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Pregalaxy formation: A nonlinear analysis of the evolution of cosmological perturbations

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A higher-order analysis of the evolution of cosmological perturbations in a Friedmann universe is given by using the PMF method. The essence of the PMF approach is to choose a gauge where all fluctuations of the density, the pressure, and the four-velocity vanish. Additionally, a planar symmetry of the perturbations is assumed. In that gauge, even in higher orders, the perturbation field equations simplify considerably; they can be decoupled and, for simple equations of state, also be solved analytically. We give the solution for the dust universe up to third order. Comparison of these solutions strongly supports the conjecture that in general unstable perturbations grow much faster than they do according to the first-order analysis. However, perturbations with very large spatial extension behave differently; they grow only moderately. Thus, an upper boundary of the region of instability seems to exist.

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I. INTRODUCTION

Because of the nonlinearity and complexity of Einstein's field equations, they have been solved analytically so far only for situations characterized by relatively high symmetry and homogeneity properties. In the other cases, one has basically two possibilities: either solve them numerically, or perform a perturbation analysis and solve the individual orders analytically. Of course, there are also mixtures of these two possibilities.

If we want to explain the origin of galaxies, we have to study the evolution of given small fluctuations in a Friedmann universe as background, and we have to investigate whether these perturbations are stable or unstable. Moreover, we have to calculate the growth rates of unstable fluctuations. A complete numerical analysis is not very useful in this context, because a possible increase does not guarantee that perturbation does not decrease somewhat later. On the other hand, a complete analytical investigation of the evolution of cosmological perturbations is not possible for reasons of complexity. Hence, we have to perform some perturbation theory. So far, that analysis was restricted, also for reasons of complexity, to the first order; the linearized field equations were studied (see, e.g., [1–5]). In the approach of Ellis *et al.* [6,7] a second-order equation is also derived but the analysis they give is only at first order. It is obvious that such a restriction is too severe. Higher orders become significant when a perturbation, unstable according to the first-order analysis, is growing. Nonlinear effects appearing thereby could be very important; they could stabilize such a perturbation, or they could change the growth rates of unstable perturbations considerably. It is a main aim of this paper to study what kind of nonlinear effects might appear.

To this end we have used a method that we invented in [5]. This so-called pure metric fluctuation (PMF) method simplified the perturbation field equations in first order dramatically (see [5]); hence, it is reasonable to expect that the same happens for higher orders, too. The PMF method is based on the gauge freedom. This is to be understood as follows.

Perturbation quantities are constructed by subtracting from the full quantity at a space-time point x^λ in the perturbed universe the background quantity at the corresponding space-time point x^λ in the fictitious Friedmann universe. The choice of such a correspondence defines a gauge. We define other gauges by performing coordinate transformations in the perturbed universe, keeping the background coordinates fixed. By means of a so-called gauge condition—a condition concerning some of the perturbation quantities—we select a definite set of coordinate systems. Unfortunately, the perturbation quantities are gauge dependent; thus, the so-called gauge problem arises: to what extent are the observed perturbations mere coordinate effects, and to what extent are they "physical"? Which gauge is suitable for judging stability (instability) of a given fluctuation? How can we find such a gauge that is "as close to the background as possible"?

In the literature one finds some gauges that were proposed in this context [2–4]. The (first-order) results are systems of coupled differential equations that can be solved only in special cases. Another treatment in order to avoid the choice of a gauge was performed by Bardeen [1]. Out of metric and matter perturbations he constructed gauge-invariant variables satisfying relatively simple equations that he could solve explicitly. Again, this analysis is performed only up to first order.

Another gauge-invariant approach is given in a recent work of Ellis and Bruni [6,7]. They do not compare two evolutions (real universe and fictitious Friedmann uni-

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verse) along the same world line like all approaches up to now; instead, they compare two neighbored world lines within the same real universe. Their basis quantity is the comoving fractional gradient of the energy density orthogonal to the fluid flow. That approach has much in favor but it is quite doubtful whether it is useful also in higher orders. The corresponding equations become very complicated quickly when the order increases, whereas they maintain their simple basic structure in the PMF approach, which will be explained now.

The essence of the PMF method is to choose a gauge such that all fluctuations of matter (i.e., perturbations in density, pressure, and velocity) vanish and only pure metric fluctuations remain. This can be achieved by choosing suitably the spacelike hypersurfaces (t =const) and by choosing appropriate coordinates onto them. Additionally, planar symmetry of the perturbations is assumed. This assumption is not too severe a restriction in first order but might be a more serious one in higher orders. However, the analysis given in this paper should be considered just as a first step towards a complete higher-orders analysis of general perturbations in a realistic universe. In first order the perturbation field equations in PMF gauge simplify considerably; they can be decoupled and, for simple equations of state, be solved [5]. Note that the PMF gauge does not claim to be "as close to the background as possible." It serves rather for the simplification of the field equations. Subsequently, one can transform the PMF solution into any desired gauge suitable for judging stability or instability. In this paper, the PMF approach shall be employed for a higher-order analysis. It turns out (see Sec. II) that there is a dramatic simplification also in higher orders. In the PMF gauge, the perturbation field equations maintain their simple basic structure which they have in first order. Therefore, they can be decoupled even in higher orders, and they can be solved analytically if the chosen equation of state is simple enough.

All approaches so far gave no hint that an upper boundary of the region of instability exists. The first-order result has always been that fluctuations larger in extension than the Jeans limit [8] are growing eternally (an opposite opinion is supported in [4], but see [5]). Thus, one major motivation for doing higher-orders analysis was the possible perspective that nonlinear effects could imply the existence of such an upper boundary. Then, fluctuations, whose extension is larger than that boundary, would cease to grow or they would grow too slowly with regard to the generation of large-scale structures of the universe. Such an existence would be in agreement with astronomical observations [9–13]. However, also an opposite point of view is supported by some astronomers (see, e.g., [14]).

The gauge problem itself is of minor interest in this paper. First of all, we want to analyze higher orders and to study the principle influence of nonlinear effects arising thereby. Therefore, we will use a very simple equation of state (dust). In future work the gauge problem in higher orders as well as more realistic equations of state shall be investigated. The plan of this paper is as follows. In Sec. II we present the PMF approach for higher orders and the corresponding perturbation field equations. These are solved in Sec. III in the case of the dust universe; a case study, which gives some insight into the regularities between the solutions of the various orders, is discussed. Section IV, which gives the main results of our work, and which discusses some other possible fields of application of the PMF approach, concludes this paper. Throughout this paper, we use Weinberg's notation [15]. Instead of $\partial f/\partial t$ we write $\dot{f}$; $f_{,1}$ means $\partial f/\partial x$.

II. NONLINEAR ANALYSIS IN THE PMF GAUGE

For reasons of simplicity, we restrict ourselves to a Friedmann universe with vanishing spatial curvature as background. This is not too severe a restriction, because in the early universe when galaxies were formed the condition $\dot{R}^2 \gg |\kappa|$ (R is the scale factor of the universe, and κ is the curvature parameter, which can adopt the values $+1$, 0, or -1) was satisfied. We choose the background coordinates such that the metric components read

$$g_{00}^{(0)} = -1, \; g_{ij}^{(0)} = R(t)^2 \delta_{ij} \tag{1}$$

(all other components vanish). The index 0 refers to the background (0th order).

Let the energy-momentum tensor be of perfect fluid form. We get for the four-velocity of the background in our Robertson-Walker coordinates

$$U^{\mu}_{(0)} = \delta^0_{\mu} \; . \tag{2}$$

Now we consider a perturbation which has two-dimensional symmetry planes, i.e., in suitable coordinates all perturbation quantities shall depend only on x ($= x^1$) and t ($= x^0$), but not on y ($= x^2$) or z ($= x^3$). Beyond that we demand that $U^2_{(n)}$ and $U^3_{(n)}$ [n is the order; see (4)] vanish in order to exclude rotational perturbations which would disturb the symmetry.

This symmetry allows the introduction of coordinates such that the metric adopts the form

$$d\tau^2 = -g_{ab}(x,t)dx^a dx^b - f(x,t)(dy^2 + dz^2) \; , \tag{3}$$

where $a, b = 0$ or 1. Note that g_{ab} and f depend only on x and t.

