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# Past-instability conjecture and cosmological attractors in generalized isotropic universes

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We conjecture that all homogeneous and isotropic physically reasonable cosmological solutions of general relativity theory are in general past unstable in the generalized framework of higher order gravity theories. In this respect we provide a detailed perturbation analysis of the most interesting of these solutions and find that the results support our conjecture. We show that, in general, radiation solutions of higher order gravity are nonperturbative (bouncing or singular) as we approach the singularity in the past. A well-known flat radiation solution of quadratic gravity is shown not to be an attractor as $t \to 0$. We prove that the quasiexponential phase in higher order gravity theories cannot be an attractor to solutions which may describe preinflationary stage in these theories. However, this last conclusion may be altered if additional conditions are imposed and this situation is similar to the stability properties of the Starobinski inflationary solution. Other examples of nongeneric-type in accordance with our conjecture are discussed and these include the Milne universe.

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## I. INTRODUCTION

There has been an extensive number of investigations relating to classical properties such as the existence and stability of cosmological models of theories of gravity with quantum corrections (higher-derivative theories of gravity) and in particular in the framework of a subset of these theories, the so-called higher order gravity theories (see below for definitions). These analyses showed that interesting solutions describing the early history of the Universe exist as self-consistent solutions in the "extended" solution space of higher-derivative gravity theories.

For example, in the context of inflationary cosmology (see [1] for a review), it is well known that Starobinski's inflation [2], which historically preceded Guth's inflation [3], implies that in theories with quantum one-loop contributions the corresponding field equations admit a certain class of nonsingular, homogeneous, and isotropic solutions which are in particular of the de Sitter-type. However, one has to add that, as in all known forms of inflationary cosmology, for Starobinski inflation to become a viable inflationary model some sort of fine-tuning must be made of the type $\kappa_1 \gg \kappa_2$, i.e., of the numerical coefficients appearing in the quantum corrections of the stress-energy tensor for free, massless, conformally invariant fields (see [4], Sec. II).

The problem of finding the complete spectrum of physically realistic solutions in higher order gravity theories is a difficult one, owing partly to the wide choice of parameters involved and partly to the mathematical complexity of the field equations. In particular, restricting to four-dimensional homogeneous and isotropic cosmological models of constant curvature $\sigma$ taking the values $\sigma = \pm 1, 0$ and with a perfect fluid stress tensor with an equation of state $p = \gamma\rho$, $\gamma \in [-1, 1]$, we see that for the natural choice of gravitational Lagrangian corresponding to higher order gravity theories,

$$f(R) = \sum_{n=1}^{m} a_n R^n \ , \tag{1}$$

where $a_n$ are constants and $R$ is the scalar curvature, there is a three-parameter solution space denoted here as $(\gamma, \sigma, m)$. [Observe that the solution space of general relativity without a cosmological constant corresponds to the simplest choice $(\gamma, \sigma, 1)$.] Since it is almost impossible to solve the general $(\gamma, \sigma, m)$-field equation obtained by varying the action associated with (1), cosmological solutions are known only for some special choices of the parameters $\gamma, \sigma$, and $m$.

Consequently very few physically realistic classical solutions are known. The special case of dust, $\gamma = 0$, $m = 4/3$, and $\sigma = \pm 1, 0$, provides an example of a set of solutions which are regular near the singularity and approach the corresponding Friedmann-Robertson-Walker (FRW) solutions of general relativity for large times [5]. Breizman *et al.* [6] found another interesting set of regular solutions corresponding to the choice $\gamma = 1/3$, $m = 4/3$, and $\sigma = \pm 1, 0$. For higher values of $m$ the general situation is less well understood. Barrow and Ottewill [7] (see also [8]) showed that there is a quasi-de Sitter solution for $\sigma = 0$, $\gamma = -1$, and $m = 2$. This result established the possibility of inflation in theories of the general Lagrangian form (1). The regular solution of Ruzmaikina and Ruzmaikin [9] lies at the point (1/3, 0, 2) in the three-space $(\gamma, \sigma, m)$. A recent rediscovery of the last two solutions together with a more complete and rigorous stability analysis covering essentially the previously unknown cases $\sigma \neq 0$ was given in [10] (see also below). This is probably all that is currently known about the existence of special exact homogeneous and isotropic

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solutions in higher order gravity theories.

The only alternative source of information about the solution space of higher order gravity theories comes from the so-called conformal equivalence theorem [11]. This has the simple consequence that all solutions of general relativity plus a scalar field matter source are necessarily solutions of higher order gravity theory derived from (1) (but *not* vice versa, cf. [12]). However, (probably all) stable solutions of general relativity are generally unstable in the enlarged solution space of higher order gravity [7,9,10], and so it becomes relevant to be able to find directly solutions to the higher order gravity field equations and analyze their stability properties.

The issue of whether or not cosmological solutions in the generalized framework of higher derivative theories of gravity are *stable in the future* has generated much controversy (and also some interesting mathematical problems of a singular perturbation character, cf. [7]) over the past decade and is still far from being solved. The picture that emerges is that FRW, flat, radiation-filled solutions of the generalized theory tend to become future stable and approach the corresponding FRW solutions of general relativity, while nonflat, radiation-filled solutions are generally future unstable and may have very different global behavior for large times from the corresponding familiar case met in general relativity (cf. [9,7,10]).

