Self-similar scalar field collapse: Naked singularities and critical behavior

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Homothetic scalar field collapse is considered in this article. By making a suitable choice of variables the equations are reduced to an autonomous system. Then using a combination of numerical and analytic techniques it is shown that there are two classes of solutions. The first consists of solutions with a nonsingular origin in which the scalar field collapses and disperses again. There is a singularity at one point of these solutions; however, it is not visible to observers at a finite radius. The second class of solutions includes both black holes and naked singularities with a critical evolution (which is neither) interpolating between these two extremes. The properties of these solutions are discussed in detail. The paper also contains some speculation about the significance of self-similarity in recent numerical studies.

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I. INTRODUCTION

That gravitational collapse leads to black hole formation is widely accepted, yet comparitively little is known about the generic features of collapse. This is well exemplified by the lack of a precise formulation of the cosmic censorship hypothesis [1]. The reason for this is simply our lack of the mathematical tools necessary to analyze the evolution of generic initial data, although some progress is being made on this front [2]. Moreover the complexity of the Einstein field equations counsels retreat to a more tractable system which may capture the essence of the physics. Spherically symmetric general relativity coupled to a massless scalar field is one such system. Indeed this model has been studied in great detail both analytically [3] and numerically [4,5]. Christodoulou has rigorously established the global existence and uniqueness of solutions to the Einstein-scalar field equations and has discussed the general properties of these solutions [3]. He has even established a sufficient condition for the formation of a trapped surface in the future evolution of a given initial data set [6]. Numerical investigations have also provided useful insight into black hole formation. The most recent of these studies, by Choptuik [7], has revealed several intriguing phenomena which were hitherto unknown.

Choptuik considered the numerical evolution of initial data sets characterized by a single parameter (p say). The resulting families of solutions, $\mathcal{S}[p]$, include both geometries containing black holes, and geometries with only slight deviations from flatness, depending on the value of p. Between these two extremes lies a critical evolution with $p = p^*$, which signals the transition between complete dispersal and black hole formation. His most interesting results pertain to near critical $p \simeq p^*$ evolutions which exhibit a particularly simple strong field behavior. In fact, his results strongly suggest that near critical evolutions may be described by a single, universal solution of the field equations. Two quantitative features also emerge from his work. (i) Near critical evolutions contain echoes in the strong field region [if $a(r,t)$ is some form invariant quantity then $a(e^{-n\Delta}r, e^{-n\Delta}t) \simeq a(r,t)$ where n is an integer and $\Delta \simeq 3\cdot 4$]. Based on his extensive investigations Choptuik has conjectured that the exactly critical solution has an infinite train of echoes in the strong field region approaching the singularity. (ii) When black holes form in near critical evolutions the mass scales as $M_{\rm BH} \sim |p-p^*|^{\beta}$, where β is an apparently universal constant determined numerically to be about 0.37.

Obtaining analytical results about solutions which exhibit a discrete self-similarity of the type discussed by Choptuik has proved to be extremely difficult [8]. On the other hand, by assuming the existence of a continuous self-similarity some progress can be made [9]. Therefore this paper is concerned with homothetic spacetimes; it is assumed that there exists a vector field ξ such that the spacetime metric satisfies $\mathcal{L}_\xi g = 2g$, where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ. The homothetic collapse of perfect fluids has received a great deal of attention during the past few years [10], where the main thrust of work was to search for naked singularities [11]. The problem of homothetic, scalar field collapse has received comparitively little attention [5], and no examples of naked singularities which evolved from regular initial data were previously known for this form of matter.

The reduction of the homothetic scalar field equations to an autonomous system allows me to give a detailed description of all the solutions. Two interesting features emerge from the analysis: first the existence of solutions with naked singularities, and second the occurrence of phase transitions in the system. The critical point behavior is of two distinct forms. The first is a phase transition from solutions which (roughly) represent dispersal to geometries representing black holes. It has been suggested elsewhere [12] that the critical solution, the existence of which is implied by the numerical results, may be both on the verge of black hole formation and being a naked singularity. We will see below that the second phase transition corresponds to such a situation—the critical evolution lies at the boundary between black holes and naked singularities. In this article the term black hole is used

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rather loosely to mean that an apparent horizon exists and precedes a central singularity. Of course it is possible to obtain an asymptotically flat solution to the Einstein-scalar equations by cutting off the self-similar evolution at some advanced time and matching it to a less symmetric (not self-similar) exterior, in this way one would obtain an asymptotically flat, black-hole spacetime.

The paper is arranged as follows. In Sec. II the field equations for the self-similar collapse are derived in terms of a retarded time coordinate u and a radial coordinate r. These equations have already been derived in [5] and used to provide initial data in a search for naked singularities. A field redefinition transforms the equations into a nonlinear autonomous system which is amenable to standard techniques [13]. The properties of solutions representing collapse are discussed in detail in Sec. III, while a summary of the salient features is presented in Table I. Section IV contains some discussion of the results, and their possible significance.

The notation of [14] is adopted throughout the paper, and detailed calculations which might distract from the main line of thought are relegated to the Appendices.