Next we make the following ansatz:

$$\begin{aligned} \rho &= \rho_0 + k\rho_1 + k^2\rho_2 + k^3\rho_3 + \cdots \; , \\ p &= p_0 + kp_1 + k^2 p_2 + k^3 p_3 + \cdots \; , \\ U^{\mu} &= U^{\mu}_{(0)} + kU^{\mu}_{(1)} + k^2 U^{\mu}_{(2)} + \cdots \; , \\ g_{\mu\nu} &= g^{(0)}_{\mu\nu} + kg^{(1)}_{\mu\nu} + k^2 g^{(2)}_{\mu\nu} + \cdots \; , \end{aligned} \tag{4}$$

where k is some dimensionless expansion parameter, ρ is th energy density, p is the pressure, U^{μ} is the four-velocity, and $g_{\mu\nu}$ is the metric. The index 0 refers to the background, the index 1 marks the perturbation quantities in first order, the index 2 those in second order, and so on. We have to insert this ansatz into the field equations and to order them according to powers of k. Subsequently, we have to solve the field equations order by order. If the sums in (4) converge, (4) is the exact solution of the field equations.

The ansatz (4) requires some explanations. First of all, it is not too obvious what is meant by perturbation quantities of orders higher than the first one. Indeed, one could replace (4) by an alternative ansatz which just contains perturbation quantities up to first order. $k\rho_1$ then simply means the difference between the total density in the real universe and the density ρ_0 in the fictitious background universe. But contrary to the linear analysis performed in [5], in this case, we also had to take into account terms proportional to k^n $(n \geq 2)$ as, e.g., the term $g^{(1)}_{\mu\nu} g^{(1)}_{\tau\lambda}$ in the field equations. Then, we could not expect that the field equations are satisfied order by order; merely the sum of all orders from the first one to the last one considered would be satisfied. We do not want to choose this approach here for, in that case, we had to solve practically the full field equations; just the zeroth order would be separated. This would not really be a perturbation analysis.

Instead of this we proceed as follows. In order to satisfy the field equations order by order we add in our ansatz a correction quantity order by order, which is chosen such that these equations hold. Doing so we arrive at the ansatz (4) and it is guaranteed that in each order n the field equations are differential equations which are linear in the unknown correction quantities ρ_n, p_n, etc. (but not in perturbation quantities of lower orders which are already known by solving the field equations in these lower orders; hence, they are no longer unknown with regard to the nth order). Note that either k can be considered as small compared with 1, or we can set $k = 1$; then, the perturbation quantities of $(n+1)$th order like $\rho_{(n+1)}$, $U^{\mu}_{(n+1)}$, etc. should be small compared with the corresponding ones of the nth order.

Now we study the influence of infinitesimal coordinate transformations

$$x'^{\mu} = x^{\mu} - k\epsilon^{\mu}_{(1)}(x^{\lambda}) - k^2\epsilon^{\mu}_{(2)}(x^{\lambda}) - k^3\epsilon^{\mu}_{(3)}(x^{\lambda}) - \cdots \quad (5)$$

in the real perturbed universe. These coordinate transformations change the correspondence between points in the background universe and points in the physical spacetime; thus, they are gauge transformations. Note that the "philosophy" of the ansatz (5) is the same as that of (4): if we would stop the sum in (5) at the term $\epsilon^{\mu}_{(1)}(x^{\lambda})$ we would get very complicated nonlinear gauge transformation laws for the quantities appearing in (4) when we would go beyond the first order. But with (5) we get in each order n transformation laws which are linear in the unknown function $\epsilon^{\mu}_{(n)}$. We give them here explicitly up to the second order.

First order:

$$\rho'_1 = \rho_1 + \dot{\rho}_0\epsilon^{t}_{(1)} , \quad (6)$$

$$p'_1 = p_1 + \dot{p}_0\epsilon^{t}_{(1)}, \quad (7)$$

$$U'^{\mu}_{(1)} = U^{\mu}_{(1)} - \dot{\epsilon}^{\mu}_{(1)} , \quad (8)$$

$$g'^{(1)}_{\mu\nu} = g^{(1)}_{\mu\nu} + \epsilon_{(1)\mu};\nu + \epsilon_{(1)\nu};\mu . \quad (9)$$

[cf. Eqs. (2.4), (2.5), (2.7), and (2.9) in [5]].

Second order:

$$\rho'_2 = \rho_2 + \dot{\rho}_0\epsilon^{t}_{(2)} + \frac{\partial\rho_1}{\partial x^{\mu}}\epsilon^{\mu}_{(1)} + \frac{\ddot{\rho}_0}{2}(\epsilon^{t}_{(1)})^2 , \quad (10)$$

$$p'_2 = p_2 + \dot{p}_0\epsilon^{t}_{(2)} + \frac{\partial p_1}{\partial x^{\mu}}\epsilon^{\mu}_{(1)} + \frac{\ddot{p}_0}{2}(\epsilon^{t}_{(1)})^2 , \quad (11)$$

$$U'^{\mu}_{(2)} = U^{\mu}_{(2)} - \dot{\epsilon}^{\mu}_{(2)} + \frac{\partial U^{\mu}_{(1)}}{\partial x^{\mu}}\epsilon^{\mu}_{(1)} - \frac{\partial\epsilon^{\mu}_{(1)}}{\partial x^{\nu}}U^{\nu}_{(1)} , \quad (12)$$

$$\begin{aligned} g'^{(2)}_{\mu\nu} = {} & g^{(2)}_{\mu\nu} + \epsilon_{(2)\mu};\nu + \epsilon_{(2)\nu};\mu - g^{(0)}_{\lambda\kappa}\frac{\partial\epsilon^{\lambda}_{(1)}}{\partial x^{\mu}}\frac{\partial\epsilon^{\kappa}_{(1)}}{\partial x^{\nu}} \\ & + g'^{(1)}_{\mu\lambda}\frac{\partial\epsilon^{\lambda}_{(1)}}{\partial x^{\nu}} + g'^{(1)}_{\nu\lambda}\frac{\partial\epsilon^{\lambda}_{(1)}}{\partial x^{\mu}} + \frac{\partial g^{(0)}_{\mu\lambda}}{\partial x^{\kappa}}\frac{\partial\epsilon^{\lambda}_{(1)}}{\partial x^{\nu}}\epsilon^{\kappa}_{(1)} \\ & + \frac{\partial g^{(0)}_{\nu\lambda}}{\partial x^{\kappa}}\frac{\partial\epsilon^{\lambda}_{(1)}}{\partial x^{\mu}}\epsilon^{\kappa}_{(1)} + \frac{g'^{(1)}_{\mu\nu}}{\partial x^{\lambda}}\epsilon^{\lambda}_{(1)} - \frac{1}{2}\frac{\partial^2 g^{(0)}_{\mu\nu}}{\partial x^{\lambda}x^{\kappa}}\epsilon^{\lambda}_{(1)}\epsilon^{\kappa}_{(1)} . \end{aligned} \quad (13)$$

According to (6) and (7) as well as (10) and (11) it is possible to transform ρ'_1, p'_1, ρ'_2, and p'_2 simultaneously to zero by a suitable choice of $\epsilon^{t}_{(1)}$ and $\epsilon^{t}_{(2)}$ provided that $\dot{\rho}_0$ is different from zero (which is satisfied in an expanding universe) and provided that the following equation holds for $i = 0$, 1, and 2

$$p_i = \omega\rho_i , \quad (14)$$

where ω is spatially and temporally constant. (14) is certainly satisfied, even for all natural numbers i, if the equation of state $p = \omega\rho$ holds [insert the ansatz (4) and decompose p according to powers of k such that this equation is satisfied order by order]. Analogously, $U'^{1}_{(1)}$ and $U'^{1}_{(2)}$ can be transformed away by a suitable choice of $\epsilon^{1}_{(1)}$ and $\epsilon^{1}_{(2)}$. All other spatial components of U^i also vanish due to our symmetry assumption. (Note that $U'^{0}_{(i)}$ is not an independent variable because it is related to $g^{(i)}_{00}$ by norm conversation.) We have now all features of the PMF gauge present: fluctuations of the density, of the pressure, and of the spatial components of the four-velocity vanish simultaneously (in first and in second order). One can show easily that this can be achieved also in higher orders. Namely, the gauge transformation laws for the perturbation quantities have in each order n the following general structure:

$$\rho'_n = \rho_n + \dot{\rho}_0\epsilon^{t}_{(n)} + f_{\rho}(\rho_i, \epsilon_{(j)}) , \quad (15)$$

$$p'_n = p_n + \dot{p}_0\epsilon^{t}_{(n)} + f_{p}(p_i, \epsilon_{(j)}) , \quad (16)$$

$$U'^{\mu}_{(n)} = U^{\mu}_{(n)} - \dot{\epsilon}^{\mu}_{(n)} + f_U(U^{\mu}_{(i)}, \epsilon_{(j)}) , \quad (17)$$

where $f_{\rho}(\rho_i, \epsilon_{(j)})$ is some function of ρ_i and $\epsilon_{(j)}$ and of their derivatives. The subscripts i and j are smaller than the considered order n. The other functions appearing in these transformation laws must be interpreted in the same way. As can be read off from (15), (16), and (17) it is possible to transform $\rho'_{(n)}$, $p'_{(n)}$, and $U'^1_{(n)}$ simultaneously to zero by a suitable choice of $\epsilon^t_{(n)}$ and $\epsilon^1_{(n)}$ provided that (14) holds for all natural numbers i.