There are two things for the "viability" of the extant cosmological solutions in the higher order field equations which are often quoted in the literature or assumed implicitly: First, since higher order quantum corrections are assumed to come from (an as yet nonexistent) quantum theory of gravity, physically interesting solutions of higher order gravity theories are expected to become *regular* (i.e., *bouncing*) near the "singularity" as $t \to 0$. Examples of regular solutions include the Ruzmaikin and Ruzmaikina solution and the Gurovich solution cited above. It is unknown whether or not inflation in higher order gravity starts off regularly, and we note that this subtle issue is not completely settled even in general relativity (see, for example, [13]). The second property for viability is of course *stability* or more accurately, the expectation that physically realistic solutions of higher order gravity must asymptotically approach the general relativistic FRW solutions as $t \to \infty$ [10].

However, one thing that is almost never mentioned is the issue of whether or not cosmological solutions of general relativity are *past stable*, i.e., whether they are stable as $t \to 0$, in higher order gravity theories. This is closely related to the question of whether the perturbation analysis used in proving stability breaks down near the singularity so that nonperturbative solutions of higher order gravity exist near $t = 0$. Stated simply, past instability of a cosmological spacetime in higher order gravity theories means that the Universe is not expected to behave in a manner known in general relativity and consequently new patterns of early evolution are generally to be expected in these frameworks.

Finally, another open issue which in our opinion deserves attention is the question of whether or not the known particular exact cosmological solutions of higher order gravity theories are attractors for all other solutions in the solution space of these theories as $t \to 0$. This is equivalent to investigating whether the above exact solutions are generic solutions of the respective theory.

In this paper we show four things. First, we provide a detailed perturbation analysis for all solutions of the type $(1/3, \sigma, m)$ as $t \to 0$, i.e., general (flat and nonflat) radiation solutions of higher order gravity. The results indicate that these solutions are in general nonperturbative (bouncing or singular) as $t \to 0$, but they may be, in particular, singular (perturbative) as $t \to 0$ if additional conditions are imposed. This analysis reinforces and extends previous studies of the problem (see [9,7]) and completes the stability analysis program announced in [10]. The important result deriving from the above perturbation analysis for $t \to 0$ is that it is sensible to attempt to construct particular exact solutions of higher order gravity since they are in general nonperturbative for $t \to 0$ and, hence may reveal as yet unknown features of the early Universe.

Second, we prove by a simple perturbation analysis that the solution (1/3, 0, 2) first discovered in [9] is not an attractor on approach to the singularity at $t = 0$. This result indicates that the behavior towards the singularity of flat, homogeneous, and isotropic radiation cosmologies may be completely different to that imagined in [9].

Third, we find a solution which has $\rho = 0$ and lies at $(\gamma, -1, 2)$ of the solution space $(\gamma, \sigma, m)$, i.e., it describes a Milne-type universe in quadratic theories of gravity. (Of course, this solution is expected in view of the conformal equivalence theorem [11].) In this case too we find that the solution $(\gamma, -1, 2)$, $\rho = 0$, is not generic in the neighborhood of $t = 0$.

Fourth, we show that the Barrow-Ottewill inflationary universe [7] which corresponds to a quasi-exponential early phase is not an attractor of solutions which may describe a preinflationary phase. However, this interesting solution may become an attractor if additional conditions are imposed. This situation is similar to the stability properties of the Starobinski inflationary solution [2].

The organization of the paper is as follows. In Sec. II we briefly review the perturbation formalism developed in [10] and establish notation. In Sec. III we give a detailed stability analysis of radiation solutions of higher order theories of gravity which establishes the fact that these solutions are in general of a nonperturbative character as $t \to 0$ and may describe regular universes which cannot be found explicitly by means of perturbation expansions. In Sec. IV we examine via perturbation the (only known) flat and radiation-filled solution in quadratic gravity of Ruzmaikin and Ruzmaikina, the Barrow-Ottewill solution, and the Milne-type solution and conclude that all three aforementioned exact particular solutions are in general *not* generic in the solution space of the respective theories. We summarize and discuss our results in Sec. V.

## II. BASICS OF THE PERTURBATION FORMALISM

In the following we consider a higher order gravity theory given by a generally covariant Lagrangian density of the form

$$L_{\rm HOG} = [f(R) + \kappa L_m](-g)^{1/2} \ , \qquad (2)$$

where $f(R)$ given by (1) is assumed to be an analytic function of the scalar curvature $R$ and $L_m$ represents possible matter couplings.

By varying the action associated with $L_{\rm HOG}$ with respect to the metric tensor $g_{ab}$ one obtains the field equations

$$f' R_{ab} - \tfrac{1}{2} f g_{ab} - \nabla_a \nabla_b f' + g_{ab} \Box f' + \kappa T_{ab} = 0 \ , \quad (3)$$

where $\Box = g_{ab}\nabla_a\nabla_b$, $\nabla_a$ is the usual covariant differential operator, a prime denotes $\partial/\partial R$, and we identify the stress-energy tensor $T_{ab}$ with the variational derivative $(2/\sqrt{-g})\delta(\sqrt{-g}L_m)/\delta g^{ab}$. We shall make the assumption that $T_{ab}$ represents the stress energy tensor of a perfect fluid with density $\rho$ and pressure $p$. Thus $p = \gamma\rho = \gamma T_{00}$ and so $\rho = T_{00} = \rho_0 a^{-3(\gamma+1)}$ with $\rho_0$ constant. Our analysis will focus on $\sigma = \pm 1, 0$ FRW solutions to the field equations (3).

