \quad (2.7)$$

e_l and f_l being the coefficients in the simple fraction decompositions of the functions $\mathcal{E}(\rho = 0, z)$ and $\Phi(\rho = 0, z)$, respectively.

Formulas (2.1)–(2.7) fully determine the electrovac solution satisfying the Ernst equations [4] and correspond to the axis data (1.1). The latter axis expressions are recovered from (2.1)–(2.7) by setting $r_n = z - \alpha_n$ [i.e., $\rho = 0$, $z \geq \max\{\mathrm{Re}(\alpha_n), n = 1, 2, 3, 4\}$].

Some remarks regarding Eqs. (2.1)–(2.7) might be relevant. First of all, the above determinant form for the potentials $\mathcal{E}$ and Φ was obtained assuming that all four roots α_n of Eq. (2.3) are of multiplicity one. If this condition is not satisfied, the numerators and denominators in the expressions for $\mathcal{E}$ and Φ may become zeros. Nevertheless, the general formulas still work, but then the application of l'Hôpital's rule is required to take the appropriate limits [such a situation arises for some specific relations between the parameters in the axis data leading to multiple roots of Eq. (2.3)]. L'Hôpital's rule also allows in principle treating the case of the double pole ($\beta_1 = \beta_2$), as well as some special cases when α_n may coincide with one of the poles β_1 or β_2. Then it is only necessary to make a common denominator in the expressions for $\mathcal{E}$ and Φ, and apply this rule to obtain the limit. However, because such a procedure turns out to be an extremely complicated and tedious one, it is by far easier to repeat the derivation of each particular solution not obtainable directly from the above general formulas, starting from a particular axis data, since Sibgatullin's method provides a straightforward recipe applicable to any special case (in the second and third papers of Ref. [10] the solutions had only one double pole).

It is not difficult to show that six arbitrary complex constants in the axis data (1.1) define 12 arbitrary relativistic Simon's multipole moments [11], namely eight gravitational and four electromagnetic multipoles. Indeed, using the notation of Hoenselaers and Perjés [12] for the gravitational moments m_i, $i = 0, 1, 2, 3$ [$\mathrm{Re}(m_i)$ and $\mathrm{Im}(m_i)$ represent, respectively, the mass and angular-momentum multipoles] and electromagnetic moments $q_j, j = 0, 1$ [$\mathrm{Re}(q_j)$ and $\mathrm{Im}(q_j)$ stand, respectively, for the electric and magnetic multipoles], the axis expressions (1.1) can be rewritten in the form

$$\mathcal{E}(\rho = 0, z) = \frac{(z - m_0 - m_1/m_0)(z + m_1/m_0 - M_{30}/M_{20}) - M_{20}/m_0^2}{(z + m_0 - m_1/m_0)(z + m_1/m_0 - M_{30}/M_{20}) - M_{20}/m_0^2} ,$$

$$\Phi(\rho = 0, z) = \frac{q_0(z + m_1/m_0 - M_{30}/M_{20}) + q_1 - q_0(m_1/m_0)}{(z + m_0 - m_1/m_0)(z + m_1/m_0 - M_{30}/M_{20}) - M_{20}/m_0^2} , \quad (2.8)$$

where

$$M_{20} = m_2 m_0 - m_1^2, \quad M_{30} = m_3 m_0 - m_2 m_1 \; . \tag{2.9}$$

It is clear that some physical restrictions may be imposed on m_i and q_j, which decrease the number of efficient arbitrary parameters. Indeed, by demanding the asymptotic flatness of solutions (2.1) i.e., $\mathrm{Im}(m_0) = 0$, the absence of the magnetic monopole moment $[\mathrm{Im}(q_0) = 0]$ and choosing the origin of coordinates in the center of mass $[\mathrm{Re}(m_1) = 0]$, we already have nine arbitrary multipoles and consequently nine efficient parameters in (1.1). If, for instance, one is interested in solutions possessing the equatorial symmetry, then one should further set $\mathrm{Re}\{m_3, q_1\} = 0$, $\mathrm{Im}(m_2) = 0$, so that solutions (2.1) with such additional symmetry are defined in general by only six arbitrary real parameters corresponding to six arbitrary multipole moments.

