Nonlinear instability of Kerr-type Cauchy horizons

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Using the general solution to the Einstein equations on intersecting null surfaces developed by Hayward we investigate the nonlinear instability of the Cauchy horizon inside a realistic black hole. Making a minimal assumption about the free gravitational data allows us to solve the field equations along a null surface crossing the Cauchy horizon. As in the spherical case, the results indicate that a diverging influx of gravitational energy, in concert with an outflux across the Cauchy horizon, is responsible for the singularity. The spacetime is asymptotically Petrov type N, the same algebraic type as a gravitational shock wave. Implications for the continuation of spacetime through the singularity are briefly discussed.

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I. INTRODUCTION

What are the generic features of gravitational collapse? We do not, as yet, have a complete answer to this question; however, most relativists are confident that gravitational collapse leads to the formation of black holes (at least within the astrophysical context). Furthermore it is plausible that the external field of such a black hole settles down to a member of the Kerr-Newman family, since these are the unique stationary solutions of the electrovac Einstein equations (see for example [1]). Externally, deviations from these solutions are expected to die away with an inverse power-law in advanced time leaving only the mass, charge, and angular momentum observable. A question which one might hope to answer is whether such a property continues to hold inside the black hole—that is, does the internal geometry also approach Kerr-Newman form? It seems not [6–8].

The known exact black-hole solutions possess Cauchy horizons—null hypersurfaces which are the boundary of the future domain of dependence for Cauchy data of the collapse problem. These horizons exhibit highly pathological behavior; small time-dependent perturbations originating outside the black hole undergo an infinite gravitational blueshift as they evolve towards the Cauchy horizon (CH). This blueshift of infalling radiation gave the first indications that these solutions, which so well describe the exterior geometry at late times, may not describe the generic internal structure. Penrose [2] pointed this out more than twenty-five years ago, and since then linear perturbations have been analyzed in detail [3]. These observations led to the conjecture that a scalar curvature singularity would form either *at* or *before* the CH once back reaction was accounted for.

Poisson and Israel [6], generalizing work of Hiscock [5], have shown that a scalar curvature singularity forms along the CH of a charged, spherical black hole in a simplified model. This singularity is characterized by the exponential divergence of the mass function with advanced time. The key ingredient producing this tremendous growth of curvature is the blueshifted radiation flux along the CH, although it is also necessary that some transverse energy flux be present. In [6] it was argued that the physics underlying the analysis was sufficiently general that similar results should hold for generic collapse (i.e., upon relaxation of the assumption of spherical symmetry). Further calculations support the conjecture that the singularity inside a generic black hole is null [8,10].

The aim of this paper is to present a detailed analysis of the CH inside a nonspherical black hole by taking advantage of the recent result due to Hayward [11]. He showed how to obtain the general solution of the Einstein equations on a pair of intersecting null surfaces. We begin by recasting the results of Poisson and Israel [6] in a suggestive form which more closely resembles the approach taken to the general problem. Our purpose in Sec. II is to stress the main points which must be taken as assumptions in the later calculations. We also feel that the analysis highlights the nonlinear nature of the effect and the merely precipitous role played by the outflux in this model [6]. With this preliminary review of the mass-inflation phenomenon out of the way, Sec. III begins the analysis of the CH singularity inside a realistic black hole. The assumptions which we make about the nature of the CH are discussed, and following Hayward [11] we formally integrate the first order Einstein equations of Appendix A. Using these results we can show that the divergences which arise are integrable, and obtain the leading behavior of all quantities of interest near to the CH. The asymptotic expressions for the Weyl scalars show that the spacetime is Petrov type N near the CH, the same algebraic type as a gravitational shock wave. These results (which are also suggested by the previous work [10]) once again raise questions about the classical continuation of spacetime through the singularity.

The equations and curvatures are relegated to the Appendixes in an effort to maintain the clarity of the presentation. Appendix A gives a summary of the notation and lists the dual null Einstein field equations in their first order form, as derived by Hayward [11]. Appendix B lists the components of the Riemann tensor necessary to analyze the algebraic type of the spacetime.

II. SPHERICAL BLACK HOLE INTERIORS

In this section we review the mass-inflation phenomenon in the context of charged, spherical black holes as studied by Poisson and Israel [6]. The presentation differs slightly from that in the literature [6,7] and closely parallels the method used later to discuss the nonspherical problem. In this way the limitations of the later analysis and of the approximations used should be more apparent.

The physics behind mass inflation is relatively simple. The CH inside a Reissner-Nordström black hole is a null hypersurface corresponding to infinite external advanced time. Time-dependent perturbations which originate in the external universe are gravitationally blueshifted as they propagate inwards near to the CH. Thus a charged black hole which deviates only slightly from Reissner-Nordström in the exterior is expected to have a barrier of radiation, with an exponentially diverging energy density, streaming along parallel to the CH. Generically some of this ingoing radiation scatters off the gravitational potential inside the black hole, leading to a flux of energy crossing the CH [12]. It is the nonlinear gravitational interaction of these two fluxes of radiation which generates a divergence of the local mass function. It is important to realize that the gravitational blueshift of time-dependent perturbations is the key ingredient producing this *mass inflation.*

A. The Poisson-Israel model

It is convenient to use null coordinates on the "radial" two spaces so that the spherical line element is

$$ds^2 = -2e^{-\lambda} du dv + r^2 (dx^2 + \sin^2 x dy^2) , \qquad (2.1)$$

where $\lambda = \lambda(u,v)$, $r = r(u,v)$ and the coordinates are such that u is a retarded time and v an advanced time. The stress-energy tensor for a radial electromagnetic field is

$$E_\mu{}^\nu = (q^2/8\pi r^4)\, \mathrm{diag}(-1,-1,1,1) , \qquad (2.2)$$

where q is the electric charge on the black hole. Poisson and Israel used crossflowing null dust to model the perturbations of the geometry, arguing that the large blueshift near to the Cauchy horizon should make the Isaacson [13] effective stress-energy desciption valid for the ingoing radiation. They also pointed out that the nature of the outflux is not important; its only purpose is to initiate the contraction of the Cauchy horizon. The stress-energy tensor is therefore

$$T_{\mu\nu} = \rho_{\rm in} l_\mu l_\nu + \rho_{\rm out} n_\mu n_\nu , \qquad (2.3)$$

where $l_\mu = -\partial_\mu v$ and $n_\mu = -\partial_\mu u$ are radial null vectors pointing inwards and outwards, respectively, and, $\rho_{\rm in}$ and $\rho_{\rm out}$ represent the energy densities of the inward and outward fluxes. Each term in Eq. (2.3) is independently conserved so that

$$\rho_{\rm in} = \frac{L_{\rm in}(v)}{4\pi r^2} , \qquad \rho_{\rm out} = \frac{L_{\rm out}(u)}{4\pi r^2} . \qquad (2.4)$$

The functions $L_{\rm in}(v)$ and $L_{\rm out}(u)$ are determined by the boundary conditions; however, it is important to note that they have no direct operational meaning since they depend on the parametrization of the null coordinates.