The standard FRW metric in polar coordinates is

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\sigma r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \ , \qquad (4)$$

where the values $\sigma = +1, 0, -1$ correspond to closed, flat, open three-surfaces (and $\alpha, \beta = 1, 2, 3$). Below we follow the sign conventions of [14]. For this metric,

$$R_{00} = 3\frac{\ddot{a}}{a}, \qquad (5)$$

$$R_{\alpha\beta} = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{\sigma}{a^2}\right] g_{\alpha\beta} \ , \qquad (6)$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\sigma}{a^2}\right) . \qquad (7)$$

Note that $\Box R = -(\ddot{R} + 3\frac{\dot{a}}{a}\dot{R})$, and the overdots denote derivatives with respect to $t$.

Then the only necessary field equation is the (00) component of the field equations (3). This reads

$$f' R_{00} + \tfrac{1}{2} f + 3 f'' \frac{\dot{a}}{a}\dot{R} + T_{00} = 0 \ , \qquad (8)$$

or, using (5)–(7),

$$f''[a^2\dot{a}\dddot{a} + a\dot{a}^2\ddot{a} - 2\dot{a}^4 - 2\dot{a}^2\sigma] + \tfrac{1}{6} f' a^3 \ddot{a} + \tfrac{1}{36} f a^4 + \tfrac{1}{18} a^4 T_{00} = 0 \ . \quad (9)$$

To write down the general stability equation for solutions of Eq. (3) we let $a_0$ be a particular *exact* solution of (3) and look for perturbed solutions of the form

$$a(t) = a_0(t)[1 + \varepsilon(t)], \quad |\varepsilon(t)| \ll 1 \ . \qquad (10)$$

Substituting the perturbed solution (1) in the (00) component of the field equations (9) and linearizing about the exact solution $a_0(t) = (t - \sigma t^2)^{1/2}$ we obtain a differential equation for the perturbation $\varepsilon(t)$ (see [15]):

$$\dddot{\varepsilon}(t) + \left[\frac{2\sigma}{\mu}\left(\frac{1}{\nu} + \frac{3\nu}{\mu}\right) + \frac{5}{2}\frac{\nu}{\mu}\right]\ddot{\varepsilon}(t) - \left[\frac{4\sigma}{\mu}\left(1 - \frac{3f_0'''}{f_0''}\frac{\nu^2}{\mu_2}\right) + \frac{f_0'}{3f_0''} + \frac{\nu^2}{\mu^2}\right]\dot{\varepsilon}(t)$$
$$-\left\{2\sigma\left(\frac{f_0'}{3f_0''}\frac{1}{\nu} + \frac{\nu}{\mu^2} + \frac{\sigma}{\mu}\left[\frac{2}{\nu} + \frac{9f_0'''}{f_0''}\frac{\nu}{\mu^2}\right]\right) + \frac{4\rho_0}{9f_0''\mu\nu}\right\}\varepsilon(t) = 0 \ , \quad (11)$$

where $\mu = t - \sigma t^2$ and $\nu = \dot{\mu}$. This third-order, linear equation describes the behavior of homogeneous, isotropic perturbations to radiation solutions of general relativity in closed ($\sigma = +1$), flat ($\sigma = 0$), and open ($\sigma = -1$) FRW universes in the context of higher order gravity theories derived from the gravitational Lagrangian (2). By solving this equation for the perturbation $\varepsilon(t)$ we can decide whether or not the $\sigma = 0, \pm 1$ FRW radiation solutions of general relativity are stable against homogeneous and isotropic perturbations in the context of the $f(R)$ theory. Note, in particular, that Eq. (11) is valid for every $t \geq 0$.

Let us briefly review the main picture that emerges for the stability properties of the solutions as $t \to +\infty$ closely following [10]. For the open universe ($\sigma = -1$), the general analytic solution for the perturbation $\varepsilon(t)$ with three (as required) arbitrary constants $c_1^{(-1)}, c_2^{(-1)}, c_3^{(-1)}$, and $C_1^{(-1)}, C_2^{(-1)}, C_3^{(-1)}$ is

$$\varepsilon(t) = c_1^{(-1)} t \int t^{-7/2} I_{3/2}(\lambda t) dt + c_2^{(-1)} t \int t^{-7/2} K_{3/2}(\lambda t) dt + c_3^{(-1)} t, \quad \lambda^2 > 0 \ , \qquad (12)$$

$$\varepsilon(t) = C_1^{(-1)} t \int t^{-7/2} J_{3/2}(kt) dt + C_2^{(-1)} t \int t^{-7/2} J_{-3/2}(kt) dt + C_3^{(-1)} t \ ,$$
$$\lambda^2 = -k^2 < 0 \ , \qquad (13)$$

with $\lambda^2 = f_0'/3f_0''$ ($\lambda > 0$ without loss of generality) and $I, K$, and $J$ denoting the modified Bessel functions and Bessel function, respectively, thus yielding the asymptotic behavior of $\varepsilon(t)$ as $t \to \infty$ as follows (cf. [10], Appendix):

$$\varepsilon(t) \sim c_1^{(-1)} t e^{\lambda t}\alpha_5 + c_2^{(-1)} t e^{-\lambda t} + c_3^{(-1)} t \,, \qquad \lambda^2 > 0, \quad t \to \infty \,, \tag{14}$$

$$\varepsilon(t) \sim t(C_1^{(-1)} + C_2^{(-1)} \sin kt + C_3^{(-1)}) \,, \qquad \lambda^2 < 0, \quad t \to \infty \,, \tag{15}$$