II. THE FIELD EQUATIONS

Using retarded Bondi coordinates $\{u, r, \theta, \phi\}$ the spherical line element may be written

$$ds^2 = -g\bar{g}du^2 - 2gdudr + r^2 d\Omega^2 , \quad (2.1)$$

where $g = g(u,r)$, $\bar{g} = \bar{g}(u,r)$, and $d\Omega^2$ is the standard line element on the unit two sphere. The origin is singular unless $\bar{g} = g$ when $r = 0$. Furthermore it is convenient to normalize the coordinate u so that it represents the proper time for an observer at the origin; thus, we write

$$\bar{g}(u,0) = g(u,0) = 1 . \quad (2.2)$$

The Einstein-scalar field equations are then

$$(\ln g)_{,r} = 4\pi r(\psi_{,r})^2 , \quad (2.3)$$

$$(r\bar{g})_{,r} = g , \quad (2.4)$$

$$(\bar{g}/g)_{,u} = 8\pi r g^{-1}\left[(\psi_{,u})^2 - \bar{g}\psi_{,u}\psi_{,r}\right] , \quad (2.5)$$

where a comma denotes partial differentiation. $\psi = \psi(u,r)$ is a massless, minimally coupled scalar field satisfying

$$(\bar{g}r^2\psi_{,r})_{,r} = 2r\psi_{,u} + 2r^2\psi_{,ru} . \quad (2.6)$$

Spherical symmetry allows the introduction of a local mass function $m(x^\alpha)$ [15] defined by

$$1 - \frac{2m(x^\alpha)}{r} := g^{\alpha\beta}r_{,\alpha}r_{,\beta} = \frac{\bar{g}}{g} , \quad (2.7)$$

where r is the function which determines the area of the two-spheres. This mass function agrees with both the Arnowitt-Deser-Misner (ADM) and Bondi masses in the appropriate limits, and is equivalent to the Hawking quasilocal mass in this case.

A. Self-similar ansatz

The existence of a homothetic symmetry in a spherical spacetime implies that the metric depend only on $x = r/|u|$, and that the scalar field evolve as

$$\psi = \bar{h}(x) - \kappa \ln|u| , \quad (2.8)$$

where $\bar{h}$ is some function to be determined and κ is constant (see Appendix A for a proof of this fact). Writing

$$\bar{h}(x) = \int_0^x \frac{\gamma(\xi)}{\xi} d\xi , \quad (2.9)$$

the self-similar equations derived from (2.3)–(2.5) are

$$(x\bar{g})' = g , \quad (2.10)$$

$$xg' = 4\pi g(\gamma)^2 , \quad (2.11)$$

$$g - \bar{g} = 4\pi\left[2\kappa^2 x - (\bar{g} - 2x)(\gamma^2 + 2\kappa\gamma)\right] , \quad (2.12)$$

TABLE I. Properties of the self-similar solutions satisfying (2.2) and (2.15), and classified according to the value of κ.

Range of κ	Class	Uniqueness	Description
$4\pi\kappa^2 \geq 1$	I	Unique for each κ	Solutions have topology $R^3 \times R$. They are singular at the point $u = 0$ on $r = 0$. This singularity is never visible to an observer at finite radius.
$0 < 4\pi\kappa^2 < 1$	II	One parameter family of solutions. For each value of κ there may be as many as three subclasses.	(a) Solutions which contain an apparent horizon which precedes a spacelike singularity at $r = 0$. (b) Critical solutions with a null singularity at $u = 0$, $r = 0$. See Fig. 3. (c) For sufficiently small values of κ there also exist solutions with a naked central singularity. The Cauchy horizon is at $u = 0$.
$4\pi\kappa^2 = 0$		One parameter family of solutions	The equations may be integrated in closed form. The solutions include both black holes and dispersal of the scalar field, along with a single critical evolution which interpolates between these two extremes [9].

where a prime denotes differentiation with respect to x. In deriving (2.12) from (2.5) it is necessary to use (2.10) and (2.11) to eliminate derivatives of g and $\overline{g}$. The scalar field evolution is determined by

$$x(\overline{g}-2x)\gamma' = 2\kappa x - \gamma(g-2x) . \tag{2.13}$$

It is now a straightforward matter to show that (2.10)–(2.12) imply (2.13) provided $\gamma \neq -\kappa$. These equations have been derived previously by Goldwirth and Piran [5] and used to provide boundary conditions in a numerical search for naked singularities.

At the origin (2.2) and (2.12) imply either $\gamma(0) = 0$ or $\gamma(0) = -2\kappa$. Directly evaluating the trace of the stress-energy tensor for the scalar field one finds

$$T^{\alpha}{}_{\alpha} = \frac{\overline{g}}{gr^2}\left[\gamma^2 - \frac{x}{\overline{g}}(\gamma^2+\kappa\gamma)\right] \stackrel{r\to 0}{\longrightarrow} \frac{\gamma^2}{r^2} . \tag{2.14}$$

Clearly the solution can have a nonsingular origin only if

$$\gamma(0) = 0 . \tag{2.15}$$

This completes the specification of the initial conditions for Eqs. (2.10)–(2.12).