The field equations can now be written in a first order form by defining the extrinsic fields

$$\theta := r^{-2} (r^2)_{,v} , \qquad (2.5)$$

$$\widetilde{\theta} := r^{-2} (r^2)_{,u} , \qquad (2.6)$$

$$\nu := \lambda_{,v} , \qquad (2.7)$$

$$\widetilde{\nu} := \lambda_{,u} , \qquad (2.8)$$

where a comma denotes partial differentiation. Along S^- there are two evolution equations

$$(r^2 \theta)_{,u} = \frac{e^{-\lambda}}{r^2}(q^2 - r^2), \qquad (2.9)$$

$$(\theta - 2\nu)_{,u} = \frac{e^{-\lambda}}{r^4}(r^2 - 3q^2) , \qquad (2.10)$$

and a focusing equation which describes the behavior of the null cones with vertices at $r = 0$:

$$(r^2 \widetilde{\theta})_{,u} + r^2(\widetilde{\nu} - \widetilde{\theta}/2)\,\widetilde{\theta} = -2L_{\rm out}(u) . \qquad (2.11)$$

The complete set of Einstein equations, including those which hold on S^+, may be obtained from the above equations (2.9), (2.10), and (2.11) via the symmetry operation

$$(u;\widetilde{\theta},\widetilde{\nu},\theta,\nu) \to (v;\theta,\nu,\widetilde{\theta},\widetilde{\nu}) , \qquad L_{\rm out} \to L_{\rm in}. \qquad (2.12)$$

It is convenient to imagine that the inflow (outflow) is turned on at some advanced (retarded) time $v = v_0$ ($u = u_0$). In the pure inflow regime the spacetime is described by a charged Vaidya solution

$$ds^2 = dw(2dr - f dw) + r^2(dx^2 + \sin^2 x dy^2), \qquad (2.13)$$

where f is given by

$$f = 1 - \frac{2m(w)}{r} + \frac{q^2}{r^2} , \qquad (2.14)$$

and w is the *standard* external advanced time coordinate. In particular it is infinite both on future null infinity and the CH. The only nonzero component of the stress-energy tensor is

$$T_{ww} = \frac{1}{4\pi r^2}\frac{dm}{dw} . \qquad (2.15)$$

In the outflow region the solution is of the same form with advanced replaced by retarded time. The structure of the spacetime is shown in Fig. 1.

B. The solution

As mentioned above our treatment differs from that in the literature [6,7]. We consider the solution to the Ein-

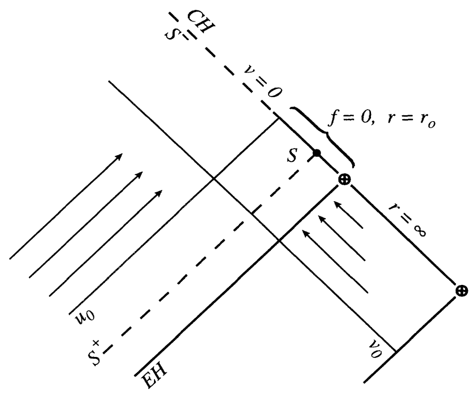

FIG. 1. A spacetime diagram showing the spherical Poisson-Israel model. The influx (outflux) of lightlike dust is turned on at advanced (retarded) time v_0 (u_0). The surface S^- coincides with the Cauchy horizon, CH. S^+ is parallel to the event horizon, EH, and intersects the Cauchy horizon at S, on which $f = 0$ and $r = r_o$.

stein equations only on the two intersecting null surfaces S^- and S^+. By choosing S^- to coincide with the CH we can show that a scalar curvature singularity must be present on the CH in the cross-flow region.

We first integrate the equations on the CH. This is readily achieved by making an appropriate choice of the retarded time coordinate u, in (2.1), such that $\widetilde{\nu} = \widetilde{\theta}/2$ along S^-. Then (2.8), (2.11), and (2.6) can be integrated, in that order, to give

$$e^{\lambda_-} = r_- \,, \tag{2.16}$$

$$\widetilde{\theta}^- = \left(r_o^2 \,\widetilde{\theta}^o - 2 \int_0^u d\tilde{u}\; L_{\rm out}(\tilde{u}) \right) \Big/ r_-^2 \,, \tag{2.17}$$

$$r_-^2 = r_o^2 + r_o^2 \,\widetilde{\theta}^o\, u - 2 \int_0^u d\hat{u} \int_0^{\hat{u}} d\tilde{u}\; L_{\rm out}(\tilde{u}) \,. \tag{2.18}$$

The (sub-) superscript $(-)$ is used to indicate that these equalities hold only on S^-, while an (o) indicates the value of the function on the two-sphere $S = S^- \cap S^+$. Of the remaining two equations we only need (2.9) which gives

$$r_-^2\, \theta^- = r_o^2\, \theta^o + \int_0^u d\tilde{u} \left(\frac{q^2}{\tilde{r}^3} - \frac{1}{\tilde{r}} \right) . \tag{2.19}$$

Equation (2.10) could be integrated in the same way.

The same construction works on S^+ (we will use this as our primary tool in the nonspherical case); however, it is more convenient to work with the exact solution (2.13) to determine $\widetilde{\theta}^0$, θ^0, and r_0 here.

C. Assumptions

The two essential features of this model will be adopted in the nonspherical case with only slight modification. We wish to emphasize them at this point.

The first assumption is that there should exist a stationary portion of the CH in the spacetime. This is achieved by turning on the outflux at some retarded time (u_0) inside the event horizon of the black hole (see Fig. 1); thus set

$$L_{\rm out}(u) = \beta H(u - u_0) \,, \tag{2.20}$$

where β is a constant, and $H(u)$ is a Heaviside step function. This is probably the most contentious issue in the Poisson-Israel [6] and subsequent analyzes [8]. It presupposes that the CH begins at finite radius, and essentially assumes that the singularity at the CH will be null. Some arguments have been advanced to suggest that this issue is important [14]; however, no convincing evidence has emerged to refute its correctness. Indeed preliminary numerical results [15] support this assumption, in contrast to the claim in [16].

The second assumption pertains to the influx of radiation; it is taken to decay with an inverse power law of advanced time. This was initially based on an extrapolation of results due to Price [9] to the event horizon of the black hole. Recently it has been verified numerically [17]. This decay is correctly reproduced by the ansatz

$$m(w) = m_o - \frac{\alpha r_o}{(p-1)} (\kappa_o w)^{-(p-1)} \,, \tag{2.21}$$

where κ_o is the surface gravity of the inner horizon, α is a dimensionless constant which depends on the luminosity of the collapsed star, and $p \geq 12$ is an integer (the numerical value derives from Price's analysis which shows that the radiative tail of the collapse decays as an inverse power $p \geq 4l + 4$ of advanced time, where l is the multipole order of the perturbation). Both linear [3,4] and nonlinear [8] perturbation analyses suggest that this inverse power-law decay of perturbations is also correct near the CH inside the black hole.

D. Mass inflation

Along S^+ the radius obeys the first order equation

$$\frac{dr}{dw} = \frac{1}{2} \left(1 - \frac{2m(w)}{r} + \frac{q^2}{r^2} \right) , \tag{2.22}$$

where $m(w)$ is given by (2.21). We are interested in calculating

$$\theta^+ = 2 r_+^{-1} \frac{dr_+}{dw} \frac{dw}{dv} , \tag{2.23}$$

where v is the coordinate which appears in (2.1) chosen so that $\lambda = \ln r$ along S^+. It is a straightforward matter to calculate dw/dv provided one notes that Eqs. (2.5)–(2.11) are form invariant under functional rescalings of the coordinates u and v. Therefore r also satisfies

$$\frac{d^2}{dw^2}(r^2) + \frac{d}{dw} \left(\ln \left[\frac{dw}{dv} \right] \right) \frac{d}{dw}(r^2) = -2 \frac{dm}{dw} \tag{2.24}$$

on S^+ in view of (2.12) and (2.11). Solving (2.22) for r, (2.24) can be integrated to get

$$v \simeq -e^{-\kappa_o w}(1+\cdots) \tag{2.25}$$

as $w \to \infty$. Thus

$$r_+ \simeq r_o + \frac{\alpha(-\ln|v|)^{-p+1}}{(p-1)\kappa_o} \times \left\{1+(p-1)(-\ln|v|)^{-1}+\cdots\right\} \tag{2.26}$$

and

$$\theta^+ \simeq \frac{2\alpha(-\ln|v|)^{-p}}{r_o\kappa_o v}\left\{1+p(-\ln|v|)^{-1}+\cdots\right\} \tag{2.27}$$

as $v \to 0$. This result means that, while the radius of the two-sphere S is finite, θ^o is actually infinite (in these coordinates).