$\alpha_5$ being a bounded constant [10]. It is clearly seen from the last two equations that the perturbation $\varepsilon(t)$ is in general unbounded at $t \to \infty$. This means that the corresponding homogeneous and isotropic $\sigma = -1$ solutions given by (4) are generally future unstable in higher order gravity *irrespective* of the sign of $\lambda^2$.

The analysis of the $\sigma = +1$ case is completely analogous to the $\sigma = -1$ case, and we obtain the asymptotic forms [10]

$$\varepsilon(t) \sim c_1^{(+1)}\alpha_4 t^2 + c_2^{(+1)} t^4(\beta_4 - \sin\lambda t) + c_3^{(+1)} t \,, \qquad \lambda^2 > 0, \quad t \to \infty \,, \tag{16}$$

whereas if $\lambda^2 < 0$ we find

$$\varepsilon(t) \sim C_1^{(+1)} t^2 e^{kt}\alpha_3 + C_2^{(+1)} t^4 e^{kt}\beta_3 + C_3^{(+1)} t \,, \qquad \lambda^2 < 0, \quad t \to \infty \,, \tag{17}$$

where we have again set $k^2 = -\lambda^2 (k > 0)$ and $\alpha_3, \alpha_4, \beta_3, \beta_4$ are finite constants. The net result is that the $\sigma = +1$ solutions are in general future unstable in our theory. Last, in the case of a flat FRW metric we find [10], where now the constant $\rho_0 = 3f_0'/4$,

$$\varepsilon(t) = \frac{c_1^{(0)}}{t} + \frac{c_2^{(0)}}{t}\int t^{3/4} I_{3/4}(\lambda t)dt + \frac{c_3^{(0)}}{t} \times \int t^{3/4} I_{-3/4}(\lambda t)dt, \quad \lambda^2 > 0, \tag{18}$$

$$\varepsilon(t) = \frac{C_1^{(0)}}{t} + \frac{C_2^{(0)}}{t}\int t^{3/4} J_{3/4}(kt)dt + \frac{C_3^{(0)}}{t} \times \int t^{3/4} J_{-3/4}(kt)dt, \quad \lambda^2 = -k^2 > 0, \quad k > 0 \,. \tag{19}$$

Then, as $t \to \infty$ the asymptotic behavior of the perturbation $\varepsilon(t)$ (cf. [10], Appendix) takes the form

$$\varepsilon(t) \sim \frac{c_1^{(0)}}{t} + \frac{\alpha_1 c_2^{(0)} + \beta_1 c_3^{(0)}}{t^{3/4}} e^{\lambda t}, \quad \lambda^2 > 0 \,, \tag{20}$$

$$\varepsilon(t) \sim \frac{C_1^{(0)}}{t} + \frac{\alpha_2 C_2^{(0)} + \beta_2 C_3^{(0)}}{t^{3/4}}, \quad \lambda^2 < 0 \,, \tag{21}$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2$ finite constants.

These results review the fact [10] that, as already pointed out in the Introduction, nonflat radiation FRW solutions of general relativity are generally unstable with respect to perturbations in our generalized gravitational theory in the large time limit. In the case of a flat FRW universe in higher order gravity, however, the region $\lambda^2 < 0$ includes regular solutions at the singularity which approach the corresponding ones in general relativity (cf. [9,10]).

## III. STABILITY OF RADIATION UNIVERSES FOR ARBITRARILY SMALL TIMES

Let us first assume that $\sigma = -1$ (open FRW model). Consider the *exact* perturbation equation (cf. [10])

$$\dddot{\varepsilon} + \left(-\frac{13}{t^3} + \frac{5}{t}\right)\ddot{\varepsilon} - \left(\frac{48 f_0'''}{f_0''}\frac{1}{t^4} + \frac{f_0'}{3f_0''}\right)\dot{\varepsilon} + \left(\frac{f_0'}{3f_0''}\frac{1}{t} - \frac{1}{t^3}\left[-\frac{2\rho_0}{9f_0''} + \frac{36 f_0'''}{f_0''} - 2\right]\right)\varepsilon = 0, \quad t \geq 0 \,, \tag{22}$$

where $\varepsilon = \varepsilon(t)$. We see that terms that involve third derivatives of $f(R)$ are the dominant terms and are coupled to terms that grow like $t^{-4}$ or $t^{-3}$ for arbitrarily small times $t$ on approach to the singularity. Keeping only such terms asymptotically as $t \to 0$, we get the equation

$$\dddot{\varepsilon} - \frac{13}{t^3}\ddot{\varepsilon} - \lambda_1^2 \frac{1}{t^4}\dot{\varepsilon} - \lambda_2^2 \frac{1}{t^3}\varepsilon = 0 \,, \tag{23}$$

where we have set $\lambda_1^2 = 48 f_0'''/f_0'', \lambda_2^2 = 36 f_0'''/f_0'' - 2\rho_0/9f_0'' - 2$.