Let us mention some known solutions that are included in formulas (2.1)–(2.7) as particular cases.

(A) To see how solutions obtained by Aleksejev [1] and by Guo and Ernst [2] are contained in our formulas, we should first make a remark on the admissible values of the parameters α_n in (2.1). These, being the roots of the algebraic equation (2.3) with real constant coefficients, can assume real values or be pairs of complex conjugate roots. So the following three general cases are possible: (i) all four α_n are different real roots of Eq. (2.3); (ii) α_n are two real roots and a pair of complex conjugate roots; (iii) α_n are two different pairs of complex conjugate roots. It is the third case that corresponds to the Aleksejev and Guo-Ernst solutions. Because α_n are complex, these solutions do not represent the exterior fields of black holes but of superextreme objects (the interpretation given by Aleksejev to his solution was the superposition of two superextreme Kerr-Newman sources). It is clear then that 12 arbitrary real parameters entering the above-mentioned solutions do not represent 12 arbitrary multipole moments, the solutions not reducing to the axis expressions (1.1) with six arbitrary complex constants (but with some restricted values of a_k, b_k, c_k). Therefore, it is natural to call the solution defined by Eqs. (2.1)–(2.7) the *extended* electrovac two-soliton solution.

(B) In the absence of an electromagnetic field ($c_1 = c_2 = 0$) we come to the vacuum solution, which is equivalent to the double Kerr solution constructed by Kramer and Neugebauer [13] with the aid of the Bäcklund transformations [14]. In this case $\Phi = 0$, and the expressions for $E_\mp$ simplify considerably since now $H_1(\alpha_n)$ and $H_2(\alpha_n)$ can be replaced by

$$H_1(\alpha_n) = \frac{1}{\alpha_n - \bar{\beta}_1}, \quad H_2(\alpha_n) = \frac{1}{\alpha_n - \bar{\beta}_2} \; . \tag{2.10}$$

It follows then that the constants α_n and β_l may be considered as arbitrary real or complex parameters, providing in the general case eight efficient real parameters, since α_n can occur only in the complex conjugate pairs.

(C) The recent new solutions from Ref. [10] are also particular cases obtainable from the general formulas (2.1)–(2.7). Some of these solutions, being the electrovac generalizations of the Kerr [15] and Kerr-Newman [16] black-hole solutions, admit a very simple representation due to the additional equatorial symmetry they possess. In the last paper of Ref. [10] a five-parameter solution for a charged, magnetized, deformed, spinning mass was constructed; it follows from the above remarks on the multipole structure of solution (2.1) that the former solution can be generalized to include only one additional parameter standing for an arbitrary angular-momentum octupole moment without breaking the equatorial symmetry.

III. THE PARTICULAR ELECTROVAC TWO MASS SOLUTIONS

Formulas (2.1)–(2.7) from the preceding section admit solutions representing the exterior fields of a single compact source or binary systems of compact objects. The well-known vacuum two mass solution contained in these formulas is the double Kerr solution [13] the detailed analysis of which was given in Ref. [17] (see also references therein). In what follows we shall point out two electrovac two-body solutions contained in the general formulas and describing the systems of two Kerr-Newman objects and of two magnetized Kerr's masses.

A. The simplest solution for a binary system of identical Kerr-Newman sources

This solution is defined by the axis data of the form

$$\mathcal{E}(\rho = 0, z) = \frac{(z - k - m - ia)(z - k + m - ia)}{(z + k - m - ia)(z + k + m - ia)} \; ,$$

$$\tag{3.1}$$

$$\Phi(\rho = 0, z) = \frac{2q(z - ia)}{(z - k + m - ia)(z + k + m - ia)} \; ,$$

where the parameters m, a, q, and k are associated, respectively, with the mass, angular momentum per unit mass, electric charge, and distance of each source from the origin of coordinates.