Using the invariant definition of the mass in a spherical geometry

$$1-\frac{2m(x^\alpha)}{r}+\frac{q^2}{r^2} := g^{\alpha\beta}r_{,\alpha}r_{,\beta} = -\frac{r^2e^\lambda\theta\widetilde{\theta}}{2} \tag{2.28}$$

we know that $\theta^+\widetilde{\theta}^+ \to 0$ as $v \to 0$, which means that $\widetilde{\theta}^o = 0$. Substituting (2.20) into (2.17) and using $\widetilde{\theta}^o = 0$

$$r_-^2\,\widetilde{\theta}^- = -2(u-u_0)\beta H(u-u_0)\ . \tag{2.29}$$

The radius of the CH is then

$$r_-^2 = r_o^2 - (u-u_0)^2\beta H(u-u_0)\ , \tag{2.30}$$

and finally θ^- is given by (2.19). The square of the Weyl curvature on the CH is

$$\mathcal{C}^- = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\big|_{S^-} = 12\left(\frac{1}{r_-^2}-\frac{q^2}{r_-^4}+\frac{r_-}{2}\theta^-\widetilde{\theta}^-\right)^2 , \tag{2.31}$$

from which we can immediately see the reason for the Cauchy horizon singularity. The product $\theta^-\widetilde{\theta}^- = 0$ at S (see Fig. 1) and hence $\mathcal{C}^-$ is finite; however, if β is nonzero $\mathcal{C}^-$ jumps to infinity at $u = u_0$ when the outflux is turned on [see Eq. (2.29)]. Furthermore Eq. (2.30) shows that the radius remains finite for some amount of retarded time, so it must be the mass function which becomes infinite at $u = u_0$.

This analysis emphasizes that the nature of the outflux is largely irrelevant; it simply serves to initiate the contraction of the CH. It is this contraction which is at the root of mass inflation [6].

III. NONSPHERICAL BLACK HOLES

Charged spherical black holes have served as a useful model in which to investigate the nonlinear instability of the CH inside a black hole; however, the question needs to be addressed in the more realistic nonspherical context. Here one expects that a black hole settles down to a member of the Kerr-Newman family at late times. (Our analysis focuses on such a situation when there is no electromagnetic field.)

The Kerr solution can be written as

$$ds^2 = -\frac{\Delta}{\rho^2}(dw-ad\phi\sin^2\theta)^2+\frac{\sin^2\theta}{\rho^2}([r^2+a^2]d\phi-adw)^2 +2dr(dw-ad\phi\sin^2\theta)+\rho^2d\theta^2, \tag{3.1}$$

where $\Delta = r^2-2mr+a^2$, $\rho^2 = r^2+a^2\cos^2\theta$, and w is standard advanced time. The CH, located at $r = m-\sqrt{m^2-a^2}$ and $w = \infty$, is a stationary null hypersurface—its lightlike generators have zero rate of expansion. Moreover, these generators are shear free since Raychaudhuri's equation implies that shear produces contraction. This is not a generic situation; a linear perturbation analysis shows that there is usually a flux of backscattered radiation crossing the CH, along with the blueshifted influx parallel to it [4]. Based on this observation and using the spherical case as a guide, we will show that the CH is the locus of a scalar curvature singularity. The leading divergence may be associated with propagating modes of the gravitational field, manifesting itself in the form of a diverging Weyl curvature of Petrov type N.

A. Formalism

We employ the dual null formalism of Hayward [11] which is based on a 2+2 null decomposition of spacetime [18]. In terms of coordinates u and v which label the intersecting null hypersurfaces that foliate the spacetime, the line element can be written as

$$ds^2 = -2e^{-\lambda}dudv+(s_as^a)du^2+2s_adx^adu+h_{ab}dx^adx^b \tag{3.2}$$

where the lower-case latin indices range over $\{1,2\}$. There remains freedom to rescale u and v, and to transform the coordinates x^a so that s^a vanishes on a chosen $v =$ constant surface, we take this to be the CH located at $v = 0$ (see Fig. 2). h_{ab} is the *two-metric*, s^a a shift *two-vector*, and λ a scaling *function* on $S(u,v)$, the spatial two-surface which is the intersection of the null surfaces labeled by u and v. The two-metric can be decomposed into a conformal factor Ω and a conformal two-metric k_{ab} so that

$$\Omega = \sqrt{h}\ , \qquad k_{ab} = \Omega^{-1}h_{ab}\ . \tag{3.3}$$

Thus k_{ab} has unit determinant. Each of the quantities defined here depend on all four variables u, v, x^a.

The vacuum Einstein equations (A7)–(A23) are written in a first order form in terms of the fields

$$\theta = \Omega^{-1}\mathcal{L}_l\Omega, \qquad \widetilde{\theta} = \Omega^{-1}\mathcal{L}_n\Omega, \tag{3.4}$$

$$\sigma_{ab} = \Omega\,\mathcal{L}_lk_{ab}, \qquad \widetilde{\sigma}_{ab} = \Omega\,\mathcal{L}_nk_{ab}, \tag{3.5}$$

$$\nu = \mathcal{L}_l\lambda, \qquad \widetilde{\nu} = \mathcal{L}_n\lambda, \tag{3.6}$$

$$\omega_a = \frac{1}{2}e^\lambda\,\Omega k_{ab}\,\mathcal{L}_ls^b\ , \tag{3.7}$$

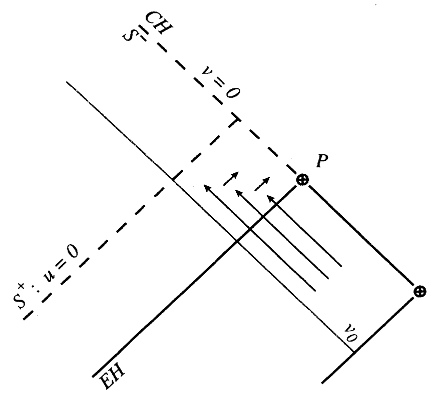

FIG. 2. A schematic representation of the generic black hole interior. The null surface S^- coincides with the Cauchy horizon, CH. The other null surface S^+ is at $u = 0$ and intersects the Cauchy horizon at S before a caustic forms. Initial values of the fields are specified on the intersection between the hypersurface $v = v_0$ and S^+. Also indicated is the ingoing gravitational wave tail, and its scattered component.

where $\mathcal{L}$ indicates the Lie derivative, and

$$l = \frac{\partial}{\partial v}, \qquad n = \frac{\partial}{\partial u} - s^a \frac{\partial}{\partial x^a} \tag{3.8}$$

are null vectors. The notation and the equations are given in Appendix A; more detail may be found in Hayward [11].

Our aim is to calculate the Weyl curvature scalars on the intersecting null hypersurfaces S^+ and S^- by making an ansatz for the free gravitational data (k_{ab}) on these surfaces. S^- is taken to coincide with the CH at $v = 0$, while S^+ is an outgoing null hypersurface, parallel to the event horizon, crossing S^- near to P in Fig. 2. The choice of the data is motivated by our understanding of the spherical situation and by the nonlinear perturbation analysis of Ori [8].