Hence, if the condition (14) is satisfied we can achieve

$$\rho = \rho_0 \ ,$$

$$p = p_0 \ ,$$

$$U^i = U^i_{(0)}, \quad i = 1,2,3 \ . \tag{18}$$

But (18) characterizes exactly the PMF gauge (see [5]). From now on we assume that (14) is satisfied and we are going to solve the field equations in PMF gauge characterized by (18). Note that because of (3) $g^{(n)}_{22} = g^{(n)}_{33}$ holds for all orders and that besides these two metric components only $g^{(n)}_{00}$, $g^{(n)}_{10}$, and $g^{(n)}_{11}$ can be nonvanishing. Hence, our field equations contain just four independent components. Like in [5] we are using instead of the components "11" and "22" the energy-momentum conservation. Clearly, for reasons of checking we have inserted the solution obtained thereby into all components of the field equations. After a very lengthy but straightforward calculation we get the field equations up to third order in the following form where the different orders are already entangled from each other.

00 component:

$$\left[8\pi G\rho_0 - 6\left(\frac{\dot{R}}{R}\right)^2\right] g^{(n)}_{00} + 2\frac{\dot{R}}{R^3} g^{(n)}_{10,1} + \frac{g^{(n)}_{22,11}}{R^4} = S^{(n-1)}_{00} \ .$$

10 component:

$$\left[3\left(\frac{\dot{R}}{R}\right)^2 + 8\pi G p_0\right] \frac{g^{(n)}_{10}}{R^2} - \frac{\dot{R}}{R^3} g^{(n)}_{00,1} - \frac{\dot{g}^{(n)}_{22,1}}{R^4} + \frac{2\dot{R}}{R^5} g^{(n)}_{22,1}$$

$$= S^{(n-1)}_{10} \ .$$

Energy conservation:

$$\frac{\dot{g}^{(n)}_{11}}{R^2} + 2\frac{\dot{g}^{(n)}_{22}}{R^2} - \frac{2\dot{R}}{R^3}(g^{(n)}_{11} + 2g^{(n)}_{22}) = S^{(n-1)}_{ec} \ .$$

Momentum conservation:

$$\frac{\dot{g}^{(n)}_{10}}{R^2} - \frac{g_{00,1}}{2R^2} + \left[\frac{\dot{p}_0}{p_0 + \rho_0}\right] \frac{g^{(n)}_{10}}{R^2} = S^{(n-1)}_{mc} \ . \tag{19}$$

The source terms $S^{(n-1)}_{\mu\nu}$ are sums of products whose factors are solutions (or their derivations) of the field equations of lower orders than n [i.e., maximally of $(n-1)$th order]. We have calculated them for $n = 1,2,3$. We give the full form of the Eqs. (19) in the Appendix, but only up to second order since for $n = 3$ they are horribly long. However, after inserting the solutions for lower orders the source terms simplify quite a lot and can be handled, e.g., in the case of the dust universe (see next section) quite easily. For $n = 1$ all source terms vanish; the system (19) is then already familiar to us from the first-order analysis [compare [5], (3.17)–(3.20), and—considering the momentum conservation equation—note that the relation $\dot{\rho}_0 + 3(p_0 + \rho_0)\dot{R}/R = 0$ (that is the energy conservation in 0th order) holds]. Because of that striking regularity it can be suspected that (19) is valid for all orders n; but so far this conjecture has been proven by us [by directly working out the full form of (19)] only for $n = 1$, 2, and 3.

Equation (19) can be decoupled like in the first order (see [5]) and in higher orders, because of its relatively simple structure. Moreover, for simple equations of state, it can be solved completely in a purely analytical manner. We did this for dust ($p = 0$) and for radiation ($p = \rho/3$) but in this paper we merely discuss the dust solution. This will be done in the next section.

III. THE DUST UNIVERSE

The dust universe is characterized by a vanishing pressure. Although such an equation of state is not very realistic concerning the evolution of galaxies, the following discussion shows important features of the application of the PMF method to higher orders. In particular, we will find new nonlinear effects. The following study of the dust universe can be considered also as some kind of training for handling the full (i.e., not restricted to the first order) PMF method. The Friedmann equations imply

$$R(t) = Kt^{2/3}, \quad K = \text{const} \ ,$$
$$\rho_0 = (6\pi G t^2)^{-1}, \quad p_0 = 0 \ . \tag{20}$$

If we insert this into our system of Eqs. (19) and decouple it in the same manner as we did it in first order in [5], we obtain the following results.

First order

The first order results are

$$g^{(1)}_{10} = (A_2 + A_1 t^{5/3})\cos(qx) \ , \tag{21}$$

$$g^{(1)}_{00} = \frac{10A_1 t^{2/3}\sin(qx)}{3q} \ , \tag{22}$$

$$g^{(1)}_{11} = \left(\frac{8A_2 k^2 t^{1/3}}{3q} + \frac{80A_1 K^4 t^{4/3}}{9q^3} + \frac{8A_1 K^2 t^2}{3q} + K^2 t^{4/3}\tau\right)\sin(qx) \ , \tag{23}$$

$$g_{22}^{(1)} = \left(\frac{-4A_2K^2t^{1/3}}{3q} - \frac{40A_1K^4t^{4/3}}{9q^3} - \frac{4A_1K^2t^2}{3q} \right) \sin(qx) \ , \quad (24)$$

where q is the wave number [for reasons of simplicity we choose solutions proportional to $\exp(iqx)$]; A_1 and A_2 are constants related in a unique way to the "history" of the universe in question (i.e., the state of the perturbed universe, see [5]); τ, however, merely reflects the remaining freedom of performing transformations within the PMF gauge, and has, hence, no deeper meaning (see again [5]). Note that the constants A_1 and A_2 are defined somewhat different as those in [5].

Second order

The decoupling of (19) in second order is performed in the same way as in first order. The second-order equations are also analytically relatively easy to solve, because the source terms $S_{\mu\nu}^{(1)}$ are—after inserting the first-order dust solutions (21), (22), (23), and (24)—just finite sums of powers of t. We obtain

$$g_{10}^{(2)} = \left[-\frac{A_2^2}{2qt} - \frac{7A_1A_2t^{2/3}}{2q} - \frac{3A_1^2t^{7/3}}{q} + B_2 + \frac{3B_1t^{5/3}}{5} \right] \sin(2qx) \ , \quad (25)$$

$$g_{00}^{(2)} = \left[-\frac{A_2^2}{2q^2t^2} + \frac{16A_1A_2}{9q^2t^{1/3}} + \frac{58A_1^2t^{4/3}}{9q^2} - \frac{B_1t^{2/3}}{q} \right] \cos(2qx) + \left[\frac{A_2^2}{2q^2t^2} - \frac{16A_1A_2}{9q^2t^{1/3}} - \frac{50A_1^2K^2t^{2/3}}{27q^4} - \frac{58A_1^2t^{4/3}}{9q^2} \right] \ , \quad (26)$$

$$\begin{aligned} g_{11}^{(2)} = \Bigg[& \frac{-A_2^2}{2} - \frac{16A_2^2K^2}{9q^2t^{2/3}} - \frac{440A_1A_2K^4t^{1/3}}{27q^4} - \frac{4B_2K^2t^{1/3}}{3q} - \frac{4A_1A_2K^2t}{3q^2} + B_3t^{4/3} \\ & -A_1A_2t^{5/3} - \frac{440A_1^2K^4t^2}{27q^4} - \frac{4B_1K^2t^2}{5q} + \frac{4A_1^2K^2t^{8/3}}{9q^2} - \frac{A_1^2t^{10/3}}{2} \Bigg] \cos(2qx) \\ & -\frac{A_2^2}{2} + \frac{16A_2^2K^2}{9q^2t^{2/3}} - F_3K^2t^{1/3} + \frac{4A_1A_2K^2t}{3q^2} + F_4K^2t^{4/3} \\ & -A_1A_2t^{5/3} + \frac{160A_1^2K^4t^2}{27q^4} - \frac{4A_1^2K^2t^{8/3}}{9q^2} - \frac{A_1^2t^{10/3}}{2} \ , \end{aligned} \quad (27)$$

$$\begin{aligned} g_{22}^{(2)} = \Bigg[& \frac{-4A_2^2K^2}{9q^2t^{2/3}} - \frac{20A_1A_2K^4t^{1/3}}{27q^4} + \frac{2B_2K^2t^{1/3}}{3q} - \frac{2A_1A_2K^2t}{q^2} \\ & + \frac{350A_1^2K^6t^{4/3}}{81q^6} + \frac{B_1K^4t^{4/3}}{3q^3} - \frac{20A_1^2K^4t^2}{27q^4} + \frac{2B_1K^2t^2}{5q} - \frac{14A_1^2K^2t^{8/3}}{9q^2} \Bigg] \cos(2qx) \\ & + \frac{4A_2^2K^2}{9q^2t^{2/3}} - F_1K^2t^{1/3} + \frac{2A_1A_2K^2t}{q^2} + F_2K^2t^{4/3} + \frac{160A_1^2K^4t^2}{27q^4} + \frac{14A_1^2K^2t^{8/3}}{9q^2} \ . \end{aligned} \quad (28)$$