Equations (2.1)–(2.7) then lead after rather laborious calculations to the following elegant expressions for the complex potentials $\mathcal{E}$ and Φ written in Kinnersley's form [18]:

$$
\begin{aligned}
&\mathcal{E}=\frac{A-B}{A+B},\quad \frac{C}{A+B}\,,\\
&A\equiv\kappa_+^2\{[a^2(k^2-4m^2+a^2+3q^2)-d(m^2-a^2-q^2)](R_-r_++R_+r_-)\\
&\quad+ia\kappa_-(3m^2-k^2-a^2-2q^2-d)(R_-r_+-R_+r_-)\}\\
&\quad-\kappa_-^2\{[a^2(k^2-4m^2+a^2+3q^2)+d(m^2-a^2-q^2)](R_-r_-+R_+r_+)\\
&\quad+ia\kappa_+(3m^2-k^2-a^2-2q^2+d)(R_-r_--R_+r_+)\}+d(m^2-k^2)(m^2-q^2)(R_+R_-+r_+r_-)\,,\\
&B\equiv m\kappa_+\kappa_-\{2d[(m^2-q^2)(R_++R_--r_+-r_-)-\kappa_+\kappa_-(R_++R_-+r_++r_-)]\\
&\quad+iad[(\kappa_++\kappa_-)(R_+-R_-)+(\kappa_+-\kappa_-)(r_--r_+)]\\
&\quad+ia(5m^2-k^2-a^2-4q^2)[(\kappa_++\kappa_-)(r_+-r_-)+(\kappa_+-\kappa_-)(R_--R_+)]\},\\
&C\equiv qB/m,\\
&R_\pm\equiv\sqrt{\rho^2+[z\pm(\kappa_++\kappa_-)]^2},\quad r_\pm\equiv\sqrt{\rho^2+[z\pm(\kappa_+-\kappa_-)]^2},\\
&\kappa_\pm\equiv\sqrt{(m^2+k^2-a^2-2q^2\pm d)/2},\quad d\equiv\sqrt{(k^2-m^2+a^2)^2+4a^2(q^2-m^2)}\,.
\end{aligned}
\tag{3.2}
$$

This solution, with q=0 (the absence of electric field), represents the nonlinear superposition of two identical Kerr objects. With $k=m$, formulas (3.2) reduce to the usual Kerr-Newman solution [16] possessing the total mass $2m$, total charge $2q$ and total angular momentum per unit mass a.

The first four relativistic multipole moments of solution (3.2) have the form

$$
\begin{aligned}
&m_0=2m,\quad m_1=2ima,\quad m_2=2m(k^2-m^2-a^2),\quad m_3=2ima(3k^2-3m^2-a^2),\\
&q_0=2q,\quad q_1=2iaq,\quad q_2=2q(k^2-m^2-a^2),\quad q_3=2iaq(3k^2-3m^2-a^2)\,,
\end{aligned}
\tag{3.3}
$$

whence it is seen that the solution is asymptotically flat and symmetric with respect to the equatorial plane. Formulas (2.1)–(2.7) also contain as a particular case a more general solution representing the superposition of two arbitrary Kerr-Newman sources (without any restrictions on the multipole moments), which does not possess, however, the equatorial symmetry in general.

B. The simplest superposition of two magnetized Kerr solutions

This solution is determined by the axis data

$$
\begin{aligned}
&\mathcal{E}(\rho=0,z)=\frac{(z-k-m-ia)\,(z+k-m-ia)}{(z-k+m-ia)\,(z+k+m-ia)}\,,\\
&\Phi(\rho=0,z)=\frac{2ib}{(z-k+m-ia)(z+k+m-ia)}\,,
\end{aligned}
\tag{3.4}
$$

where the parameters m, a, and k again stand, respectively, for the mass, angular momentum per unit mass, and location of each source on the symmetry axis, while b is the magnetic dipole parameter.