B. Assumptions

The focusing equation on S^+ is [Eq. (A7)]

$$\frac{\partial \theta}{\partial v} = -\frac{1}{2}\theta^2 - \nu\theta - \frac{1}{4}\sigma_{ab}\sigma^{ab} \,. \tag{3.9}$$

Hayward noticed that making use of coordinate freedom on S^+ (rescaling v), this equation may be linearized. Thus choosing v such that

$$\nu^+ = -\frac{1}{2}\theta^+ \tag{3.10}$$

it is possible to obtain the solution to the Einstein equations on S^+. Each of the equations (A7)–(A15) can be integrated (in the order they appear) provided one knows k_{ab} on this hypersurface, since

$$\sigma^+_{ab}\sigma^{ab}_+ = (k^{ab}_+ \partial_v k^+_{bc})(k^{cd}_+ \partial_v k^+_{da}) \,. \tag{3.11}$$

We cannot evolve generic initial data from the event horizon of a nonspherical, rotating black hole so we make an ansatz for this conformal two-metric. It is inferred from the work of Ori [8] that the conformal metric may be written as

$$k^+_{ab} = k^o_{ab}(x^a) + K^+_{ab}(\ln|v|, x^a) \tag{3.12}$$

where $\det(k^o_{ab}) = 1$ and asymptotically

$$K^+_{ab} \simeq \mathcal{F}_{ab}(x^c)(-\ln|v|)^{-n} + \cdots \tag{3.13}$$

as $v \to 0$. The functions $\mathcal{F}_{ab}(x^c)$ are well behaved for all x^a, and n is an integer. (The value implied by Ori's work is $n \geq 6$.) This form is based on the observation that nonlinear metric perturbations decay according to an inverse power-law of advanced time [8], and the expectation that Hayward's coordinate is related to external advanced time by

$$v \simeq -e^{-\kappa_o w}(1 + \cdots) \tag{3.14}$$

as $v \to 0$ (or $w \to \infty$). While this cannot be proven to hold, it seems reasonable as long as the surface S^+ does not encounter a caustic ($\Omega = 0$) at or before the CH—this condition is satisfied *a posteriori* implying, at least, a self-consistent treatment.

As in the spherical analysis, we give no detail of the gravitational data on S^- save to point out that the gravitational energy flux crossing the CH should not diverge. In fact, on physical grounds, it is expected that any radiation from the star will be exponentially redshifted near P in Fig. 2. Furthemore once the incoming radiation gets scattered by the gravitational potential inside the black hole, it should only produce a slow contraction of the CH [12].

To summarize, we assume that a portion of the CH exists which is caustic free near to P in Fig. 2. Furthermore, the gravitational perturbations propagating into the hole decay according to an inverse power law of advanced time which is related to v by (3.14).

C. Solution on S^+

With this aspect in hand we can proceed to the solutions of Eqs. (A7)–(A15). Integrating (3.9)

$$\theta^+ = \theta^o(x^a) - \frac{1}{4}\int_{v_0}^{v} d\tilde{v}\; \sigma^+_{ab}\sigma^{ab}_+ . \tag{3.15}$$

A (sub-) superscript o indicates the initial value of the function at v_0 in S^+ (see Fig. 2). On this two-surface we expect all the fields to be well behaved, bounded (and in some cases nonzero) functions of x^a.

Based on the assumption (3.12) and (3.13) about k^+_{ab} we can estimate the behavior of this integral near to the CH. First note that the inverse of the conformal metric can be written

$$k^{ab}_+ = k^{ab}_o + \epsilon^{ac}\epsilon^{bd}K^+_{cd}, \tag{3.16}$$

where ϵ^{bc} is the two-dimensional, antisymmetric matrix with $\epsilon^{12}=1$. The functional form of K^+_{ab} implies that

$$\partial_v K^+_{ab} = \frac{1}{v}\frac{\partial K^+_{ab}}{\partial \ln|v|} \tag{3.17}$$

will diverge since

$$\frac{\partial K^+_{ab}}{\partial \ln|v|} \simeq n\mathcal{F}_{ab}(-\ln|v|)^{-n-1} \tag{3.18}$$

as $v\to 0$. Since $\sigma_{ab}\sigma^{ab}$ is v^{-2} times a slowly varying function (near the CH), θ^+ is asymptotically

$$\theta^+ \simeq \theta^0(x^a) + \frac{n^2 k_o^{ab}k_o^{cd}\mathcal{F}_{bc}\mathcal{F}_{da}}{4v(-\ln|v|)^{2(n+1)}} + \cdots , \tag{3.19}$$

which diverges at the CH for the same reason that $\partial_v K^+_{ab}$ does.

It is now straightforward to formally integrate the remaining equations; however, our main objective is to show which quantities diverge and which remain bounded at the CH. Observing that the integral

$$I = \int^v \tilde v^{-1}(-\ln|\tilde v|)^{-n} d\tilde v = \frac{1}{n-1}\,(-\ln|v|)^{-n+1} \tag{3.20}$$

is finite as $v\to 0$ (provided $n>1$), it is clear that

$$\int^v d\tilde v\ \left(\tilde v^{-1}K^+_{ab}\right) \to \frac{1}{n-1}\,\mathcal{F}_{ab}(-\ln|v|)^{-n+1} \tag{3.21}$$

as $v\to 0$, and remains bounded at the CH. Each of the equations must be treated in turn, examining the divergent part of the source and its integral near the CH. We quickly see that Ω is bounded at the CH, and so all the intrinsic quantities on the two-dimensional surfaces of foliation remain finite all the way to the CH. To see this, first integrate (A8) to get

$$\lambda_+ = \lambda_o - \frac{1}{2}\int_{v_o}^v d\tilde v\ \theta^+ , \tag{3.22}$$

then substituting in the asymptotic form for θ^+ we find

$$\begin{aligned}\int^v d\tilde v\,\theta^+ &\simeq \frac{n^2}{4}k_o^{ab}k_o^{cd}\mathcal{F}_{bc}\mathcal{F}_{da}\int^v d\tilde v\ \tilde v^{-1}(-\ln|\tilde v|)^{-2(n+1)}\\ &\simeq \frac{n^2}{4(2n+1)}k_o^{ab}k_o^{cd}\mathcal{F}_{bc}\mathcal{F}_{da}(-\ln|v|)^{-(2n+1)}\end{aligned} \tag{3.23}$$

and hence $\lambda_+\to$ const as $v\to 0$. The gauge choice (3.10) implies that

$$\Omega^+ = \Omega_o e^{-2(\lambda_+-\lambda_o)} \tag{3.24}$$

and so Ω and k_{ab} are both bounded at the CH. Now continuing the integration

$$\omega_b^+ = \omega_b^o e^{2(\lambda_+-\lambda_o)} + \tfrac{1}{2}e^{2\lambda_+}\int_{v_0}^v d\tilde v\ e^{-2\lambda_+}(\triangle^a\sigma^+_{ab} - \tfrac{3}{2}\triangle_b\theta^+ - \theta^+\triangle_b\lambda_+) , \tag{3.25}$$

$$s^b_+ = s^b_o + 2\int_{v_0}^v d\tilde v\ e^{-\lambda_+}\omega^b_+ \tag{3.26}$$

and

$$\begin{aligned}\widetilde\theta^+ &= \widetilde\theta_o e^{2(\lambda_+-\lambda_o)} + e^{2\lambda_+}\int_{v_o}^v d\tilde v\ e^{-3\lambda_+}(-\tfrac{1}{2}{}^{(2)}R + \omega^a_+\omega^+_a - \tfrac{1}{2}\triangle^a\triangle_a\lambda_+\\ &\quad + \tfrac{1}{4}\triangle^a\lambda_+\triangle_a\lambda_+ + \omega^a_+\triangle_a\lambda_+ - \triangle_a\omega^a_+) ,\end{aligned} \tag{3.27}$$

from which it is clear that only finite terms appear in the equation for $\widetilde\theta$, which therefore cannot diverge on CH.

$$\widetilde\sigma^+_{ab} \simeq \text{(finite)} - \tfrac{1}{2}e^{\lambda_+}h^+_{ac}\int_{v_0}^v d\tilde v\ e^{-\lambda_+}\widetilde\theta h^{cd}\sigma_{db} , \tag{3.28}$$

the shear of the ingoing null rays orthogonal to $S(u=0,v)$, is also bounded at the CH. Finally

$$\widetilde\nu_+ = \widetilde\nu_o + \int_{v_o}^v d\tilde v\ \left\{\tfrac{1}{4}\sigma^+_{ab}\widetilde\sigma^{ab}_+ - \tfrac{1}{2}\theta^+\widetilde\theta^+ + e^{-\lambda_+}(-\tfrac{1}{2}{}^{(2)}R + 3\omega^+_a\omega^a_+ - \tfrac{1}{4}\triangle_a\lambda_+\triangle^a\lambda_+ - \omega^a_+\triangle_a\lambda_+)\right\} , \tag{3.29}$$

which is finite despite the presence of divergent terms in the integrand [which look like $v^{-1}\times$ inverse powers of $(-\ln|v|)$, see Eq. (3.20)]. This analysis implies that θ^+ and σ^+_{ab} diverge inside the black hole, and the divergence is of the same nature as in the spherical model once $\sigma_{ab}\sigma^{ab}$ is interpreted as the gravitational energy crossing S^+.