The new parameters B_i and F_i arise by solving the homogeneous part of the system of differential equations (19); the terms that contain the parameters A_i, which are already known from our first-order analysis, are generated as special inhomogeneous solutions by the source terms $S_{\mu\nu}^{(1)}$. The integration constant τ, which can be transformed to zero within the PMF gauge, has been omitted. While the B_i can be chosen freely, the parameters F_i have to satisfy the following equation:

$$160A_1A_2K^2 + 18F_1q^4 + 9F_3q^4 = 0 \ . \quad (29)$$

Note that the transition from the first order to the second one causes a doubling of the wave number q! Additionally, terms which are spatially constant arise.

Third order

Again, the source terms turn out to be—after inserting the just obtained dust solutions in first and second order—finite sums of powers of t. Hence, (19) is also in third order easy to solve. We get

$$g_{10}^{(3)} = \left[C_2 + \frac{A_2^3}{8q^2t^2} + \frac{5A_1A_2^2K^2}{2q^4t} + \frac{3A_2B_2}{4qt} - \frac{109A_1A_2^2}{36q^2t^{1/3}} + \frac{40A_1^2A_2K^2t^{2/3}}{3q^4} \right.$$
$$\left. + \frac{29A_2B_1t^{2/3}}{20q} + \frac{17A_1B_2t^{2/3}}{6q} - \frac{763A_1^2A_2t^{4/3}}{72q^2} + \frac{3C_1t^{5/3}}{5} + \frac{40A_1^3K^2t^{7/3}}{21q^4} + \frac{27A_1B_1t^{7/3}}{10q} - \frac{67A_1^3t^3}{9q^2} \right] \cos(3qx)$$
$$+ \left[C_4 - \frac{A_2^3}{8q^2t^2} - \frac{A_2F_1}{4t} + \frac{A_2F_3}{4t} + \frac{25A_1A_2^2K^2}{18q^4t} - \frac{A_2B_2}{4qt} + \frac{109A_1A_2^2}{36q^2t^{1/3}} - \frac{3A_1F_1t^{2/3}}{2} \right.$$
$$+ \frac{A_1F_3t^{2/3}}{4} + \frac{25A_1^2A_2K^2t^{2/3}}{27q^4} - \frac{3A_2B_1t^{2/3}}{20q} - \frac{3A_1B_2t^{2/3}}{2q} + \frac{763A_1^2A_2t^{4/3}}{72q^2}$$
$$\left. + \frac{3C_3t^{5/3}}{5} + \frac{250A_1^3K^2t^{7/3}}{63q^4} - \frac{9A_1B_1t^{7/3}}{10q} + \frac{67A_1^3t^3}{9q^2} \right] \cos(qx) \ , \tag{30}$$

$$g_{00}^{(3)} = \left[\frac{-5A_1A_2^2K^2}{3q^5t^2} - \frac{A_2B_2}{2q^2t^2} + \frac{25A_1A_2^2}{27q^3t^{4/3}} + \frac{160A_1^2A_2K^2}{27q^5t^{1/3}} + \frac{8A_2B_1}{15q^2t^{1/3}} + \frac{8A_1B_2}{9q^2t^{1/3}} - \frac{200A_1^2A_2t^{1/3}}{27q^3} + \frac{2C_1t^{2/3}}{3q} \right.$$
$$\left. + \frac{80A_1^3K^2t^{4/3}}{27q^5} + \frac{58A_1B_1t^{4/3}}{15q^2} - \frac{350A_1^3t^2}{27q^3} \right] \sin(3qx)$$
$$+ \left[\frac{-25A_1A_2^2K^2}{9q^5t^2} + \frac{A_2B_2}{2q^2t^2} + \frac{A_2F_1}{2qt^2} - \frac{A_2F_3}{2qt^2} - \frac{25A_1A_2^2}{9q^3t^{4/3}} - \frac{8A_2B_1}{15q^2t^{1/3}} - \frac{8A_1B_2}{9q^2t^{1/3}} - \frac{2A_1F_1}{qt^{1/3}} + \frac{A_1F_3}{3qt^{1/3}} \right.$$
$$\left. + \frac{200A_1^2A_2t^{1/3}}{9q^3} + \frac{2C_3t^{2/3}}{q} + \frac{1400A_1^3K^2t^{4/3}}{81q^5} - \frac{58A_1B_1t^{4/3}}{15q^2} + \frac{350A_1^3t^2}{9q^3} \right] \sin(qx) \ , \tag{31}$$

$$g_{11}^{(3)} = \left[-(A_2B_2) - \frac{55A_2^3K^2}{81q^3t^{5/3}} + \frac{A_2^3}{2qt} - \frac{1120A_1A_2^2K^4}{81q^5t^{2/3}} - \frac{16A_2B_2K^2}{9q^2t^{2/3}} \right.$$
$$- \frac{1000A_1^2A_2K^6t^{1/3}}{27q^7} - \frac{4A_2B_1K^4t^{1/3}}{9q^4} - \frac{200A_1B_2K^4t^{1/3}}{27q^4} + \frac{4A_2B_3t^{1/3}}{3q}$$
$$+ \frac{8C_2K^2t^{1/3}}{9q} + \frac{19A_1A_2^2t^{2/3}}{6q} - \frac{20A_1^2A_2K^4t}{3q^5} - \frac{2A_2B_1K^2t}{5q^2} - \frac{2A_1B_2K^2t}{3q^2}$$
$$+ C_6t^{4/3} - \frac{3A_2B_1t^{5/3}}{5} - A_1B_2t^{5/3} + \frac{5A_1^2A_2K^2t^{5/3}}{27q^3} - \frac{1000A_1^3K^6t^2}{27q^7} - \frac{44A_1B_1K^4t^2}{9q^4}$$
$$+ \frac{4A_1B_3t^2}{3q} + \frac{8C_1K^2t^2}{15q} + \frac{29A_1^2A_2t^{7/3}}{6q} - \frac{440A_1^3K^4t^{8/3}}{567q^5} + \frac{4A_1B_1K^2t^{8/3}}{15q^2}$$
$$\left. - \frac{3A_1B_1t^{10/3}}{5} - \frac{40A_1^3K^2t^{10/3}}{81q^3} + \frac{13A_1^3t^4}{6q} \right] \sin(3qx)$$

$$+ \left[-(A_2B_2) + \frac{55A_2^3K^2}{27q^3t^{5/3}} + \frac{A_2^3}{2qt} + \frac{800A_1A_2^2K^4}{81q^5t^{2/3}} + \frac{16A_2B_2K^2}{9q^2t^{2/3}} \right.$$
$$+ \frac{4A_2F_1K^2}{3qt^{2/3}} - \frac{2A_2F_3K^2}{qt^{2/3}} - \frac{25\,000A_1^2A_2K^6t^{1/3}}{243q^7} + \frac{4A_2B_1K^4t^{1/3}}{3q^4}$$
$$+ \frac{40A_1B_2K^4t^{1/3}}{27q^4} + \frac{40A_1F_1K^4t^{1/3}}{9q^3} - \frac{40A_1F_3K^4t^{1/3}}{3q^3} - \frac{4A_2B_3t^{1/3}}{3q}$$
$$+ \frac{8C_4K^2t^{1/3}}{3q} + \frac{8A_2F_4K^2t^{1/3}}{3q} + \frac{19A_1A_2^2t^{2/3}}{6q} + \frac{4540A_1^2A_2K^4t}{27q^5} + \frac{2A_2B_1K^2t}{5q^2}$$
$$+ \frac{2A_1B_2K^2t}{3q^2} + \frac{18A_1F_1K^2t}{q} + \frac{8A_1F_3K^2t}{q} + C_5t^{4/3} - \frac{3A_2B_1t^{5/3}}{5}$$
$$- A_1B_2t^{5/3} - \frac{5A_1^2A_2K^2t^{5/3}}{9q^3} - \frac{12\,200A_1^3K^6t^2}{243q^7} + \frac{20A_1B_1K^4t^2}{9q^4}$$