B. An equivalent autonomous system

The analysis of the above equations is facilitated by the field redefinitions

$$y = \overline{g}/g , \quad z = x/\overline{g} , \tag{2.16}$$

and the introduction of a new coordinate

$$\xi = \ln x . \tag{2.17}$$

Upon substitution into (2.10), (2.11), and (2.13) one obtains the three-dimensional, nonlinear autonomous system

$$\dot{z} = z\left(2 - y^{-1}\right) , \tag{2.18}$$

$$\dot{y} = 1 - (4\pi\gamma^2+1)y , \tag{2.19}$$

$$(1-2z)\dot{\gamma} = 2\kappa z - \gamma(y^{-1}-2z) . \tag{2.20}$$

The system is effectively two dimensional, however, since γ is determined by the algebraic relation

$$\gamma = -\kappa \pm \sqrt{\frac{(1+4\pi\kappa^2)-y^{-1}}{4\pi(1-2z)}} , \tag{2.21}$$

provided $y \neq 1/(1+4\pi\kappa^2)$. Further discussion is therefore couched in terms of a projection into the yz plane. Consistent with the initial condition (2.15) the positive square root is taken in (2.21); however, it must be emphasized that solutions may still evolve continuously onto the other leaf of the surface defined by taking the negative square root above. Indeed it is solutions of this type which have naked singularities.

Requiring that the mass function, defined in (2.7), should be positive or zero implies $y \leq 1$, while γ is real only if

$$\frac{(1+4\pi\kappa^2)-y^{-1}}{1-2z} \geq 0 . \tag{2.22}$$

Black hole formation is signaled by $y \to 0$ (technically this is the condition which locates an apparent horizon in the spacetime). The continuation of the solution to negative values of y will not be considered in the sequel, thus the integral curves of interest lie in the strip $0 \leq y \leq 1$.

It should be noted that γ is not continuous at

$$z = 1/2 , \quad y = 1/(1+4\pi\kappa^2) , \tag{2.23}$$

thus invalidating the usual existence and uniqueness theorems for systems of ordinary differential equations at this point. This has the important consequence that integral curves of the differential equations may intersect at (2.23). Furthermore, the continuation of such solutions is not always uniquely defined, in some cases there exists an infinite family of possibilities. This is discussed in more detail below and in Appendix C.

The system (2.18)–(2.20) has two stationary points, one on either leaf of (2.21), given by

$$y_{\pm} = \frac{1}{2} , \quad z_{\pm} = \frac{1}{1\pm\sqrt{4\pi}\kappa} , \quad \gamma_{\pm} = \pm\frac{1}{\sqrt{4\pi}} . \tag{2.24}$$

The nature of these points depends on the value of κ. It is discussed below and in Appendix B.

III. SELF-SIMILAR SOLUTIONS

The character of the solutions shows a strong dependence on the value of κ. Solving (2.10)–(2.13), subject to the regularity conditions on $r = 0$, gives rise to two distinct classes of solution according as $4\pi\kappa^2$ is greater than or less than unity. Class I solutions are neither black holes nor naked singularities although they are singular at one point on $r = 0$. The critical evolution ($4\pi\kappa^2 = 1$) also belongs to the first class. The transition from one class to the other as one adjusts κ is similar to the behavior discussed by Choptuik [7]. For each value of κ in the range $0 < 4\pi\kappa^2 < 1$ there exists a one-parameter family of solutions. For sufficiently small κ a second type of phase transition from solutions representing black holes to naked singularities occurs. A single critical evolution having a null singularity (which is *not* naked) interpolates between these two extremes. A similar phenomenon has been observed in Tolman-Bondi collapse [16]. For completeness, at the end of this section it is shown how the results of [9] fit into the overall picture.

A. Class I: $4\pi\kappa^2 > 1$

For κ in this range only the stationary point (y_+, z_+, γ_+) is of interest. This exact solution is equivalently written as

$$g = 2x(1+\sqrt{4\pi}\kappa) , \quad \overline{g} = x(1+\sqrt{4\pi}\kappa) . \tag{3.1}$$

It has a singular origin ($r = 0$), in fact there are two sheets of this singularity. Future directed, ingoing light rays terminate on the sheet located at $u = 0$, while outgoing light rays originate on the past sheet (the solution may be obtained by setting $\alpha = \beta = 0$ in Eq. (9) of [9]).

Locally, (3.1) is a positive attractor (see Appendix B). The global structure of the class I solutions is easily determined by examining the behavior of the integral curves in the region

$$\mathcal{A} = \{1/(1 + 4\pi\kappa^2) < y < 1,\ 0 < z < 1/2\} . \quad (3.2)$$

Noting that integral curves enter $\mathcal{A}$ across the lines $y = 1/(1 + 4\pi\kappa^2)$, $y = 1$, and $\{z = 0, y \geq 1/2\}$ it is evident that the solution originating at $z = 0$, $y = 1$ either terminates at the stationary point or leaves $\mathcal{A}$ across $z = 1/2$. In Appendix C it is shown that integral curves only cross $z = 1/2$ at $y = 1/(1+4\pi\kappa^2)$, and that the solution passing through this point is unique when $4\pi\kappa^2 > 1$. A direct consequence of this is that the solution with a nonsingular origin approaches the asymptotic form (3.1), its evolution being characterized by a sequence of decaying oscillations in y about the value $y = 1/2$. (See Fig. 1.)