The treatment of (23) is rather lengthy, and the details are given in the Appendix. The essential result consists in the asymptotic behavior as $t \to 0$ of the general solution $\varepsilon(t)$ which is

$$\varepsilon(t) \sim A_1 t^n + A_2 + A_3, \quad t \to 0 \,, \tag{24}$$

with the necessary three arbitrary constants $A_1, A_2, A_3$, while $n$ is a real number depending on $\lambda_1^2$.

The conclusion from this analysis is that solutions which describe open FRW radiation-filled universes in higher order gravity are in general of a nonperturbative character irrespective of the sign of $n$, since in general $A_2 + A_3 \neq 0$. Hence, they may be singular, but not obtainable from the FRW radiation solutions of general relativity via perturbation theory, or bouncing at the singularity. The analysis of (22) for $t \to 0$ in conjunction with the results obtained for $t \to \infty$ in [10], i.e., Eqs. (14) and (15), lead for $\sigma = -1$ in this kind of perturbation formalism to the general pattern: future-unstable and past-nonperturbative (bouncing or singular) solutions.

There exist, however, some special cases which *a priori* cannot be ruled out and, furthermore, allow us to extract rather interesting results. In particular, we deduce for $\lambda^2 > 0$ from Eq. (14) the existence of future-stable solutions, provided $c_1^{(-1)} = c_3^{(-1)} = 0$. On the other hand, from (24) we can obtain singular universes from perturbation theory if $A_1 = 0, A_2 + A_3 = 0$. Consequently, we may conclude that there exists a one-parameter family of solutions, $a_1(t)$, in the three-parameter solution space of the fundamental equation (9) ($\sigma = -1$), which is future stable and a one-parameter family of singular solutions, $a_2(t)$, which is past perturbative. Evidently, these two families of solutions $a_1(t)$ and $a_2(t)$ cannot, barring of course exceptional circumstances, have a common member, $a_c(t)$, since to determine $a_c(t)$ completely one needs four conditions, according to the results of the analysis of Eq. (22) for $t \to 0$, $t \to \infty$, whereas the general solution of (9), or (22) for that matter, contains three constants. Similar results hold for $\lambda^2 < 0$ [cf. Eqs. (15) and (24)].

The important conclusion that can be drawn from the above reasoning and which concerns the central issue of viability of cosmological solutions to the higher order gravity equations, as noted in the Introduction, is that there exist *nonperturbative*, singular or bouncing at $t = 0$ solutions which are future stable, i.e., the family $a_1(t)$. Therefore it is sensible to attempt the construction of particular exact solutions to (9), of this "viable" type, for example, for quadratic gravity, as already pointed out in the Introduction of the present paper.

We proceed in the following to the $\sigma = +1$ case in precisely the same manner as for the case $\sigma = -1$. We obtain, for the relevant perturbation equation [10], that the asymptotic behavior of $\varepsilon(t)$ for $t \to 0$ is

$$\varepsilon(t) \sim B_1 t^2 + (B_2 + B_3) t^{-3/11}, \quad t \to 0 , \tag{25}$$

$B_1, B_2$, and $B_3$ being arbitrary constants. Evidently, from (25), it is obvious yet again that in general we have higher order gravity solutions which are of a nonperturbative nature and bouncing or singular at $t = 0$.

Now, from Eqs. (16) and (17), a fact emerges which is probably worth mentioning, in particular if it is considered in connection with (25). Indeed, from (16) and (17) we observe that in both cases $\lambda^2 > 0$ and $\lambda^2 < 0$ all three constants have to be set equal to zero to force $\varepsilon(t) \to 0$ as $t \to \infty$. This implies that at the most only *one* future-stable solution $a(t)$ of (9) exists in each one of the cases $\lambda^2 > 0$ and $\lambda^2 < 0$. This contrasts the case of an open universe, where a one-parameter family of future-stable solutions exists and might be expected in view of $a(t) \to 0$ as $t \to \infty$. Such as future-stable solution most probably will be nonperturbative as $t \to 0$.

In general we have future-unstable and nonperturbative, singular or bouncing at $t = 0$, solutions of (9). The special case $B_2 + B_3 = 0$ simply shows the existence of a two-parameter family of singular solutions $a(t)$ which are perturbative as $t \to 0$ and future unstable (with the one exception mentioned earlier) for $t \to \infty$. In conclusion, we have shown the existence of nonperturbative solutions near the beginning which are, however, future unstable.

Finally in the flat FRW case we get [cf. (18) and (19)]

$$\varepsilon(t) \sim \frac{c_1^{(0)}}{t} + c_3^{(0)}, \quad t \to 0, \quad \lambda^2 > 0 , \tag{26}$$

$$\varepsilon(t) \sim \frac{C_1^{(0)}}{t} + C_3^{(0)}, \quad t \to 0, \quad \lambda^2 < 0 , \tag{27}$$

thus deducing, in general, the nonperturbative features of the respective solutions. By invoking (20), (21), (26), and (27) we can easily prove the existence of the physically interesting past nonperturbative and future-stable solutions in this case too.