$$-\frac{4A_1B_3t^2}{3q}+\frac{8C_3K^2t^2}{5q}+\frac{8A_1F_4K^2t^2}{3q}+\frac{29A_1^2A_2t^{7/3}}{6q}-\frac{4360A_1^3K^4t^{8/3}}{567q^5}$$
$$\left.-\frac{4A_1B_1K^2t^{8/3}}{15q^2}-\frac{3A_1B_1t^{10/3}}{5}+\frac{40A_1^3K^2t^{10/3}}{27q^3}+\frac{13A_1^3t^4}{6q}\right]\sin(qx)\ , \tag{32}$$

$$g_{22}^{(3)}=\left[\frac{4A_1A_2^2K^2}{3q^3}+\frac{7A_2^3K^2}{162q^3t^{5/3}}-\frac{40A_1A_2^2K^4}{81q^5t^{2/3}}-\frac{4A_2B_2K^2}{9q^2t^{2/3}}\right.$$
$$+\frac{200A_1^2A_2K^6t^{1/3}}{243q^7}-\frac{20A_1B_2K^4t^{1/3}}{27q^4}-\frac{4C_2K^2t^{1/3}}{9q}-\frac{110A_1^2A_2K^4t}{27q^5}-\frac{3A_2B_1K^2t}{5q^2}$$
$$-\frac{A_1B_2K^2t}{q^2}-\frac{68\,000A_1^3K^8t^{4/3}}{6561q^9}+\frac{80A_1B_1K^6t^{4/3}}{243q^6}-\frac{40A_1B_3K^2t^{4/3}}{81q^3}-\frac{8C_1K^4t^{4/3}}{81q^3}$$
$$+\frac{187A_1^2A_2K^2t^{5/3}}{54q^3}+\frac{200A_1^3K^6t^2}{243q^7}-\frac{4A_1B_1K^4t^2}{9q^4}-\frac{4C_1K^2t^2}{15q}$$
$$\left.+\frac{220A_1^3K^4t^{8/3}}{567q^5}-\frac{14A_1B_1K^2t^{8/3}}{15q^2}+\frac{176A_1^3K^2t^{10/3}}{81q^3}\right]\sin(3qx)$$
$$+\left[\frac{-4A_1A_2^2K^2}{q^3}-\frac{7A_2^3K^2}{54q^3t^{5/3}}-\frac{280A_1A_2^2K^4}{81q^5t^{2/3}}+\frac{4A_2B_2K^2}{9q^2t^{2/3}}\right.$$
$$+\frac{2A_2F_1K^2}{3qt^{2/3}}-\frac{A_2F_3K^2}{3qt^{2/3}}+\frac{800A_1^2A_2K^6t^{1/3}}{27q^7}-\frac{4A_2B_1K^4t^{1/3}}{9q^4}+\frac{100A_1B_2K^4t^{1/3}}{27q^4}$$
$$+\frac{20A_1F_1K^4t^{1/3}}{9q^3}+\frac{20A_1F_3K^4t^{1/3}}{9q^3}-\frac{4C_4K^2t^{1/3}}{3q}-\frac{4A_2F_2K^2t^{1/3}}{3q}-\frac{130A_1^2A_2K^4t}{9q^5}$$
$$+\frac{3A_2B_1K^2t}{5q^2}+\frac{A_1B_2K^2t}{q^2}+\frac{7A_1F_1K^2t}{3q}-\frac{A_1F_3K^2t}{3q}+\frac{82\,000A_1^3K^8t^{4/3}}{729q^9}$$
$$-\frac{40A_1B_1K^6t^{4/3}}{27q^6}-\frac{8C_3K^4t^{4/3}}{3q^3}-\frac{40A_1F_2K^4t^{4/3}}{9q^3}$$
$$-\frac{40A_1F_4K^4t^{4/3}}{9q^3}-\frac{187A_1^2A_2K^2t^{5/3}}{18q^3}+\frac{800A_1^3K^6t^2}{243q^7}+\frac{16A_1B_1K^4t^2}{9q^4}$$
$$\left.-\frac{4C_3K^2t^2}{5q}-\frac{4A_1F_2K^2t^2}{3q}-\frac{9580A_1^3K^4t^{8/3}}{567q^5}+\frac{14A_1B_1K^2t^{8/3}}{15q^2}-\frac{176A_1^3K^2t^{10/3}}{27q^3}\right]\sin(qx)\ . \tag{33}$$

The new parameters (or integration constants) C_i arise when we solve the homogeneous part of the differential equation system (19). Terms, which contain the already known A_i, B_i, or F_i, are generated as special inhomogeneous solutions of (19) by the source terms $S_{\mu\nu}^{(2)}$. Note that in third order the wave number has tripled compared with the first order; additionally, terms proportional to $\sin(qx)$ or $\cos(qx)$ arise.

The following, general structure of the PMF solution for the dust universe now becomes visible:

$$g_{\mu\nu}^{(n)}=f_0^{(n)}[t]\mathrm{trig}(nqx)+f_2^{(n)}[t]\mathrm{trig}((n-2)qx)+f_4^{(n)}[t]\mathrm{trig}((n-4)qx)+\cdots+f_{2\mathrm{Int}[n/2]}^{(n)}[t]\mathrm{trig}(n-2\mathrm{Int}[n/2])qx)\ . \tag{34}$$

Here, the $f_i^{(n)}[t]$ are sums of powers of t; trig is either sin or cos (according to the order and the component in equation), and Int is the integer function. Thus, in nth order we get the wave numbers nq, $(n-2)q$, $(n-4)q,\ldots$; this sequence ends with q or $0q$. Hence, a harmonic (i.e., proportional to sin or cos) fluctuation is, in higher orders, necessarily accompanied by corrections of equal and lesser extension. It must be mentioned that (34) still has to be proved for general n. Comparison of the functions $f_i^{(n)}[t]$ for $n=1,2,3$ with each other shows regularities that we are going to discuss now by means of the following case study.

To that end we set the free parameters of orders higher than the first one equal to zero; hence, the only remaining nonvanishing parameters are those given by the first order, namely the A_i's. Let us write down here merely the 10 component:

$$g_{10}^{(1)}=(A_2+A_1t^{5/3})\cos(qx)\ , \tag{35}$$

$$g_{10}^{(2)} = \left[-\frac{A_2^2}{2qt} - \frac{7A_1A_2t^{2/3}}{2q} - \frac{3A_1^2t^{7/3}}{q}\right]\sin(2qx) , \tag{36}$$

$$g_{10}^{(3)} = \left[\frac{A_2^3}{8q^2t^2} - \frac{109A_1A_2^2}{36q^2t^{1/3}} - \frac{763A_1^2A_2t^{4/3}}{72q^2} - \frac{67A_1^3t^3}{9q^2} + \frac{5A_1A_2^2K^2}{2q^4t} + \frac{40A_1^2A_2K^2t^{2/3}}{3q^4} + \frac{40A_1^3K^2t^{7/3}}{21q^4}\right]\cos(3qx)$$
$$+ \left[-\frac{A_2^3}{8q^2t^2} + \frac{109A_1A_2^2}{36q^2t^{1/3}} + \frac{763A_1^2A_2t^{4/3}}{72q^2} + \frac{67A_1^3t^3}{9q^2} + \frac{25A_1A_2^2K^2}{18q^4t} + \frac{25A_1^2A_2K^2t^{2/3}}{27q^4} + \frac{250A_1^3K^2t^{7/3}}{63q^4}\right]\cos(qx) . \tag{37}$$

First of all, the results (35), (36), and (37) show again very clearly that in higher orders *necessarily* corrections to the first order arise (observe, that the $g_{10}^{(n)}$ are different from zero for $n \geq 1$).

We have written down the 10 component in each order such that within that order the sums are ordered according to different powers of $1/q$; and within these powers they are ordered according to increasing powers of A_1. Now we see the following: within the same power of $1/q$ the exponent of t increases by 5/3 when the power of A_1 increases by 1. From one order to the next the lowest power of $1/q$ that appears increases always by $1/q$, the corresponding sequence of exponents of t starting from a value reduced by 1 (compared with the starting value of the previous order): A_2t^0, $A_2^2t^{-1}$, $A_2^3t^{-2}$, etc. The highest time exponent appearing in each order increases order by order by 2/3: $A_1t^{5/3}$, $A_1^2t^{7/3}$, $A_1^3t^{9/3}$. In the third order we have additionally to the terms proportional to $1/q^2$ such proportional to K^2/q^4. They have the same time exponents as the terms of the second order where the one with the highest time exponent (proportional to $A_1^3K^2t^{7/3}/q^4$) corresponds to that one with the highest time exponent in second order (proportional to $A_1^2t^{7/3}/q$). Since in second order there are only three such terms available, we have no contribution proportional to $A_2^3K^2/q^4$. This holds as well for the terms proportional to $\cos(3qx)$ as for those proportional to $\cos(qx)$. Thus, we can expect the following terms in fourth order.