These solutions do not contain black holes, yet there is a singularity at the origin when $u \to 0$. Is this singularity naked? Consider ingoing, radial null geodesics

$$\frac{dr}{du} = -\overline{g}/2 . \quad (3.3)$$

Since $x = -r/u$ the geodesics are given by

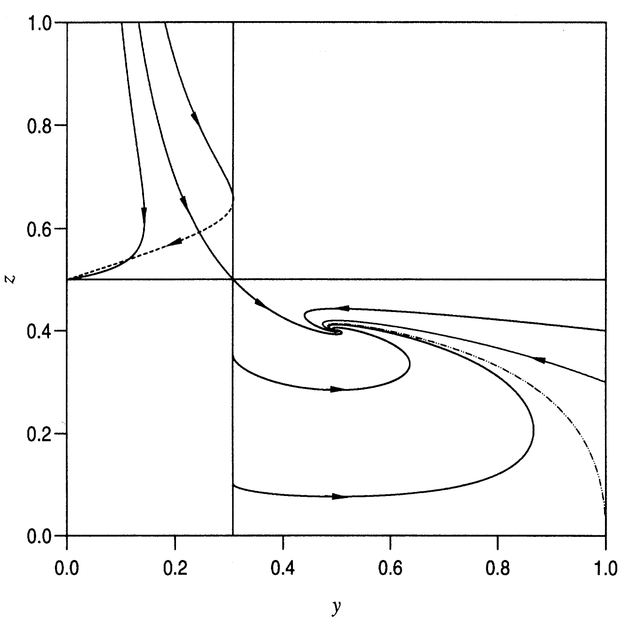

FIG. 1. Integral curves when $4\pi\kappa^2 = 2.25$. The solutions are seen to spiral towards the stationary solution. The dot-dashed curve from $y = 1$, $z = 0$ is the solution with a nonsingular origin. The dashed integral curve is on the negative leaf of (2.21) and is the continuation of the solid curve which reaches $y = 1/(1 + 4\pi\kappa^2)$.

$$\ln|u| = 2\int_{x_1}^{x} (\overline{g} - 2x')^{-1} dx' , \quad (3.4)$$

where x_1 is the radius at the point of intersection between the ingoing geodesic and the null cone $u = -1$. Since $z = x/\overline{g} < 1/2$ throughout the entire evolution the integrand in (3.4) is always bounded, and x decreases with increasing u. Thus ingoing light rays must reach $r = 0$ before $u = 0$, and they never reach the singularity provided $x_1 < \infty$.

In view of the asymptotic solution (3.1) one also finds that an observer at a (large) fixed radius takes an infinite proper time to reach $u = 0$. A central observer, on the other hand, reaches the singularity at $u = 0$ in finite proper time. It is, however, only after he has seen the entire history of the Universe in a tremendous flash. Thus these spacetimes have trivial topology, with a singularity only at $r = 0$ as $u \to 0$.

Similar arguments apply when $4\pi\kappa^2 = 1$, although in this case $\overline{g} - 2x \to 0$ as $x \to \infty$. It must be emphasized that this solution is unstable, in the sense that an arbitrarily small change in the value of κ drastically changes the character of the resulting solution. In particular, for smaller values the spacetime contains a black hole.

B. Class II: $4\pi\kappa^2 < 1$

Containing two subclasses of solutions and offering another example of a phase transition in gravitational collapse, from black hole spacetimes to naked singularities, these solutions are more interesting than those in class I. In fact for each κ in the range $0 < 4\pi\kappa^2 < 1$ there appears to be a continuous infinity of solutions with a regular origin. This is due to the failure of uniqueness at the singular point (2.23). Naked singularities develop only for sufficiently small values of κ.

Solutions with a nonsingular origin contain a null hypersurface, Γ say, on which $x =$ const. This corresponds to the point $y = 1/(1 + 4\pi\kappa^2)$, $z = 1/2$ in phase space. As mentioned earlier the standard uniqueness theorems break down at this point, and there is a one-parameter family of self-similar continuations beyond it. More general extensions which produce asymptotically flat spacetimes have been considered in [5].

Figure 2 shows various solutions to the system (2.18)–(2.20) when $4\pi\kappa^2 = 0.49$. It is clear from the diagram that all solutions (except one) evolve into black holes (signaled by the formation of an apparent horizon, $y \to 0$). Some of the solutions evolve onto the negative leaf of (2.21); y undergoes a single oscillation before decreasing monotonically to $y = 0$. While I have no analytic proof that all of these solutions contain black holes, the numerical results (as illustrated in Fig. 2) suggest this is true. The single exceptional solution exhibits a behavior which has been discussed elsewhere [9]; beyond Γ no black hole forms, instead the solution asymptotically approaches one of the stationary points (2.24). The spacetime is singular at the null surface $u = 0$, $r = 0$ where $y = 1/2$. This singularity lies at infinite redshift for ob-