## IV. COSMOLOGICAL ATTRACTORS

Let us concentrate on the case of a quadratic Lagrangian theory given by $L_2 = R + \alpha R^2$. In this case and for the metric (4) the field equation (9) becomes [7]

$$y''(x) + y^{-5/3}(x) x^{-2/3}\sigma^2 - \frac{1}{\sqrt{3}} y^{-1/3}(x) x^{-4/3}\sigma + \alpha^{-1}[y^{-1/3}(x) x^{-2/3} - \rho_1 y^{-5/3}(x) x^{-\gamma-1/3} + 2\sqrt{3}\sigma y^{-5/3}(x)] = 0 , \tag{28}$$

where $y^{2/3} = \dot{a}a, x(t) = ka^3$, $k = (2\sqrt{3})^{-3/2}$, and $\rho_1 = [2^{(1-3\gamma)/2}]3^{-3(\gamma+1)/4}\rho_0$. Now setting

$$y(x) = (v - \sigma a^2)^{3/4}, \quad v = v(a) \tag{29}$$

and $\sigma = -1$ the field equation (28) gives, by requiring also that $\rho_1 = 0$ for a Milne universe [16],

$$\frac{v''}{12k^2a^4} + \frac{1}{6k^2a^5} - \frac{v' + 2a}{6k^2a^5} - \frac{v'^2 + 4a^2 + 4v'a}{48k^2a^4(v + a^2)}$$
$$+ \frac{1}{k^{2/3}a^2(v + a^2)} + \frac{k^{-4/3}}{a^4\sqrt{3}} + \frac{\alpha^{-1}k^{-2/3}}{a^2} - \frac{2\alpha^{-1}\sqrt{3}}{v + a^2} = 0 . \tag{30}$$

A particular solution of (30) is $v = 0$; therefore, because of $y^{2/3} = a\dot{a}$ and Eq. (29) we find

$$a(t) = t + C , \tag{31}$$

where $C$ is a constant. This describes the radius of a Milne universe in higher order gravity.

We now turn to the issue of whether or not this Milne universe is generic on approach to the singularity in the sense that it attracts neighboring spacetimes as $t \to 0$. Substituting

$$v = \delta(a), \quad |\delta(\alpha)| \ll 1 , \tag{32}$$

in (30) we find the following perturbation equation, with $\delta = \delta(a)$:

$$\delta'' - \frac{3\delta'}{a} - \frac{\delta'^2}{4a^2} = 0 . \tag{33}$$

This has the solution

$$\delta(a) = c_2 - 4a^2 - 32c_1 \ln|a^2 - 8c_1| , \tag{34}$$

where $c_1$ and $c_2$ are constants.

Equation (34) implies that $\delta$ is generally nonzero on approach to the singularity at $t = 0$, and so the Milne kinematical universe is not generic on approach to the singularity in higher order gravity and does not attract neighboring universes in the small time limit.

Next we consider (28) which by means of the substitutions (29) becomes, for $\sigma = 0, \gamma = 1/3$,

$$\frac{v''}{12k^2a^4} - \frac{v'}{6k^2a^5} - \frac{v'^2}{48k^2a^4v} + \frac{\alpha^{-1}}{k^{2/3}a^2} - \frac{\alpha^{-1}\rho_1}{a^2vk^{2/3}} = 0 . \tag{35}$$

Equation (35) possesses the particular solution $v = \rho_1$, thus due to (29) and $y^{2/3} = a\dot{a}$ we obtain

$$a(t) = (2t\sqrt{\rho_1} + C)^{1/2} , \tag{36}$$

with $C$ a constant. Solution (36) is none other than the Ruzmaikin and Ruzmaikina solution [9] which, being obviously bouncing at $t = 0$ and future stable, is one of the physically important solutions, as we have pointed out in the $\sigma = 0$ case in Sec. III. Upon inserting

$$v = \rho_1[1 + \delta(a)], \quad |\delta(a)| \ll 1 \tag{37}$$

in (35) we obtain, with $\delta = \delta(a)$,

$$\delta'' - \frac{2\delta'}{1} - \frac{\delta'^2}{4} = 0 . \tag{38}$$

This has the solution

$$\delta(a) = C_2 - 4 \ln |a^3 - C_1| , \tag{39}$$

$C_1$ and $C_2$ being integration constants. From (39) it becomes obvious that $\delta(a)$ is in general nonzero as $t \to 0$ and, therefore, (36) is not an attractor in the space of solutions of the differential equation (35). Let it be noted here that by choosing $C_1$ to satisfy the condition $C_1 = a^3(0) - 1$ and $|C_2| \ll 1$ we conclude that there exists a two-parameter family of solutions remaining close to (36) for small times precisely as for the solution found by Starobinski [2].