Proportional to $1/q^3$:

$$A_2^4t^{-3}/q^3, A_2^3A_1t^{-4/3}/q^3, A_2^2A_1^2t^{1/3}/q^3, A_2A_1^3t^2/q^3, A_1^4t^{11/3}/q^3 ,$$

Proportional to K^2/q^5:

$$A_2^3A_1t^{-2}K^2/q^5, A_2^2A_1^2t^{-1/3}K^2/q^5, A_2A_1^3t^{4/3}K^2/q^5, A_1^4t^3K^2/q^5 ,$$

Proportional to K^4/q^7:

$$A_2^2A_1^2t^{-1}K^4/q^7, A_2A_1^3t^{2/3}K^4/q^7, A_1^4t^{7/3}K^4/q^7 .$$

(The appearance of the terms proportional to K^4/q^7 is especially suggested by doing an analogous consideration for the other metric components.)

This consideration shows that fluctuations with large spatial extension (i.e., $q \to 0$) are governed within each order n by terms $A_1^nK^{2(n-2)}t^{7/3}/q^{3n-5}$, if we consider here only the growing mode (to this end we set $A_2 = 0$), for it possesses the highest power of $1/q$. For smaller fluctuations also terms with smaller powers of $1/q$ are important. But those contain higher powers of t. The one with the highest power of t is proportional to $A_1^nt^{2n/3+1}/q^{n-1}$. This means that such fluctuations are growing faster provided that we are far beyond the Jeans limit (this lower boundary of the instability region is for the case of the dust universe simply being $q = \infty$). Let us stress that while in first order small extended fluctuations grow as fast as large extended ones (namely, proportional to $A_1t^{5/3}$), a higher-order analysis shows that they have different growth rates. The reason is that order by order terms with constantly increasing time exponents are appearing; however, these "growth terms" are dominated in the case of perturbations with large extension by others that are growing only moderately. It is true that this dominance is disappearing when t becomes sufficiently big enough but this boundary can be shifted arbitrarily towards the future if we merely take q small enough. This is a clear hint for the existence of an upper boundary of the region of instability, and this existence is caused by nonlinear effects. To sum up: it seems that nonlinear effects do not stabilize a perturbation which is unstable according to the first-order analysis but they let large extended fluctuations just moderately grow while they cause small extended ones to grow much faster.

The other metric components show an analogous behavior. It must be mentioned that all those conjectures have not been strictly proven yet; they have just been verified for $n \leq 3$. This should be done in the future; similarly, we have to find the algorithm for the numerical coefficients in our sums. Especially for small extended fluctuation the whole sum is important and we have to calculate its limit in order to know what functions of t the metric components are. However, since those "growth terms," which are proportional to $\cos(qx)$, namely $A_1t^{5/3}$ in $g_{10}^{(1)}$ and $67A_1^3t^3/9q^2$ in $g_{10}^{(3)}$, both have the same sign "+," one might expect that by summing up all those contributions we get a rapidly increasing perturbation (provided, of course, that q is not too small). Moreover, our analysis has been performed so far merely within the PMF gauge. It remains to study whether the results found in that gauge, in particular to what extent the nonlinear effects, survive if we transform the PMF solution into a gauge which is "close to the background."

One can also infer from (35), (36), and (37) the following feature. Let us assume that the first-order perturbation quantities $g_{\mu\nu}^{(1)}$ are small compared to the background quantities (i.e., the parameters A_i have to be very small). Then, the necessary correction terms of higher orders like $g_{\mu\nu}^{(2)}, g_{\mu\nu}^{(3)}$, etc. are automatically small compared to those of lower orders provided that q and t are moderate. The reason is that they contain terms proportional to $A_1^r A_2^s$, where $r + s = n$. Hence, for small times and for not too largely extended fluctuations a first-order analysis is justified. This is what we expect.

IV. CONCLUSIONS AND PERSPECTIVES

Let us now summarize the main results of our higher-order analysis. First of all higher-order ("nonlinear") effects cause, in principle, perturbations to grow much faster than they grow according to the first-order analysis. The reason for that behavior is that higher-order contributions contain "growth terms," and it seems that they all have the same sign (i.e., they are all acting along the same direction). However, for very large perturbations those growth terms are dominated within each other by others which grow only moderately. Thus, we get the following picture: fluctuations with large extension (beyond super clusters of galaxies?) grow only moderately (more or less with a similar small rate they grow according to the first-order analysis) but the other perturbations grow much faster provided their extension is beyond the Jeans limit. It must be mentioned that this interesting result has to be taken for the moment just as a conjecture which is supported by some tendencies observed from the results in PMF gauge in first, second, and third order. We still have to work out the solution in nth order. Moreover, our equation of state used here (dust universe) is not too realistic with regard to the formation of galaxies. And, finally, in order to be in the position for judging stability/instability of a given perturbation we should transform our PMF solution into some appropriate gauge. Then, we can infer from the density contrast within this gauge whether that perturbation is stable or not. However, since in the PMF gauge instabilities show up in the metric components alone rather than in the density contrast, one can expect that in an appropriate gauge we will observe a similar behavior for the density contrast like that which the metric components show in the PMF gauge. As was suggested by one of the referees of this paper in his report it can be avoided to transform the PMF solution into some appropriate gauge. Instead, we can construct out of the PMF metric perturbations physically meaningful quantities and measures of the instability at the given perturbative order. If, after performing all these improvements, the observed tendencies survive at least in principle, we have found a possible explanation for the breaking off of the hierarchy (clusters of stars, galaxies, clusters of galaxies, super-clusters) at super-clusters or at super-super-clusters [9–13]. In this case, general relativity would imply an upper boundary of the instability region.

Another result is that the transition to higher orders is connected with a multiplication of the wave numbers. Maximally, we obtain in nth order a contribution proportional to $\sin(nqx)$ or $\cos(nqx)$, but there are also terms with smaller wave numbers [see (34)]. Altogether, we observe a kind of fragmentation which is increasing with the order (i.e., with the evolution time). That means that more and more parts of the fluctuations evolve differently, and the extension of those parts becomes smaller and smaller when the order increases. Hence, the perturbation is fraying more and more when time passes by.

In our analysis we have considered only fluctuations with a spatial shape proportional to sine or cosine. However, the realistic fluctuations are those whose spatial shape is, e.g., something like a Gaussian. This is not at all a problem in first order because we can obtain the solution for any shape by means of a Fourier synthesis. But in the nonlinear theory, a sum of solutions of the field equations is not necessarily also a solution of these equations. Nevertheless, we can use our solutions obtained in this paper also in such a case. We just have to perform the Fourier analysis "order by order." This expression is to be understood as follows. First of all, we perform the Fourier analysis in first order. Our solutions proportional to $\exp(iqx)$ are solutions of the individual Fourier components. Subsequently, we compose them and obtain the solution of the given perturbation in first order. That solution must then be inserted into the source terms in (19) for $n = 2$. We form the products and sums of $S_{\mu\nu}^{(1)}$, and, after that, we decompose the source terms into their Fourier components. Then, our solutions proportional to $\exp(iqx)$ are again solutions of the individual Fourier components. Since the perturbation field equations (19) are also in second-order linear in the unknown functions $g_{\mu\nu}^{(2)}$, the Fourier synthesis out of our solutions proportional to sine and cosine yields the full solution in second order which is to be inserted into the source terms in third order. In this way we proceed order by order.

Moreover, one should rid oneself of the concept of perturbations with two-dimensional symmetry planes, if the aim is a theory of the formation of galaxies as realistic as possible. Instead, one should consider perturbations that are (approximately) spherically symmetric. The transition to these fluctuations should not be any problem because the starting metric (in spherical coordinates) is not much more complicated than (1). Our PMF methods should, hence, be applicable also in this case.

The PMF method can be used also on a larger scale. If our conjecture about the solution in an arbitrary order turns out to be true, we can generate by means of the PMF method exact solutions (presented as infinite power series in t) of Einstein's field equations. Then, it is not necessary to assume that the fluctuations are small compared to the corresponding background quantities; however, if we drop that assumption, all orders are equally important. An investigation about the structure of these solutions and their classification could give useful information about classical general relativity.