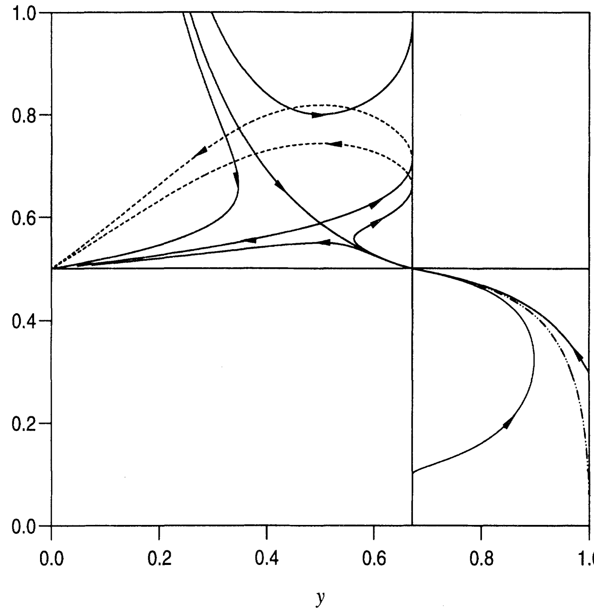

FIG. 2. Integral curves when $4\pi\kappa^2 = 0.49$. Here we see the integral curve representing the solution with a nonsingular origin reach the point $z = 1/2$, $y = 1/(1 + 4\pi\kappa^2)$. There is a one-parameter family of continuations past this point. All solutions of interest terminate at $y = 0$ (an apparent horizon in spacetime) except one which approaches the stationary point. Here also the dashed lines are integral curves on the negative leaf of γ in (2.21).

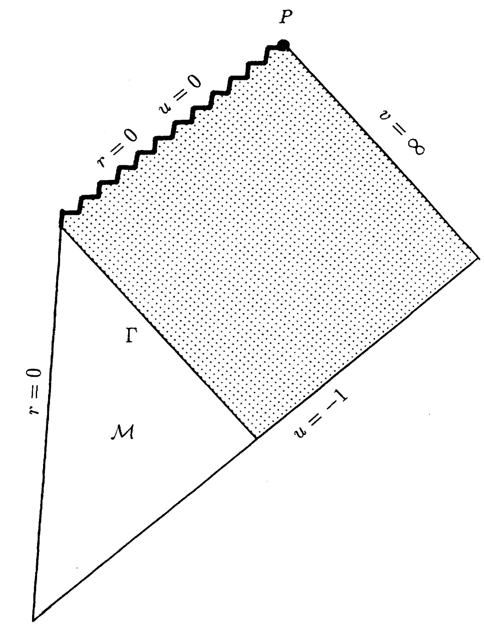

FIG. 3. A spacetime diagram for the critical evolution when $4\pi\kappa^2 < 1$. The unshaded region corresponds to the dot-dashed curves in Fig. 2 and Fig. 4. The shaded region shows a continuation past Γ [where $z = 1/2$, $y = 1/(1+4\pi\kappa^2)$] which approaches a stationary point. The singularity at $r = 0$ is null.

servers at large radius. Figure 3 is a spacetime diagram for this exceptional case.

For smaller values of κ more complicated behavior is possible, and indeed another type of phase transition is apparent—from black holes to naked singularities. When $z > 1/2$ the $x =$ const surfaces are spacelike. If no black hole forms then $x \to \infty$ as $u \to 0$ [see Eq. (3.4)], corresponding to a Cauchy horizon in the spacetime. For sufficiently small κ there are solutions which evolve from the singular point onto the negative leaf of (2.21), asymptoting to $y = 1/(1 + 4\pi\kappa^2)$ as $z \to \infty$. A typical solution of this type is shown in Fig. 4, for $4\pi\kappa^2 = 0.25$. It is not difficult to obtain an approximate solution in the large z limit, and hence to show that the Cauchy horizon is nonsingular. For large z Eq. (2.19) becomes

$$\dot{y} \simeq 1 - (1 + 4\pi\kappa^2)y \; . \tag{3.5}$$

Integrating and inserting the result in (2.18) implies

$$y \simeq \frac{1}{1 + 4\pi\kappa^2} + cx^{-(1+4\pi\kappa^2)}, \tag{3.6}$$

$$z = \frac{x}{g} \simeq dx^{1-4\pi\kappa^2} \; , \tag{3.7}$$

where c and d are arbitrary constants of integration. It is now straightforward to verify the regularity of the Ricci scalar on the Cauchy horizon, using (2.21) and (2.14):

$$T^{\alpha}{}_{\alpha}|_{u=0} = \frac{\kappa^2}{(1 + 4\pi\kappa^2)r^2}. \tag{3.8}$$

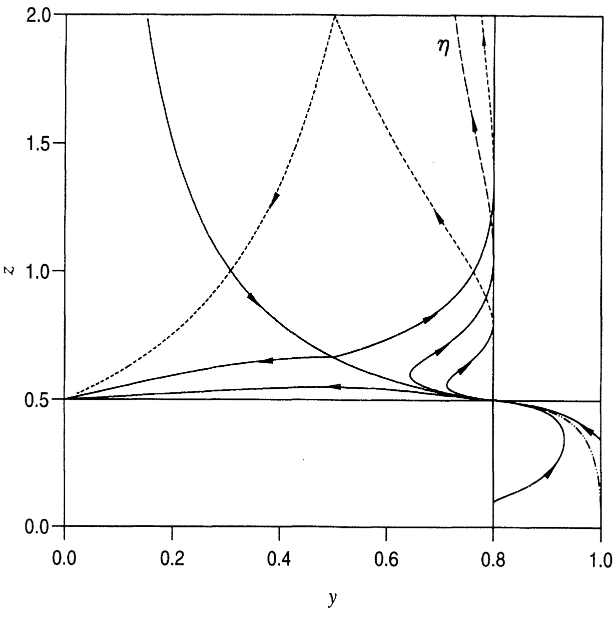

FIG. 4. Integral curves when $4\pi\kappa^2 = 0.25$. η is an example of an integral curve which evolves onto the negative leaf of γ in (2.21) but does not reach $y = 0$, instead it attains a minimum value of y and then asymptotes to $y = 1/(1 + 4\pi\kappa^2)$. Such solutions have a naked singularity at $r = 0$.