The corresponding potentials $\mathcal{E}$ and Φ can be shown to have the form

$$
\begin{aligned}
&\mathcal{E}=\frac{A-B}{A+B},\quad \Phi=\frac{C}{A+B}\,,\\
&A\equiv\kappa_+^2\{[a^2(k^2-4m^2+a^2)+b^2-d(m^2-a^2)](R_+r_-+R_-r_+)\\
&\quad+ia\kappa_-(k^2-3m^2+a^2+d)(R_+r_--R_-R_+)\}\\
&\quad-\kappa_-^2\{[a^2(k^2-4m^2+a^2)+b^2+d(m^2-a^2)](R_+r_++R_-r_-)\\
&\quad+ia\kappa_+(k^2-3m^2+a^2-d)(R_+r_+-R_-r_-)\}+d(b^2-k^2m^2+m^4)(R_+R_-+r_+r_-),\\
&B\equiv2m\kappa_+\kappa_-\{d[m^2(R_++R_--r_+-r_-)-\kappa_+\kappa_-(R_++R_-+r_++r_-)]\\
&\quad+ia[(\kappa_++\kappa_-)(k^2-2m^2-\kappa_+\kappa_-)(r_--r_+)+(\kappa_+-\kappa_-)(k^2-2m^2+\kappa_+\kappa_-)(R_+-R_-)]\},\\
&C\equiv2b\kappa_+\kappa_-\{i[(\kappa_++\kappa_-)(k^2-a^2-\kappa_+\kappa_-)(r_+-r_-)\\
&\quad+(\kappa_+-\kappa_-)(k^2-a^2+\kappa_+\kappa_-)(R_--R_+)]+ad(r_++r_--R_+-R_-)\},\\
&R_\pm\equiv\sqrt{\rho^2+[z\pm(\kappa_++\kappa_-)]^2},\quad r_\pm\equiv\sqrt{\rho^2+[z\pm(\kappa_+-\kappa_-)]^2},\\
&\kappa_\pm\equiv\sqrt{(m^2+k^2-a^2\pm d)/2},\quad d\equiv\sqrt{(k^2-m^2+a^2)^2+4(b^2-a^2m^2)}\,.
\end{aligned}
\tag{3.5}
$$

In the absence of magnetic field ($b = 0$) this solution, like solution (3.2), represents a nonlinear superposition of two identical Kerr masses. Because of the additional parameter b, formulas (3.5) describe a binary system of magnetized Kerr objects, which is symmetric about the equatorial plane. The first four multipole moments of solution (3.5) have the form

$$m_0 = 2m, \quad m_1 = 2ima, \quad m_2 = 2m(k^2 - m^2 - a^2), \quad m_3 = 2ima(3k^2 - 2m^2 - a^2),$$
$$q_0 = 0, \quad q_1 = 2ib, \quad q_2 = -4ab, \quad q_3 = 2ib(k^2 - m^2 - 3a^2) \;, \tag{3.6}$$

thus supporting the above interpretation of the parameters m, a, k, and b in the axis data.

Formulas (2.1)–(2.7) are also appropriate to treat the case of two arbitrary Kerr masses possessing an electric charge and magnetic dipole moment. However, the more general case does not admit a simple representation similar to Eqs. (3.2) or (3.5), since the latter have been obtained supposing an additional, equatorial symmetry of the respective solutions which is absent when the sources are not identical.

IV. CONCLUSIONS

Therefore, the extended family of the electrovac two-soliton solutions defined by Eqs. (2.1)–(2.7) is a rather vast family of exact solutions of the Einstein-Maxwell equations, which contains the particular members with already very well-established physical reputation (the Schwarzschild, Reissner-Nordström [19], Kerr and Kerr-Newman black-hole solutions, the double Kerr solution of Kramer and Neugebauer), as well as some other solutions recently obtained, which may exhibit potential physical importance. The exterior fields of charged spinning superextreme compact objects are described by particular solutions from Refs. [1,2]; at the same time; the general formulas also allow treating the "normal" case due to the correspondence of the parameters in the axis data (1.1) to arbitrary relativistic multipole moments.

In our future paper Sibgatullin's method will be applied for the construction of the extended N-soliton electrovac solution, which will involve already $6N$ arbitrary real parameters corresponding to $6N$ arbitrary multipole moments (the solution will generalize the known N-soliton solution [1,20] which is characterized by a restricted set of multipoles). We believe that the extended N-soliton family may become a powerful tool in treating the problem of the correct description of exterior fields of real astrophysical objects, even though the "extraction" of particular physically interesting solutions out of the general formulas will need much more efforts than in the $N = 2$ case.

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