IV. THE CAUCHY HORIZON SINGULARITY

The solution obtained in the previous section can be used to show that a scalar curvature singularity is present on the CH due to the blueshifted influx of radiation. On S^- the shift vector s^a is set to zero by a coordinate transformation $\tilde x^a = \tilde x^a(u,x^b)$ without altering the form of the metric (3.2). [These coordinates exist provided s^a is regular on S^- in the original coordinate system. Equation (3.26) suggests that this is true.] It is assumed that this transformation was made in Sec. III; this in no way alters the previous analysis.

Hayward's scheme to integrate the equations on S^- can now be completed. Making the gauge choice

$$\widetilde\nu^- = -\tfrac{1}{2}\widetilde\theta^- , \tag{4.1}$$

Eq. (A16) implies that

$$\widetilde{\theta}^- = \widetilde{\theta}^+\Big|_{v=0} - \tfrac{1}{4}\int_0^u d\tilde{u}\,(k_-^{ab}\partial_{\tilde{u}}k_{bc}^-)(k_-^{cd}\partial_{\tilde{u}}k_{da}^-)\,. \quad (4.2)$$

As mentioned in the introduction $\widetilde{\theta}^-$ (and all other quantities) are formally known once $k_{ab}^-(u,x^c)$ is specified. Our analysis assumes that k_{ab}^- is a slowly varying function of u which tends to a nondegenerate value as $u \to -\infty$; that is, near to P (in Fig. 2) k_{ab} should approach the value obtained on the CH in Kerr spacetime. Furthermore $\widetilde{\theta}^- \to 0$ and the spacetime should be asymptotically Kerr in the limit $u \to -\infty$. It is not a generic situation to have $\widetilde{\theta}^- = 0$ and/or $\widetilde{\sigma}_{ab}^- = 0$ on the CH, since some radiation (both from the star, and from backscattering of the gravitational wave tail inside the black hole [8,12]) will usually be present. This discussion implies that there should exist a caustic free portion of the CH near to P on which our analysis is valid.

The integration now proceeds as it did for S^+, and in particular we arrive at expressions for θ^- and σ_{ab}^-

$$\theta^- = e^{2\lambda_-}\left[e^{-2\lambda_+}\theta^+\right]_{v=0} + e^{2\lambda_-}\int_0^u d\tilde{u}\,e^{-3\lambda}\left(-\tfrac{1}{2}{}^{(2)}R + \omega_a\omega^a - \tfrac{1}{2}\triangle^a\triangle_a\lambda + \tfrac{1}{4}\triangle_a\lambda\triangle^a\lambda - \omega^a\triangle_a\lambda + \triangle_a\omega^a\right) \quad (4.3)$$

and

$$\begin{aligned}\sigma_{ab}^- = {}& e^{\lambda_-}h_{fa}^-\left[e^{-\lambda_+}h_+^{fd}\sigma_{db}^+\right]_{v=0} - \tfrac{1}{2}h_{fa}^- e^{\lambda_-}\int_0^u d\tilde{u}\,\big(e^{-\lambda}\theta h^{df}\widetilde{\sigma}_{db} - 4e^{-2\lambda}h^{df}\{\omega_{(d}\,\omega_{b)}\\ & -\tfrac{1}{2}\triangle_{(d}\triangle_{b)}\lambda + \tfrac{1}{4}\triangle_{(d}\lambda\triangle_{b)}\lambda - \omega_{(d}\triangle_{b)}\lambda + \triangle_{(d}\omega_{b)}\}\big)\\ & -h_{ab}^- e^{\lambda_-}\int_0^u d\tilde{u}\,e^{-2\lambda}\left(\omega^c\omega_c - \tfrac{1}{2}\triangle_c\triangle^c\lambda + \tfrac{1}{4}\triangle_c\lambda\triangle^c\lambda - \omega^c\triangle_c\lambda + \triangle_c\omega^c\right).\end{aligned} \quad (4.4)$$

We saw in the previous section how

$$\lim_{v\to 0}\left\{\begin{matrix}\sigma_{ab}^+\\ \theta^+\end{matrix}\right\} \to \infty\,; \quad (4.5)$$

thus θ^- and σ_{ab}^- are generically unbounded on the CH. Of the remaining fields, only ν contains divergent quantities. From Eq. (A23) we see that

$$\nu_- \simeq \int_0^u d\tilde{u}\,\left(\frac{1}{4}\sigma_{ab}^-\widetilde{\sigma}_-^{ab} - \frac{1}{2}\theta^-\widetilde{\theta}_-\right) + \cdots\,, \quad (4.6)$$

which is actually infinite on CH. Imposing the field equations on the Weyl scalars one finds that all but Ψ_4 contain divergent quantities, and hence the CH is singular.

It is possible to say a little more about the nature of the singularity by examining the rate of growth of the curvature along S^+. Once again, imposing the field equations in Eq. (B11) gives

$$\Psi_0^+ = \tfrac{1}{4}e^{2\lambda}m^a m^b\,\{2\mathcal{L}_l\sigma_{ab} + 2\nu\sigma_{ab} - \sigma_{am}h^{mn}\sigma_{nb}\}|_+\,. \quad (4.7)$$

Using σ_{ab}^+, etc., the asymptotic expression is

$$\Psi_0^+ \simeq (\text{finite}) \times \left\{v^2(-\ln|v|)^{n+1}\right\}^{-1} \quad (4.8)$$

as $v \to 0$. This is the contribution from $\mathcal{L}_l\sigma_{ab}^+$, which is damped by the smallest number of powers of $(-\ln|v|)$. In a similar manner the leading behavior can be extracted from Ψ_1 and Ψ_2 giving

$$\Psi_1^+ \simeq (\text{finite}) \times \left\{v\,(-\ln|v|)^{n+1}\right\}^{-1}, \quad (4.9)$$

$$\Psi_2^+ \simeq (\text{finite}) \times \left\{v\,(-\ln|v|)^{n+1}\right\}^{-1}, \quad (4.10)$$

where these leading contributions arise due to the presence of terms involving the shear (σ_{ab}^+) of the surface S^+. A little more work shows that Ψ_3^+ is finite on the CH; notice that Eq. (3.26) combined with the choice $s^a \to 0$ on S^-, and knowledge that λ^+ and ω_a^+ are finite on $S^+\cap S^-$, implies that $s_a^+ \sim v$ as $v \to 0$, hence $s_+^m\Sigma_{mn}^+ s_+^n \to 0$ on S^-. Finally Ψ_4 is independent of θ^+ and σ_{ab}^+ and therefore is finite. These results imply that the square of the Weyl tensor is dominated by Ψ_0,

$$\begin{aligned}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} &\simeq 8\,(\Psi_0\,\Psi_4 + \overline{\Psi}_0\,\overline{\Psi}_4) + \cdots\\ &\simeq (\text{finite}) \times \left\{v^2(-\ln|v|)^{n+1}\right\}^{-1} + \cdots\end{aligned} \quad (4.11)$$

as $v \to 0$.

V. DISCUSSION

The treatment of the nonspherical gravitational collapse is generally very difficult; however, it seems likely that the external field settles down to that of a Kerr black hole. Inside the black hole some progress may also be made by advancing two assumptions: (1) There exists a caustic free portion of the CH, which is a null hypersurface. (2) There is an ingoing gravitational wave tail which gets blueshifted by an infinite amount at the CH (this is achieved here by our ansatz for k_{ab}^+). Indeed using Hayward's formalism the nature of the CH can be investigated in some detail.