Only as $r \to 0$ is this quantity singular, indicating the existence of a naked singularity at $u = 0$, $r = 0$ in these solutions. The Cauchy horizon is also a null orbit of the homothetic Killing vector; therefore, the work of Lake and Zannias [17] implies that the singularity is strong in the sense that tidal forces diverge at it. Notice also that the metric can be made manifestly regular at the Cauchy horizon by transforming to the new coordinate given by

$$dU = (-u)^{-4\pi\kappa^2} du \,. \tag{3.9}$$

These spacetimes confirm that naked singularities, which evolve from regular initial data, also exist for scalar field sources. Whether they are stable to nonhomothethic, not to mention nonspherical, perturbations is an open question although the work of Goldwirth and Piran [5] suggests that they are not.

There are two critical points where phase transitions are observed. Each of the exactly critical solutions, which interpolates between naked singularities and black holes, asymptotically approaches one of the stationary points (2.24). Therefore these spacetimes have the structure shown in Fig. 3, with a null singularity at $u = 0$, $r = 0$.

As κ decreases further, the topology of the phase space changes slightly; only a single phase transition occurs going from black holes to naked singularities. The solutions (including the single critical evolution) are the same as those already discussed.

C. The Roberts solution: $\kappa = 0$

When $\kappa = 0$ the scalar field is a function of $x = -r/u$ only and Eqs. (2.10)–(2.12) can be exactly integrated [9]. The resulting solution was first discovered by Roberts [18] and may be written as

$$\overline{g} = \{(p^2 + x^2)^{1/2} - 2p^2\}/x, \tag{3.10}$$

$$g = 2\overline{g}\left(1 \pm \sqrt{1 - 4p^2(2x\overline{g} - \overline{g}^2)}\right)^{-1}, \tag{3.11}$$

and

$$\gamma = p/\sqrt{4\pi(p^2 + x^2)} \,. \tag{3.12}$$

The integral curves are plotted in Fig. 5. The solutions are labeled by p and exhibit critical point behavior. When $p = 1/2$ the integral curve leaves $z = 1/2$, $y = 1$ and approaches the stationary point (2.24). This exactly critical evolution lies between solutions which contain black holes ($y \to 0$, $z \to 1/2$ along the integral curves) and those which evolve back to flat space [$y \to 1/(1 + 4\pi\kappa^2)$, $z \to \infty$ along the integral curves]. Since this is a saddle point we see that near critical evolutions (sub- or supercritical) can approach this point arbitrarily closely before moving away in their respective directions.

The case $\kappa = 0$ is therefore exceptional because the subcritical evolutions have zero mass on the Cauchy horizon, $u = 0$, as is readily seen by taking the limit $x \to \infty$ in the above solution. This behavior is also apparent in the asymptotic approach of the integral curves to the line $y = 1$. As discussed in [9] the natural continuation past $u = 0$ is Minkowski space since there is no material flux across this surface on which $m = 0$. Notice however that there is no self-similar extension of these solutions to $r = 0$ which is nonsingular.

This discussion places the results obtained in [9] within the more general context of self-similar spacetimes with scalar field matter sources.

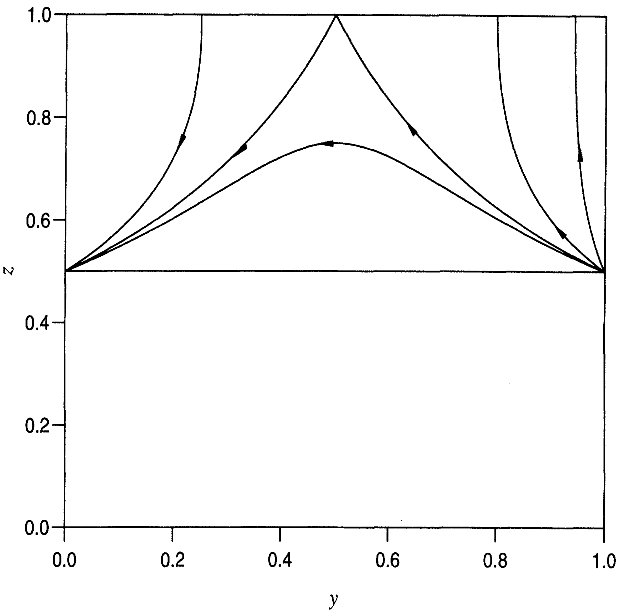

FIG. 5. The exact expression for the solutions when $\kappa = 0$ is given in the text, however, this diagram is for comparison with the other cases. Note that y=1 is now a solution to the equations and corresponds to flat space. For this reason no naked singularities exist for any value of p when $\kappa = 0$.