Finally we look into the possibility of the inflationary stage in the quadratic theory $L_2$, which is described by the quasiexponential solution [7] $a_0(t) = \exp(Bt - t^2/12\alpha)$, being an attractor. This solution is obtained as a self-consistent solution of the field equation (9) for $\sigma = 0, \gamma = -1, \alpha < 0$, where $B$ is an integration constant. In terms of the Hubble parameter $H = \dot{a}/a$, (4) can be written as

$$\frac{H^2}{2} + 2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2 = \frac{-1}{36\alpha^2} . \tag{40}$$

To write down the general stability equation for the Barrow-Ottewill inflationary solution, we set $H_0 = B - (1/6\alpha)t$ which corresponds to the $a_0(t)$ above via $H = \dot{a}/a$, for the unperturbed phase and look for solutions of the form

$$H = H_0[1 + \delta(t)], \quad \delta(t) \ll 1 , \tag{41}$$

for some small perturbation $\delta(t)$. Substituting this into (40) we arrive at the perturbation equation

$$2\ddot{\delta} + 6H_0\dot{\delta} - \dot{\delta}^2 = 0, \quad H_0 = B - (1/6\alpha)t, \quad \delta = \delta(t) . \tag{42}$$

We can solve this equation after some manipulation and we present the final result which is

$$\delta(t) = \delta(0) + \int_0^t \frac{d\tau}{e^{-(\tau-\lambda)^2/4\alpha} f(B,\alpha) - \sqrt{-\alpha}\sum_{n=0}^{\infty} 2^n(\tau-\lambda)^{2n+1}/[(2\sqrt{-\alpha})^{2n+1}(2n+1)!!]} \tag{43}$$

with $\lambda = 6B\alpha$ and

$$f(B,\alpha) = e^{-18B^2\alpha}\sqrt{-\alpha}\sum_{n=0}^{\infty} \frac{(-\lambda/\sqrt{-\alpha})^{2n+1}}{2^{n+1}(2n+1)!!} + C_1 , \tag{44}$$

$C_1$ being an integration constant. We note that for $t \to 0$ the integrand in Eq. (43) apart from specific values of the arbitrary integration constants $B$ and $C$ is finite, and therefore the integral exists for $t \to 0$. Hence, we see that the perturbation cannot be naively set to zero as we approach the singularity at $t = 0$ because of the presence of the generally nonvanishing integration constant $\delta(0)$. We thus arrive at the interesting conclusion that the quasiexponential solution $a_0(t) = \exp(Bt - t^2/12\alpha)$ is clearly *not* a transient attractor as $t \to 0$. This, at first may seem surprising (and probably even discouraging) since it puts doubts on a generic birth of the inflationary universe in higher order gravity theories. However, setting $\delta(0) = 0$, which is tantamount to imposing an initial condition for $t = 0$ on the solution of (40) [or (9) for that matter], implies that there exists a two-parameter family of solutions $a(t)$ attracted as $t \to 0$ by the Barrow-Ottewill quasiexponential solution $a_0(t)$. In our perturbation setting the above result implies that generic inflation is "improbable," but we believe that this result could be improved if quantum effects are taken into account and one studies the associated Wheeler-DeWitt equation. On considering now the large time limit and noting, as it can be easily shown, that the integrand in (43) for $t \to \infty$ behaves like $\exp\{(\tau - \lambda)^2/4\alpha\}\tau^{-1}$, $\alpha < 0$, we observe that the integral in (43) exists. Hence by setting now

$$\delta(0) = -\int_0^\infty \frac{d\tau}{e^{-(\tau-\lambda)^2/4\alpha} f(B,\alpha) - \sqrt{-\alpha}\sum_{n=0}^{\infty} 2^n(\tau-\lambda)^{2n+1}/[(2\sqrt{-\alpha})^{2n+1}(2n+1)!!]} = -I \ , \tag{45}$$

$I$ denoting the integral in (45), which is yet again equivalent to requiring that the solution of (40) [i.e., Eq. (9)] satisfy an initial condition, the Barrow-Ottewill solution becomes an attractor of a two-parameter family of solutions in the three-parameter solution space of (9) (cf. [17]), precisely as in the $t \to 0$ case. We may conclude from the above that there exists a one-parameter family of solutions of Eq. (9) generated by two appropriate initial conditions, corresponding to $\delta(0) = 0$ and $I = 0$ [by virtue of (45) $I = 0$ is simply a defining equation for the constant $C_1$ included in $f(B, \alpha)$], which is attracted by $a_0(t) = \exp(Bt - t^2/12\alpha)$ for both $t \to 0$ and $t \to \infty$.

## V. DISCUSSION

We have analyzed the past-stability properties of homogeneous and isotropic, radiation cosmologies in the generalized framework of higher order gravity theories. The conclusion that follows clearly from this analysis is that, in general, the relevant solutions are past unstable or, in other words, of a nonperturbative nature and therefore may provide new insight into the early evolution of the universe in $f(R)$ gravity. The stability analysis of Sec. III confirms this and leads us to suspect that this situation is even more general and so we propose the following.

*Conjecture.* All homogeneous and isotropic, physically reasonable cosmological solutions of general relativity are, in general, past unstable against perturbations in higher order gravity.

Furthermore we have shown that all the known-up-to-now exact particular solutions of higher order gravity, the Ruzmaikin and Ruzmaikina solution [9] and the Barrow-Ottewill solution [7], are *not* attractors as we approach the singularity in the solution space of the relevant field equations. This result implies the nongeneric character of the aforementioned solutions and justifies future attempts aimed at the construction of new particular exact solutions of higher order gravity. We note that the Milne-type solution [16] of general relativity, which is, according to the conformal equivalence theorem [11,15] and as verified here, a solution of the higher order gravity theory, is also not an attractor on approach to the singularity.