Following from the analysis in Secs. III and IV we conclude that the CH is the locus of a scalar curvature singularity, and the curvature is an integrable function of the advanced time. Ori [7,8] has argued that the latter property of the curvature suggests an extension of the spacetime through the singularity may be possible. It is worth commenting a little further on this point. We have demonstrated the asymptotic values of each of the Weyl scalars; in particular, it has been shown that Ψ_0 contains the leading divergences. The algebraic classification of the Weyl curvature is obtained by solving the quartic

$$\Psi_0 a^4 + \Psi_1 a^3 + \Psi_2 a^2 + \Psi_3 a + \Psi_4 = 0 \tag{5.1}$$

for a and examining the degeneracy of its roots [19]. Based on Eqs. (4.8)–(4.10) there is a fourfold degeneracy as $v \to 0$ implying that the gravitational field is asymptotically type N, with repeated principal null direction $\boldsymbol{k} = \boldsymbol{n} + a\boldsymbol{m} + a^*\overline{\boldsymbol{m}} + aa^*\boldsymbol{l}$ (where $a \to 0$ as $v \to 0$). It is worth noting that gravitational shock waves are of this algebraic type (and are the only curvatures which may be confined to a thin skin without a surface layer of matter being present [20]); therefore, the intuitive picture of the singularity which emerges is that of a gravitational shock propagating along the CH. These results might be taken to support arguments for the existence of a classical continuation of the geometry beyond the CH singularity. However, this is a highly speculative point and it should be noticed that the Weyl curvature contains contributions which are not present in an exactly type N spacetime. Indeed these contributions diverge at the CH and are manifest in the square of the Weyl tensor (4.11) (which vanishes for a pure gravitational shock). Thus a classical continuation is unlikely, as pointed out in [21], since nonclassical matter is needed to confine the divergent curvature to a thin layer along the CH.

To conclude, the *hairy* singularity inside a generic black hole is dominated by the propagating gravitational wave tail of the collapse (which decays in the exterior of the black hole).

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APPENDIX A: NOTATION

In this appendix we summarize the dual null formalism of Hayward [11]. The reader is referred to his articles for more technical details. This approach, based on a 2+2 decomposition, assumes that the spacetime can be foliated locally by compact orientable two-surfaces S. Therefore given a Lorentzian manifold $\mathcal{M}$ (with signature $- + ++$) we assume that there exists a smooth embedding $\phi : S \times [0,U) \times [0,V) \to \mathcal{M}$ with the induced metric, h_{ab}, on S being spatial. Latin indices range over $\{1,2\}$. In terms of a basis which is Lie-propagated along $\partial/\partial u$ and $\partial/\partial v$ the spacetime line element may be written as in Eq. (3.2). The Lie derivative along a vector ξ is denoted $\mathcal{L}_\xi$. The natural covariant derivative of the metric h_{ab} is $\triangle_a$, and the Ricci scalar is denoted by ${}^{(2)}R$. Tensor valued quantities on the two-surface S are denoted $X^{a\cdots}{}_{b\ldots}$ [sometimes we will include a superscript (2) to emphasize the two-dimensional covariant nature]. Introducing the null vectors

$$l^\mu = (1,0,0,0), \qquad n^\mu = (0,1,-s^a) \tag{A1}$$

we define

$$\Sigma_{ab} = \mathcal{L}_l h_{ab}\,, \qquad \widetilde{\Sigma}_{ab} = \mathcal{L}_n h_{ab}\,. \tag{A2}$$

To write the Einstein field equations in a first order form Hayward introduces the extrinsic fields:

$$\theta = \tfrac{1}{2}h^{ab}\Sigma_{ab}, \qquad \widetilde{\theta} = \tfrac{1}{2}h^{ab}\widetilde{\Sigma}_{ab}, \tag{A3}$$

$$\sigma_{ab} = \Sigma_{ab} - \theta h_{ab}, \qquad \widetilde{\sigma}_{ab} = \widetilde{\Sigma}_{ab} - \widetilde{\theta} h_{ab}, \tag{A4}$$

$$\nu = \mathcal{L}_l\lambda, \qquad \widetilde{\nu} = \mathcal{L}_n\lambda\,, \tag{A5}$$

$$\omega_a = \tfrac{1}{2}e^{\lambda}h_{ab}\mathcal{L}_l s^b\,. \tag{A6}$$

These quantities are readily interpreted geometrically; $(\theta,\widetilde{\theta})$ are the expansions in the null directions normal to the foliation S. The tensors $(\sigma_{ab},\widetilde{\sigma}_{ab})$ are the shears, which are traceless by definition; the vector ω_a is the twist. The two functions $(\nu,\widetilde{\nu})$ are called the inaffinities and measure the nonaffine quality of the coordinates u and v. Then the Einstein equations and contracted Bianchi identities are the $\boldsymbol{l}$ equations,

$$\mathcal{L}_l\theta = -\tfrac{1}{2}\theta^2 - \nu\theta - \tfrac{1}{4}\sigma_{ab}\sigma^{ab}, \tag{A7}$$

$$\mathcal{L}_l\lambda = \nu, \tag{A8}$$

$$\mathcal{L}_l\Omega = \theta\,\Omega, \tag{A9}$$

$$\mathcal{L}_l k_{ab} = \Omega^{-1}\sigma_{ab}, \tag{A10}$$
$$\mathcal{L}_l \omega_a = -\theta\,\omega_a + \tfrac{1}{2}\triangle^b\sigma_{ba} + \tfrac{1}{2}\triangle_a(\nu-\theta) - \tfrac{1}{2}\theta\triangle_a\lambda, \tag{A11}$$
$$\mathcal{L}_l s^a = 2e^{-\lambda}\omega^a, \tag{A12}$$
$$\mathcal{L}_l\widetilde{\theta} = -\theta\widetilde{\theta} + e^{-\lambda}(-\tfrac{1}{2}{}^{(2)}R + \omega_a\omega^a - \tfrac{1}{2}\triangle^a\triangle_a\lambda + \tfrac{1}{4}\triangle_a\lambda\,\triangle^a\lambda + \omega^a\,\triangle_a\lambda - \triangle_a\omega^a), \tag{A13}$$
$$\begin{aligned}\mathcal{L}_l\widetilde{\sigma}_{ab} = {}& \sigma_{c(a}\widetilde{\sigma}_{b)d}h^{cd} + \tfrac{1}{2}\theta\,\widetilde{\sigma}_{ab} - \tfrac{1}{2}\widetilde{\theta}\,\sigma_{ab},\\ &+ 2e^{-\lambda}(\omega_a\omega_b - \tfrac{1}{2}\triangle_{(a}\triangle_{b)}\lambda + \tfrac{1}{4}\triangle_{(a}\lambda\triangle_{b)}\lambda + \omega_{(a}\triangle_{b)}\lambda - \triangle_{(a}\omega_{b)}),\\ &- e^{-\lambda}\,h_{ab}\,(\omega_c\omega^c - \tfrac{1}{2}\triangle^c\triangle_c\lambda + \tfrac{1}{4}\triangle_c\lambda\,\triangle^c\lambda + \omega^c\triangle_c\lambda - \triangle_c\omega^c),\end{aligned} \tag{A14}$$
$$\mathcal{L}_l\widetilde{\nu} = \tfrac{1}{4}\sigma_{ab}\widetilde{\sigma}^{ab} - \tfrac{1}{2}\theta\widetilde{\theta} + e^{-\lambda}(-\tfrac{1}{2}{}^{(2)}R + 3\omega_a\omega^a - \tfrac{1}{4}\triangle_a\lambda\,\triangle^a\lambda - \omega^a\triangle_a\lambda), \tag{A15}$$