IV. DISCUSSION

Spherically symmetric, homothetic spacetimes have received a great deal of attention over the last few years due to the ease with which it is possible to construct naked singularities in such spacetimes. Recent numerical studies of spherical collapse [7,19] suggest that self-similarity may play an important role in describing the approach to the singularity in gravitational collapse. This study of scalar field collapse was in fact motivated by the work of Choptuik, where he observed discrete self-similarity in solutions on the verge of black hole formation. However it is not clear how a continuous self-similarity, as discussed here, could be at the center of the results which he has obtained. Nevertheless some interesting features do emerge from the study of spacetimes with a homothetic symmetry; the properties of the solutions are summarized in Table I.

One of the most intriguing results obtained by Choptuik was the scaling law for black hole mass, unfortunately self-similar spacetimes cannot have finite mass black holes. In order to obtain an asymptotically flat spacetime it is necessary to cut off the self-similar evolution at some advanced time, and consider a suitable continuation (which is not self-similar). Goldwirth and Piran [5] did exactly this, although they did not examine the behavior of black hole mass as the critical point $4\pi\kappa^2 = 1$ was approached. This question is currently under active investigation [20], and it will be interesting to see if the mass exhibits the same behavior which has been observed elsewhere [7,19,21].

We have in self-similar scalar field collapse further examples of spacetimes which violate cosmic censorship. One might be surprised about this were it not for the plethora of examples which now exist. What emerges from these examples (generally) is that there do exist initial data sets which when evolved according to the Einstein equations lead to naked singularities; however, the genericity of these data is far from clear. It therefore seems that the thrust of any attempt to formulate (and prove) cosmic censorship must address this issue directly. Some interesting preliminary results have been obtained by Lake [22] where he has shown that (spherically symmetric) spacetime in the neighbourhood of a naked singularity may be approximately self-similar. It therefore seems that future work on naked singularities must consider deviations from the symmetric situations treated to date.

Finally in searching for a theoretical understanding of the results obtained by Choptuik [7] and Abrahams and Evans [21] one might consider the obvious generalization of hometheticity to a conformal symmetry; that is to suppose the existence of a vector field ξ such that

$$\mathcal{L}_\xi g = \Omega(x) g \; . \tag{4.1}$$

If the dependence on position in Ω is weak, solutions might behave like self-similar solutions with some sort of superimposed periodicity. It would seem interesting to investigate this possibility.

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APPENDIX A: SCALAR FIELD EVOLUTION IN SPHERICAL, SELF-SIMILAR SPACETIMES

Results of Defrise-Carter [23] imply that a spherical spacetime with a homothetic symmetry (i.e., there exists a vector ξ such that $\mathcal{L}_\xi g = 2g$) can be written in the form

$$ds^2 = e^{2t}[g_1(x)dt^2 + g_2(x)dx^2 + e^{2x}d\Omega^2], \tag{A1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, and the similarity vector is $\xi = \partial/\partial t$.

The exponential dependence of the metric on t guarantees that the Christoffel symbols and hence the Ricci tensor $R_{\mu\nu}$ are independent of this coordinate. This may be expressed covariantly as

$$\mathcal{L}_\xi R_{\mu\nu} = 0 \; . \tag{A2}$$

Before examining the implication of this for a self-similar spacetime which satisfies Einstein's equations with scalar field matter, let me show that (A1) can be recast into the form used in Sec. II.

Introduce new coordinates r and u defined by

$$r = \exp(t + x), \quad u = rG(x), \tag{A3}$$

where $G(x)$ is to be determined. Substituting them into the line element (A1) and requiring u to be null one obtains the ordinary differential equation

$$\frac{dG}{dx} = G\left(-1 \pm \sqrt{-g_2(x)/g_1(x)}\right) \tag{A4}$$

which determines $G(x)$. Furthermore the line element reduces to

$$ds^2 = -\bar{g}(r/u)g(r/u)du^2 - 2g(r/u)dudr + r^2 d\Omega^2 \; , \tag{A5}$$

where r/u is related to x by (A3). Clearly this means that g and $\bar{g}$ are functions of r/u as stated in Sec. II. The similarity vector is

$$\xi = r\partial_r + u\partial_u \tag{A6}$$

in these coordinates.

Now in view of (A2) and the Einstein field equations

$$R_{\mu\nu} = 8\pi\psi_{,\mu}\psi_{,\nu} \tag{A7}$$

the scalar field must satisfy

$$\mathcal{L}_\xi(\psi_{,\mu}) = 0 \; . \tag{A8}$$

Assuming that ψ is independent of θ and ϕ, Eq. (A8) is readily integrated to

$$\frac{\partial\psi}{\partial r} = \frac{\gamma(r/u)}{r} \, , \tag{A9}$$

$$\frac{\partial\psi}{\partial u} = \frac{-\gamma(r/u)}{u} \, , \tag{A10}$$

where γ is an arbitrary function of r/u. (Integrability was used to reduce the number of arbitrary functions to one.) The general solution of these coupled equations is, therefore,

$$\psi = \bar{h}(r/u) - \kappa \ln|u| - \beta \ln|r| \; . \tag{A11}$$

In particular β may be set to zero by absorbing $\beta(\ln|u| - \ln|r|)$ into $\bar{h}$, thus reducing (A11) to (2.8).