If one insists on continuing this type of research, some next steps could be as follows.

(1) Past and future (in)stability and attractor property of homogeneous and isotropic solutions against anisotropic or inhomogeneous perturbations.

(2) Past and future (in)stability and attractor property of inhomogeneous and anisotropic solutions against perturbations in higher order gravity. The first obvious candidate here should be the Bianchi type-I solution discovered in [18].

(3) Study solutions of the Wheeler-DeWitt equations for $R + \alpha R^2$ actions and anisotropic metrics and analyze their stability and attractor properties.

In conclusion we remark that the problems discussed in this paper constitute a field which certainly deserves further investigation and which could lead to new and interesting results. We leave these matters to future papers.

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## APPENDIX

In this Appendix we provide the necessary details for the derivation of (24). To effect the solution of (23) we make the ansatz

$$\varepsilon_p = t^n \ , \tag{A1}$$

and find that (A1) is a particular solution of (23) provided

$$n = \frac{13 - \lambda_1^2}{13} \ , \tag{A2}$$

$$\frac{\lambda_1^2(169 - \lambda_1^4)}{169} = 13\lambda_2^2 \ . \tag{A3}$$

Condition (A3) simply implies that [cf. after (23)] the constant $\rho_0$ has to be fixed accordingly. Therefore no loss of generality is incurred through the requirement that (A1) be a solution to (23). Setting now

$$\varepsilon(t) = t^n y(t), \quad y(t) = \phi'(t) \ , \tag{A4}$$

(23) yields

$$t^4\phi''(t) + t(3nt^2 - 13)\phi'(t) - \{26n + \lambda_1^2 - 3n(n-1)t^2\}\phi(t) = 0 \ . \tag{A5}$$

Further, by means of a change of variables

$$\phi(t) = u(z), \quad z = \tfrac{1}{4}t^2 \ , \tag{A6}$$

(A5) becomes

$$z^2u''(z) + z(a_1 + b_1 z)u'(z) + (a_2 + b_2 z)u(z) = 0 \ , \tag{A7}$$

where

$$a_1 = 3(1-n)/2, \quad b_1 = 26, \quad a_2 = 3n(n-1)/4, \quad b_2 = -(26n + \lambda_1^2) . \tag{A8}$$

Finally, on inserting the substitutions

$$u(z) = z^k w(z), \quad w(z) = v(z_1), \quad z_1 = -b_1 z, \quad k^2 + (a_1 - 1)k + a_2 = 0 , \tag{A9}$$

into (A7) we obtain

$$z_1 v''(z_1) + (2k + a_1 - z_1)v'(z_1) - \left(k + \frac{b_2}{b_1}\right) v(z_1) = 0 . \tag{A10}$$

Equation (A10) is the confluent hypergeometric equation [19] with the general solution

$$v(z_1) = A_2\, {}_1F_1\left(k + \frac{b_2}{b_1}, 1 + \frac{s}{2}; z_1\right) + A_3 z_1^{-s/2}\, {}_1F_1\left(k + \frac{b_2}{b_1} - \frac{s}{2}, 1 - \frac{s}{2}; z_1\right) , \quad s = \sqrt{-3n^2 + 6n + 1} . \tag{A11}$$

Hence, by utilizing (A4), (A6), (A8), (A9), and (A11) we deduce

$$\varepsilon(t) = A_1 t^n + A_2 t^n \int^{(t)} x^{(1-3n-s)/2}\, {}_1F_1\left[\frac{n-3+s}{4}, \frac{2+s}{2}, \frac{-13}{2x^2}\right] dx + A_3 t^n \int^{(t)} x^{(1-3n+s)/2}\, {}_1F_1\left[\frac{n-3-s}{4}, \frac{2-s}{2}, \frac{-13}{2x^2}\right] dx , \tag{A12}$$

where ${}_1F_1[\frac{n-3\pm s}{4}, \frac{2\pm s}{2}, \frac{-13}{2x^2}]$ is the confluent hypergeometric function. Since now [19]

$${}_1F_1\left[\frac{n-3\pm s}{4}, \frac{2\pm s}{2}, \tau\right] \sim \tau^{(-n+3\mp s)/4}, \tau \to -\infty , \tag{A13}$$

where $\tau = -13/2x^2$ and $x \to 0$, we conclude from (A12) and (A13) that

$$\varepsilon(t) \sim A_1 t^n + A_2 + A_3, \quad t \to 0 , \tag{A14}$$

and all numerical factors have been absorbed into the integration constants $A_2, A_3$. Consequently, Eq. (24) of Sec. III has been proved. It is perhaps interesting to note here that a straightforward numerical investigation of (23) shows clearly that without assuming (A1) be a particular solution of (23), i.e., by droping condition (A3), which fixes the arbitrary constant $\rho_0$, the general solution of (23) is *not* determined at $t = 0$. This evidently shows that the perturbation in general breaks down at $t = 0$, as noted in Sec. III, and this is predicted by (A14) for all $n < 0$.

The proof of Eq. (25), as well as the respective numerical investigation, runs precisely along the same lines as for the case of an open universe, whereas the verification of (26) and (27) is immediately carried out by introducing the series expansions of the Bessel functions into (18) and (19) and letting $t \to 0$ after carrying out the elementary integrations.

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