and the $\boldsymbol{n}$ equations,

$$\mathcal{L}_n\widetilde{\theta} = -\tfrac{1}{2}\widetilde{\theta}^2 - \widetilde{\nu}\widetilde{\theta} - \tfrac{1}{4}\widetilde{\sigma}_{ab}\widetilde{\sigma}^{ab}, \tag{A16}$$
$$\mathcal{L}_n\lambda = \widetilde{\nu}, \tag{A17}$$
$$\mathcal{L}_n\Omega = \widetilde{\theta}\,\Omega, \tag{A18}$$
$$\mathcal{L}_n k_{ab} = \Omega^{-1}\widetilde{\sigma}_{ab}, \tag{A19}$$
$$\mathcal{L}_n\omega_a = -\widetilde{\theta}\,\omega_a - \tfrac{1}{2}\triangle^b\widetilde{\sigma}_{ba} - \tfrac{1}{2}\triangle_a(\widetilde{\nu}-\widetilde{\theta}) + \tfrac{1}{2}\widetilde{\theta}\triangle_a\lambda, \tag{A20}$$
$$\mathcal{L}_n\theta = -\theta\widetilde{\theta} + e^{-\lambda}(-\tfrac{1}{2}{}^{(2)}R + \omega_a\omega^a - \tfrac{1}{2}\triangle^a\triangle_a\lambda + \tfrac{1}{4}\triangle_a\lambda\,\triangle^a\lambda - \omega^a\,\triangle_a\lambda + \triangle_a\omega^a), \tag{A21}$$
$$\begin{aligned}\mathcal{L}_n\sigma_{ab} = {}& \tilde{\sigma}_{c(a}\sigma_{b)d}h^{cd} + \tfrac{1}{2}\widetilde{\theta}\,\sigma_{ab} - \tfrac{1}{2}\theta\,\widetilde{\sigma}_{ab}\\ &+ 2e^{-\lambda}(\omega_a\omega_b - \tfrac{1}{2}\triangle_{(a}\triangle_{b)}\lambda + \tfrac{1}{4}\triangle_{(a}\lambda\triangle_{b)}\lambda - \omega_{(a}\triangle_{b)}\lambda + \triangle_{(a}\omega_{b)})\\ &- e^{-\lambda}\,h_{ab}\,(\omega_c\omega^c - \tfrac{1}{2}\triangle^c\triangle_c\lambda + \tfrac{1}{4}\triangle_c\lambda\,\triangle^c\lambda - \omega^c\triangle_c\lambda + \triangle_c\omega^c),\end{aligned} \tag{A22}$$
$$\mathcal{L}_n\nu = \tfrac{1}{4}\sigma_{ab}\widetilde{\sigma}^{ab} - \tfrac{1}{2}\theta\widetilde{\theta} + e^{-\lambda}(-\tfrac{1}{2}{}^{(2)}R + 3\omega_a\omega^a - \tfrac{1}{4}\triangle_a\lambda\,\triangle^a\lambda + \omega^a\triangle_a\lambda). \tag{A23}$$

APPENDIX B: RIEMANN TENSOR AND WEYL SCALARS

1. Riemann tensor

Here we list all the components of the Riemann tensor for the metric (3.2):

$$\begin{aligned}R^v_{\ vuv} = {}& -\mathcal{L}_l\,\widetilde{\nu} - s^a\triangle_a\nu + \tfrac{1}{2}e^\lambda s^a\Sigma_{ab}s^b\nu + 2s^a\mathcal{L}_l\,\omega_a - e^{-\lambda}\omega_a\triangle^a\lambda + 3e^{-\lambda}\omega^a\omega_a + s^a\Sigma_{ab}\omega^b\\ &+ \tfrac{1}{2}e^\lambda s^a\,(\mathcal{L}_l\,\Sigma_{ab})\,s^b + \tfrac{1}{2}s^a\Sigma_{ab}\,\triangle^b\lambda - \tfrac{1}{4}e^{-\lambda}\triangle^a\lambda\,\triangle_a\lambda - \tfrac{1}{4}e^\lambda s^a\Sigma_{ab}h^{bc}\Sigma_{cd}s^d,\end{aligned} \tag{B1}$$
$$R^v_{\ vua} = -\tfrac{1}{2}\triangle_a\nu + \tfrac{1}{2}e^\lambda\nu s^m\Sigma_{ma} + \tfrac{1}{2}\omega^m\Sigma_{ma} + \mathcal{L}_l\,\omega_a + \tfrac{1}{2}e^\lambda s^m\mathcal{L}_l\,\Sigma_{ma} - \tfrac{1}{4}e^\lambda\Sigma_{am}h^{mn}\Sigma_{nq}s^q + \tfrac{1}{4}\triangle^m\lambda\,\Sigma_{ma}, \tag{B2}$$
$$R^v_{\ aub} = \tfrac{1}{2}e^\lambda\mathcal{L}_l\,\Sigma_{ab} + \tfrac{1}{2}e^\lambda\nu\Sigma_{ab} - \tfrac{1}{4}e^\lambda\Sigma_{am}h^{mn}\Sigma_{nb}, \tag{B3}$$
$$R^v_{\ vab} = \tfrac{1}{2}e^\lambda s^m\Sigma_{m[b}\triangle_{a]}\lambda + e^\lambda s^m\triangle_{[a}\Sigma_{b]m} - e^\lambda s^m\Sigma_{m[a}\omega_{b]} + 2\triangle_{[a}\omega_{b]} + \tfrac{1}{2}e^\lambda\Sigma_{n[a}\widetilde{\Sigma}_{b]m}h^{nm}, \tag{B4}$$
$$R^c_{\ abd} = -e^\lambda s^c\triangle_{[b}\Sigma_{d]a} - \tfrac{1}{2}e^\lambda s^c\Sigma_{a[d}\triangle_{b]}\lambda + \tfrac{1}{2}e^\lambda h^{cm}(\Sigma_{a[d}\,\widetilde{\Sigma}_{b]m} - \Sigma_{m[d}\,\widetilde{\Sigma}_{b]a}) - e^\lambda s^c\Sigma_{a[d}\omega_{b]} + {}^{(2)}R^c{}_{abd}, \tag{B5}$$
$$R^v_{\ acd} = \tfrac{1}{2}e^\lambda\triangle_{[c}\lambda\Sigma_{d]a} + e^\lambda\triangle_{[c}\Sigma_{d]a} + e^\lambda\Sigma_{a[d}\omega_{c]}, \tag{B6}$$
$$\begin{aligned}R^v_{\ avb} = {}& \tfrac{1}{2}e^\lambda\mathcal{L}_n\,\Sigma_{ab} + \tfrac{1}{2}e^\lambda s^m(\triangle_m\Sigma_{ab} - \triangle_b\Sigma_{am}) + \tfrac{1}{2}\triangle_a\triangle_b\lambda - \tfrac{1}{4}\triangle_a\lambda\,\triangle_b\lambda - \triangle_b\omega_a\\ &+ \tfrac{1}{2}e^\lambda\Sigma_{ab}w^c s_c + \tfrac{1}{4}e^\lambda s^c\triangle_c\lambda\,\Sigma_{ab} - \tfrac{1}{4}e^\lambda\Sigma_{mb}h^{mn}\widetilde{\Sigma}_{na} + \tfrac{1}{2}\omega_a\triangle_b\lambda + \tfrac{1}{2}\omega_b\triangle_a\lambda - \omega_a\omega_b\\ &- \tfrac{1}{2}e^\lambda\omega_b s^m\Sigma_{am} - \tfrac{1}{4}e^\lambda\triangle_b\lambda\,\Sigma_{ma}s^m,\end{aligned} \tag{B7}$$
$$\begin{aligned}R^v_{\ vva} = {}& \tfrac{1}{2}e^\lambda s^m\mathcal{L}_n\,\Sigma_{ma} + \mathcal{L}_n\,\omega_a + \tfrac{1}{2}e^\lambda s^m\,(s^n\triangle_n)\,\Sigma_{ma} - \tfrac{1}{2}e^\lambda\Sigma_{ma}h^{mn}\widetilde{\Sigma}_{nq}s^q + \tfrac{1}{2}\triangle_a\widetilde{\nu} + \tfrac{1}{2}\,(s^m\triangle_m)\,\triangle_a\lambda\\ &- 2s^m\triangle_a\omega_m - \tfrac{1}{4}e^\lambda\triangle_a\lambda\,s^m\Sigma_{mn}s^n - \tfrac{1}{2}e^\lambda s^m\,(\triangle_a\Sigma_{mn})\,s^n - \tfrac{1}{4}s^m\triangle_m\lambda\,\triangle_a\lambda + \tfrac{1}{4}e^\lambda s^m\Sigma_{ma}\,s^n\triangle_n\lambda\\ &+ \tfrac{1}{2}s^m\triangle_m\lambda\omega_a - \tfrac{1}{4}\widetilde{\Sigma}_{am}\triangle^m\lambda - \tfrac{1}{2}e^\lambda\,(s^m\Sigma_{mn}s^n)\,\omega_a + \tfrac{1}{4}e^\lambda s^m\Sigma_{mn}h^{nq}\widetilde{\Sigma}_{aq} + \tfrac{1}{2}\,(s^m\omega_m)\,\triangle_a\lambda\\ &+ \tfrac{1}{2}e^\lambda\,(s^n\omega_n)\,s^m\Sigma_{ma} - (s^m\omega_m)\,\omega_a + \tfrac{1}{2}\omega^m\widetilde{\Sigma}_{ma} + s^m\triangle_m\omega_a,\end{aligned} \tag{B8}$$
$$\begin{aligned}R^u_{\ avb} = {}& \tfrac{1}{2}e^\lambda\widetilde{\Sigma}_{ab}\widetilde{\nu} + \tfrac{1}{4}e^\lambda s^m\triangle_m\lambda\,\widetilde{\Sigma}_{ab} + \tfrac{1}{2}e^\lambda\mathcal{L}_n\,\widetilde{\Sigma}_{ab} + \tfrac{1}{2}e^\lambda s^m\triangle_m\widetilde{\Sigma}_{ab} - \tfrac{1}{4}e^\lambda\triangle_b\lambda\,s^m\widetilde{\Sigma}_{am} - \tfrac{1}{2}e^\lambda s^m\,\triangle_b\widetilde{\Sigma}_{am}\\ &- \tfrac{1}{2}e^\lambda\,(s_m w^m)\,\widetilde{\Sigma}_{ab} - \tfrac{1}{4}e^\lambda\widetilde{\Sigma}_{bm}h^{mn}\widetilde{\Sigma}_{an} + \tfrac{1}{2}e^\lambda s^m\widetilde{\Sigma}_{am}\omega_b,\end{aligned} \tag{B9}$$