APPENDIX B: THE STATIONARY POINTS

The discussion in Sec. III relies on the properties of the stationary points of Eqs. (2.18)–(2.20). In general there exists two such points given by

$$y_\pm = \frac{1}{2}\,, \quad z_\pm = \frac{1}{1 \pm \sqrt{4\pi}\kappa}\,, \quad \gamma_\pm = \pm\frac{1}{\sqrt{4\pi}}\,. \qquad \text{(B1)}$$

Provided $4\pi\kappa^2 \neq 1$ it is straightforward to linearize about each of these points and hence to analyze the topology of the phase space in their neighborhoods. The eigenvalues are

$$\lambda_{1,2} = \frac{1 - z_\pm}{2z_\pm - 1} \pm \sqrt{\frac{(1 - z_\pm)^2 + 4(2z_\pm - 1)}{(2z_\pm - 1)^2}}\,, \qquad \text{(B2)}$$

where $z_\pm$ may be chosen independently of the sign of the square root.

When $4\pi\kappa^2 > 1$ only z_+ is relevant for the discussion in Sec. III. Both eigenvalues are real and have the same sign, $\lambda_{1,2} < 0$, when $1 < 4\pi\kappa^2 \leq 4/3$ so that the stationary point is an attractive node. Once $4\pi\kappa^2 > 4/3$ the eigenvalues become complex conjugate, and since

$$2z_+ - 1 < 0\,, \quad 1 - z_+ > 0 \qquad \text{(B3)}$$

they have a negative real part. This is a positive attractor, with spiral behavior.

When $0 \leq 4\pi\kappa^2 < 1$ both stationary points are of interest. Simply noting that

$$2z_\pm - 1 > 0\,, \quad 1 - z_+ > 0\,, \quad 1 - z_- < 0 \qquad \text{(B4)}$$

the discussion of both is easily combined. Clearly the eigenvalues are real and have opposite signs since

$$|1 - z_\pm| < \sqrt{(1 - z_\pm)^2 + 4(2z_\pm - 1)}\,. \qquad \text{(B5)}$$

Thus they are saddle points.

APPENDIX C: THE SINGULAR LINE $Z = 1/2$

In the above analysis it is important that integral curves cannot cross $z = 1/2$ except at $y = 0$ or at $y = 1/(1 + 4\pi\kappa^2)$. We now show that this is so, and derive the solution in the neighborhood of $z = 1/2$, $y = 1/(1 + 4\pi\kappa^2)$. The analysis is split into two cases.

(i) Suppose an integral curve crosses $z = 1/2$ at $y_0 \neq 1/(1 + 4\pi\kappa^2)$. Writing $z = 1/2 + \zeta$ and considering the $\zeta \to 0$ limit of (2.18), (2.19), and (2.21) it is a straightforward matter to derive

$$\frac{dy}{d\zeta} \simeq \frac{(1 + 4\pi\kappa^2)y^2 - y}{(2y - 1)\zeta}\,. \qquad \text{(C1)}$$

Integrating this equation gives

$$\ln|\zeta| \simeq \text{const} + \frac{(1 - 4\pi\kappa^2)}{(1 + 4\pi\kappa^2)} \ln|(1 + 4\pi\kappa^2)y^2 - y|\,. \qquad \text{(C2)}$$

Examining this expression shows that $y_0 = 0$ is the only place where integral curves may intersect $z = 1/2$.

(ii) It is necessary to treat the case when $\zeta|_{y=1/(1+4\pi\kappa^2)} = 0$ separately since the limit $z \to 1/2$ in (2.21) is more delicate. For this purpose we introduce ζ as above and write

$$y = \frac{1}{1 + 4\pi\kappa^2} + \eta, \qquad \text{(C3)}$$

where $\eta \ll 1/(1 + 4\pi\kappa^2)$; thus,

$$\gamma \simeq -\kappa \pm \frac{1 + 4\pi\kappa^2}{\sqrt{8\pi}}\sqrt{\frac{\eta}{-\zeta}}\,. \qquad \text{(C4)}$$

Substituting this approximate expression for γ into (2.19), and using ζ as the independent variable we arrive at the equation

$$\frac{d\eta^{1/2}}{d\zeta} - \frac{1 + 4\pi\kappa^2}{2\zeta(1 - 4\pi\kappa^2)}\eta^{1/2} \simeq \pm\frac{\kappa\sqrt{8\pi}}{(-\zeta)^{1/2}(1 - 4\pi\kappa^2)} \qquad \text{(C5)}$$

and find

$$\eta^{1/2} \simeq \frac{\sqrt{8\pi}}{4\pi\kappa}(-\zeta)^{1/2} + c(-\zeta)^{(1+4\pi\kappa^2)/(1-4\pi\kappa^2)}\,, \qquad \text{(C6)}$$

where c is a constant of integration. Now the initial condition is $\eta = 0$ when $\zeta = 0$ so that there are two distinct possibilities; when $4\pi\kappa^2 > 1$ the integration constant must vanish, implying that a single integral curve passes through this point. On the other hand if $4\pi\kappa^2 < 1$ the constant is not fixed by the initial conditions and there is a one-parameter family of curves passing through $\eta = 0 = \zeta$. It is exactly this fact which gives rise to the variety of solutions in class II.

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