A further application is obvious. If we give up our separation ansatz $g^{(n)}_{\mu\nu} = f(t)h(x)$, we can investigate also gravitational waves propagating in a Friedmann-Robertson-Walker universe, i.e., propagation through matter. It should be possible to obtain solutions (with the help of the PMF method) even without such a separation ansatz, because our system of differential equations (19) can be decoupled independently from that ansatz [see, e.g., [5], (3.17)–(3.22)].

Finally, we want to emphasize that the main aim of this paper is not to supply a theory of galaxy formation, which is as realistic as possible. These investigations should rather be understood as a first step towards such a theory satisfying astrophysicists. First of all, we are interested in the development of a method powerful enough for solving the field equations also in the case of space-times, which are less homogeneous than, e.g., the Friedmann universe. We wanted to understand what kind of principle problems arise, and how they can be handled. Additionally, we were interested in what kind of new effects caused by higher orders appear. The most essential new nonlinear effect is that perturbations grow much faster than they do according to a first-order analysis, but those perturbations with an extremely large extension do not.

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APPENDIX A

Here we give the full perturbation field equations up to second order for the relevant components.

1. First order

00 component:

$$8G\pi g^{(1)}_{00}(x,t)\rho_0(t) + \frac{4g^{(1)}_{22}(x,t)\dot{R}(t)^2}{R(t)^4} + \frac{2g^{(1)}_{11}(x,t)\dot{R}(t)^2}{R(t)^4} - \frac{6g^{(1)}_{00}(x,t)\dot{R}(t)^2}{R(t)^2}$$
$$-\frac{2\dot{R}(t)\frac{dg^{(1)}_{22}(x,t)}{dt}}{R(t)^3} - \frac{\dot{R}(t)\frac{dg^{(1)}_{11}(x,t)}{dt}}{R(t)^3} + \frac{2R(t)\frac{dg^{(1)}_{10}(x,t)}{dx}}{R(t)^3} + \frac{\frac{d^2g^{(1)}_{22}(x,t)}{dx^2}}{R(t)^4} = 0\ . \tag{A1}$$

10 component:

$$\frac{8G\pi g^{(1)}_{10}(x,t)p_0(t)}{R(t)^2} + \frac{3g^{(1)}_{10}(x,t)\dot{R}(t)^2}{R(t)^4} - \frac{\dot{R}(t)\frac{dg^{(1)}_{00}(x,t)}{dx}}{R(t)^3} + \frac{2\dot{R}(t)\frac{dg^{(1)}_{22}(x,t)}{dx}}{R(5)^5} - \frac{\frac{d^2g^{(1)}_{22}(x,t)}{dxdt}}{R(t)^4} = 0\ . \tag{A2}$$

Energy conservation:

$$\frac{-4g^{(1)}_{22}(x,t)\dot{R}(t)}{R(t)^3} - \frac{2g^{(1)}_{11}(x,t)\dot{R}(t)}{R(t)^3} + \frac{2\frac{dg^{(1)}_{22}(x,t)}{dt}}{R(t)^2} + \frac{\frac{dg^{(1)}_{11}(x,t)}{dt}}{R(t)^2} = 0\ . \tag{A3}$$

Momentum conservation:

$$\frac{g^{(1)}_{10}(x,t)\dot{p}_0(t)}{R(t)^2} + \frac{p_0(t)\frac{dg^{(1)}_{10}(x,t)}{dt}}{R(t)^2} + \frac{\rho_0(t)\frac{dg^{(1)}_{10}(x,t)}{dt}}{R(t)^2} - \frac{p_0(t)\frac{dg^{(1)}_{00}(x,t)}{dx}}{2R(t)^2} - \frac{\rho_0(t)\frac{dg^{(1)}_{00}(x,t)}{dx}}{2R(t)^2} = 0\ . \tag{A4}$$

2. Second order

00 component:

$$8G\pi g_{00}^{(2)}(x,t)\rho_0(t)+\frac{4g_{22}^{(2)}(x,t)\dot{R}(t)^2}{R(t)^4}+\frac{2g_{11}^{(2)}(x,t)\dot{R}(t)^2}{R(t)^4}-\frac{6g_{00}^{(2)}(x,t)\dot{R}(t)^2}{R(t)^2}$$
$$-\frac{2\dot{R}(t)\frac{dg_{22}^{(2)}(x,t)}{dt}}{R(t)^3}-\frac{\dot{R}(t)\frac{dg_{11}^{(2)}(x,t)}{dt}}{R(t)^3}+\frac{2\dot{R}(t)\frac{dg_{10}^{(2)}(x,t)}{dx}}{R(t)^3}+\frac{\frac{d^2g_{22}^{(2)}(x,t)}{dx^2}}{R(t)^4}$$
$$+\frac{8G\pi g_{10}^{(1)}(x,t)^2p_0(t)}{R(t)^2}+8G\pi g_{00}^{(1)}(x,t)^2\rho_0(t)-\frac{5g_{22}^{(1)}(x,t)^2\dot{R}(t)^2}{R(t)^6}$$
$$-\frac{2g_{22}^{(1)}(x,t)g_{11}^{(1)}(x,t)\dot{R}(t)^2}{R(t)^6}-\frac{2g_{11}^{(1)}(x,t)^2\dot{R}(t)^2}{R(t)^6}+\frac{6g_{10}^{(1)}(x,t)^2\dot{R}(t)^2}{R(t)^4}$$
$$+\frac{8g_{00}^{(1)}(x,t)g_{22}^{(1)}(x,t)\dot{R}(t)^2}{R(t)^4}+\frac{4g_{00}^{(1)}(x,t)g_{11}^{(1)}(x,t)\dot{R}(t)^2}{R(t)^4}$$
$$-\frac{9g_{00}^{(1)}(x,t)^2\dot{R}(t)^2}{R(t)^2}+\frac{3g_{22}^{(1)}(x,t)\dot{R}(t)\frac{dg_{22}^{(1)}(x,t)}{dt}}{R(t)^5}+\frac{g_{11}^{(1)}(x,t)\dot{R}(t)\frac{dg_{22}^{(1)}(x,t)}{dt}}{R(t)^5}$$
$$-\frac{4g_{00}^{(1)}(x,t)\dot{R}(t)\frac{dg_{22}^{(1)}(x,t)}{dt}}{R(t)^3}-\frac{\frac{dg_{22}^{(1)}(x,t)}{dt}^2}{4R(t)^4}+\frac{g_{22}^{(1)}(x,t)\dot{R}(t)\frac{dg_{11}^{(1)}(x,t)}{dt}}{R(t)^5}$$
$$+\frac{g_{11}^{(1)}(x,t)\dot{R}(t)\frac{dg_{11}^{(1)}(x,t)}{dt}}{R(t)^5}-\frac{2g_{00}^{(1)}(x,t)\dot{R}(t)\frac{dg_{11}^{(1)}(x,t)}{dt}}{R(t)^3}-\frac{\frac{dg_{22}^{(1)}(x,t)}{dt}\frac{dg_{11}^{(1)}(x,t)}{dt}}{2R(t)^4}$$
$$-\frac{2g_{22}^{(1)}(x,t)\dot{R}(t)\frac{dg_{10}^{(1)}(x,t)}{dx}}{R(t)^5}-\frac{2g_{11}^{(1)}(x,t)\dot{R}(t)\frac{dg_{10}^{(1)}(x,t)}{dx}}{R(t)^5}+\frac{4g_{00}^{(1)}(x,t)\dot{R}(t)\frac{dg_{10}^{(1)}(x,t)}{dx}}{R(t)^3}$$
$$+\frac{\frac{dg_{22}^{(1)}(x,t)}{dt}\frac{dg_{10}^{(1)}(x,t)}{dx}}{R(t)^4}+\frac{2g_{10}^{(1)}(x,t)\dot{R}(t)\frac{dg_{22}^{(1)}(x,t)}{dx}}{R(t)^5}-\frac{\frac{dg_{22}^{(1)}(x,t)}{dx}^2}{4R(t)^6}$$
$$-\frac{g_{10}^{(1)}(x,t)\dot{R}(t)\frac{dg_{11}^{(1)}(x,t)}{dx}}{R(t)^5}-\frac{\frac{dg_{22}^{(1)}(x,t)}{dx}\frac{dg_{11}^{(1)}(x,t)}{dx}}{2R(t)^6}-\frac{g_{22}^{(1)}(x,t)\frac{d^2g_{22}^{(1)}(x,t)}{dx^2}}{R(t)^6}$$
$$-\frac{g_{11}^{(1)}(x,t)\frac{d^2g_{22}^{(1)}(x,t)}{dx^2}}{R(t)^6}+\frac{g_{00}^{(1)}(x,t)\frac{d^2g_{22}^{(1)}(x,t)}{dx^2}}{R(t)^4}=0\ . \tag{A5}$$