$$
\begin{aligned}
R^{c}_{avb} = {}& \tfrac{1}{2}h^{cm}(\triangle_a\widetilde{\Sigma}_{mb} - \triangle_m\widetilde{\Sigma}_{ab}) - h^{cn}s_m{}^{(2)}R^m{}_{ban} - \tfrac{1}{2}e^{\lambda}s^c\mathcal{L}_n\,\Sigma_{ab} - \tfrac{1}{2}e^{\lambda}s^cs^m\triangle_m\Sigma_{ab} - \tfrac{1}{2}s^c\triangle_b\triangle_a\lambda \\
&+ \tfrac{1}{4}e^{\lambda}s^c\triangle_b\lambda\, s^m\Sigma_{am} + \tfrac{1}{2}e^{\lambda}s^cs^m\triangle_b\Sigma_{am} + s^c\triangle_b\omega_a + \tfrac{1}{4}s^c\triangle_a\lambda\,\triangle_b\lambda - \tfrac{1}{2}s^c\omega_b\triangle_a\lambda + \tfrac{1}{4}\triangle_a\lambda\, h^{cm}\widetilde{\Sigma}_{mb} \\
&+ \tfrac{1}{2}e^{\lambda}s^m\Sigma_{am}s^cw_b - \frac{1}{4}e^{\lambda}s^n\Sigma_{an}h^{cm}\widetilde{\Sigma}_{mb} - \tfrac{1}{2}s^c\omega_a\triangle_b\lambda + s^c\omega_a\omega_b - \tfrac{1}{2}h^{cm}\widetilde{\Sigma}_{mb}\omega_a \\
&- \tfrac{1}{2}e^{\lambda}\omega^m s_m\, s^c\Sigma_{ab} + \tfrac{1}{4}e^{\lambda}h^{cm}s^n\widetilde{\Sigma}_{mn}\Sigma_{ab} + \tfrac{1}{2}e^{\lambda}\widetilde{\Sigma}_{ab}\left(\tfrac{1}{2}h^{cm}s^n\Sigma_{mn} + e^{-\lambda}w^c - \tfrac{1}{2}e^{-\lambda}\triangle^c\lambda\right) \\
&- \tfrac{1}{4}e^{\lambda}s^m\triangle_m\lambda\,\Sigma_{ab}s^c + \tfrac{1}{4}e^{\lambda}s^c\Sigma_{nb}h^{nm}\widetilde{\Sigma}_{ma} - \tfrac{1}{4}e^{\lambda}s^n\widetilde{\Sigma}_{an}h^{cm}\Sigma_{bm}.
\end{aligned} \tag{B10}
$$

2. The Weyl scalars

We choose a complex-null tetrad $\{e^{\lambda}\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ such that $2m^{(\mu}\overline{m}^{\nu)} = h^{\mu\nu}$ and $\boldsymbol{l}$ and $\boldsymbol{n}$ are given by (3.8). In this tetrad the five Newman-Penrose Weyl scalars are in a Ricci flat spacetime

$$\Psi_0 = \tfrac{1}{4}e^{2\lambda}(2\mathcal{L}_l\,\Sigma_{ab} + 2\nu\Sigma_{ab} - \Sigma_{am}h^{mn}\Sigma_{bn})m^am^b, \tag{B11}$$

$$\Psi_1 = \tfrac{1}{4}e^{\lambda}(-2\triangle_a\nu + 2\omega^m\Sigma_{am} + 4\mathcal{L}_l\,\omega_a + \triangle^m\lambda\Sigma_{ma})m^a, \tag{B12}$$

$$
\begin{aligned}
\Psi_2 = {}& -\tfrac{1}{4}(2e^{\lambda}\mathcal{L}_n\,\Sigma_{ab} + 2\triangle_a\triangle_b\lambda - \triangle_a\lambda\triangle_b\lambda - 4\triangle_b\omega_a - e^{\lambda}\Sigma_{mb}h^{mn}\widetilde{\Sigma}_{an} + 2\omega_a\triangle_b\lambda \\
&+ 2\omega_b\triangle_a\lambda - 4\omega_a\omega_b + e^{\lambda}\triangle_b\lambda s^m\Sigma_{am} + 2e^{\lambda}s^m\Sigma_{am}\omega_b)m^a\bar{m}^b,
\end{aligned} \tag{B13}
$$

$$\Psi_3 = \tfrac{1}{4}(-4\mathcal{L}_n\,\omega_a - 2\triangle_a\widetilde{\nu} + e^{\lambda}s^m\Sigma_{mn}s^n\triangle_a\lambda + \triangle^m\lambda\widetilde{\Sigma}_{ma} + 2e^{\lambda}s^m\Sigma_{mn}s^n\omega_a - 2\omega^m\widetilde{\Sigma}_{ma})\bar{m}^a, \tag{B14}$$

$$\Psi_4 = -\tfrac{1}{4}(-2\widetilde{\Sigma}_{ab}\widetilde{\nu} - 2\mathcal{L}_n\,\widetilde{\Sigma}_{ab} + \widetilde{\Sigma}_{am}h^{mn}\widetilde{\Sigma}_{bn})\bar{m}^a\bar{m}^b. \tag{B15}$$

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