10 component:

$$\frac{8G\pi g_{10}^{(2)}(x,t)p_0(t)}{R(t)^2}+\frac{3g_{10}^{(2)}(x,t)\dot{R}(t)^2}{R(t)^4}-\frac{\dot{R}(t)\frac{dg_{00}^{(2)}(x,t)}{dx}}{R(t)^3}$$
$$+\frac{2\dot{R}(t)\frac{dg_{22}^{(2)}(x,t)}{dx}}{R(t)^5}-\frac{\frac{d^2g_{22}^{(2)}(x,t)}{dxdt}}{R(t)^4}-\frac{8G\pi g_{10}^{(1)}(x,t)g_{11}^{(1)}(x,t)p_0(t)}{R(t)^4}$$
$$+\frac{8G\pi g_{00}^{(1)}(x,t)g_{10}^{(1)}(x,t)p_0(t)}{R(t)^2}-\frac{4g_{10}^{(1)}(x,t)g_{22}^{(1)}(x,t)\dot{R}(t)^2}{R(t)^6}-\frac{5g_{10}^{(1)}(x,t)g_{11}^{(1)}(x,t)\dot{R}(t)^2}{R(t)^6}$$
$$+\frac{6g_{00}^{(1)}(x,t)g_{10}^{(1)}(x,t)\dot{R}(t)^2}{R(t)^4}+\frac{2g_{10}^{(1)}(x,t)\dot{R}(t)\frac{dg_{22}^{(1)}(x,t)}{dt}}{R(t)^5}+\frac{g_{10}^{(1)}(x,t)\dot{R}(t)\frac{dg_{11}^{(1)}(x,t)}{dt}}{R(t)^5}$$
$$+\frac{g_{22}^{(1)}(x,t)\dot{R}(t)\frac{dg_{00}^{(1)}(x,t)}{dx}}{R(t)^5}+\frac{g_{11}^{(1)}(x,t)\dot{R}(t)\frac{dg_{00}^{(1)}(x,t)}{dx}}{R(t)^5}-\frac{2g_{00}^{(1)}(x,t)\dot{R}(t)\frac{dg_{00}^{(1)}(x,t)}{dx}}{R(t)^3}$$
$$-\frac{\frac{dg_{22}^{(1)}(x,t)}{dt}\frac{dg_{00}^{(1)}(x,t)}{dx}}{2R(t)^4}-\frac{3g_{22}^{(1)}(x,t)\dot{R}(t)\frac{dg_{22}^{(1)}(x,t)}{dx}}{R(t)^7}-\frac{3g_{11}^{(1)}(x,t)\dot{R}(t)\frac{dg_{22}^{(1)}(x,t)}{dx}}{R(t)^7}$$
$$+\frac{2g_{00}^{(1)}(x,t)\dot{R}(t)\frac{dg_{22}^{(1)}(x,t)}{dx}}{R(t)^5}+\frac{\frac{dg_{22}^{(1)}(x,t)}{dt}\frac{dg_{22}^{(1)}(x,t)}{dx}}{2R(t)^6}+\frac{\frac{dg_{11}^{(1)}(x,t)}{dt}\frac{dg_{22}^{(1)}(x,t)}{dx}}{2R(t)^6}$$
$$+\frac{g_{22}^{(1)}(x,t)\frac{d^2g_{22}^{(1)}(x,t)}{dxdt}}{R(t)^6}+\frac{g_{11}^{(1)}(x,t)\frac{d^2g_{22}^{(1)}(x,t)}{dxdt}}{R(t)^6}-\frac{g_{00}^{(1)}(x,t)\frac{d^2g_{22}^{(1)}(x,t)}{dxdt}}{R(t)^4}=0\ . \tag{A6}$$

Energy conservation:

$$\frac{-4g_{22}^{(2)}(x,t)\dot{R}(t)}{R(t)^3}-\frac{2g_{11}^{(2)}(x,t)\dot{R}(t)}{R(t)^3}+\frac{2\frac{dg_{22}^{(2)}(x,t)}{dt}}{R(t)^2}+\frac{\frac{dg_{11}^{(2)}(x,t)}{dt}}{R(t)^2}$$
$$+\frac{4g_{22}^{(1)}(x,t)^2\dot{R}(t)}{R(t)^5}+\frac{2g_{11}^{(1)}(x,t)^2\dot{R}(t)}{R(t)^5}-\frac{2g_{10}^{(1)}(x,t)^2\dot{R}(t)}{R(t)^3}$$
$$-\frac{4g_{00}^{(1)}(x,t)g_{22}^{(1)}(x,t)\dot{R}(t)}{R(t)^3}-\frac{2g_{00}^{(1)}(x,t)g_{11}^{(1)}(x,t)\dot{R}(t)}{R(t)^3}-\frac{6g_{10}^{(1)}(x,t)^2\dot{p}_0(t)\dot{R}(t)}{R(t)^3\dot{\rho}_0(t)}$$
$$+\frac{4g_{10}^{(1)}(x,t)\frac{dg_{10}^{(1)}(x,t)}{dt}}{R(t)^2}-\frac{2g_{22}^{(1)}(x,t)\frac{dg_{22}^{(1)}(x,t)}{dt}}{R(t)^4}+\frac{2g_{00}^{(1)}(x,t)\frac{dg_{22}^{(1)}(x,t)}{dt}}{R(t)^2}$$
$$-\frac{g_{11}^{(1)}(x,t)\frac{dg_{11}^{(1)}(x,t)}{dt}}{R(t)^4}+\frac{g_{00}^{(1)}(x,t)\frac{dg_{11}^{(1)}(x,t)}{dt}}{R(t)^2}-\frac{g_{10}^{(1)}(x,t)\frac{dg_{00}^{(1)}(x,t)}{dx}}{R(t)^2}=0\;. \tag{A7}$$

Momentum conservation:

$$\frac{g_{10}^{(2)}(x,t)\dot{p}_0(t)}{R(t)^2}+\frac{p_0(t)\frac{dg_{10}^{(2)}(x,t)}{dt}}{R(t)^2}+\frac{\rho_0(t)\frac{dg_{10}^{(2)}(x,t)}{dt}}{R(t)^2}-\frac{p_0(t)\frac{dg_{00}^{(2)}(x,t)}{dx}}{2R(t)^2}-\frac{\rho_0(t)\frac{dg_{00}^{(2)}(x,t)}{dx}}{2R(t)^2}$$
$$-\frac{g_{10}^{(1)}(x,t)g_{11}^{(1)}(x,t)\dot{p}_0(t)}{R(t)^4}+\frac{g_{00}^{(1)}(x,t)g_{10}^{(1)}(x,t)\dot{p}_0(t)}{R(t)^2}+\frac{g_{10}^{(1)}(x,t)p_0(t)\frac{dg_{00}^{(1)}(x,t)}{dt}}{2R(t)^2}$$
$$+\frac{g_{10}^{(1)}(x,t)\rho_0(t)\frac{dg_{00}^{(1)}(x,t)}{dt}}{2R(t)^2}-\frac{g_{11}^{(1)}(x,t)p_0(t)\frac{dg_{10}^{(1)}(x,t)}{dt}}{R(t)^4}+\frac{g_{00}^{(1)}(x,t)p_0(t)\frac{dg_{10}^{(1)}(x,t)}{dt}}{R(t)^2}$$
$$-\frac{g_{11}^{(1)}(x,t)\rho_0(t)\frac{dg_{10}^{(1)}(x,t)}{dt}}{R(t)^4}+\frac{g_{00}^{(1)}(x,t)\rho_0(t)\frac{dg_{10}^{(1)}(x,t)}{dt}}{R(t)^2}$$
$$+\frac{g_{11}^{(1)}(x,t)p_0(t)\frac{dg_{00}^{(1)}(x,t)}{dx}}{2R(t)^4}-\frac{g_{00}^{(1)}(x,t)p_0(t)\frac{dg_{00}^{(1)}(x,t)}{dx}}{2R(t)^2}+\frac{g_{11}^{(1)}(x,t)\rho_0(t)\frac{dg_{00}^{(1)}(x,t)}{dx}}{2R(t)^4}$$
$$-\frac{g_{00}^{(1)}(x,t)\rho_0(t)\frac{dg_{00}^{(1)}(x,t)}{dx}}{2R(t)^2}=0\;. \tag{A8}$$

Note that these equations transform into the system (19) if we insert in each order the energy conservation equation into the 00 component, and if we divide the momentum conservation equation by (ρ_0+p_0).

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