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Kaplan-Narayanan-Neuberger lattice fermions pass a perturbative test

S. Aoki*
Institute of Physics, University of Tsukuba, Tsukuba Ibaraki-305, Japan

R. B. Levien†
Department of Physics, Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
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We test perturbatively a recent scheme for implementing chiral fermions on the lattice, proposed by Kaplan and modified by Narayanan and Neuberger, using as our testing ground the chiral Schwinger model. The scheme is found to reproduce the desired form of the effective action, whose real part is gauge invariant and whose imaginary part gives the correct anomaly in the continuum limit, once technical problems relating to the necessary infinite extent of the extra dimension are properly addressed. The indications from this study are that the Kaplan-Narayanan-Neuberger scheme has a good chance at being a correct lattice regularization of chiral gauge theories.

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I. INTRODUCTION

There has been much progress recently in an old problem in the understanding of gauge theories, namely, the regularization of chiral gauge theories. The goal is a gauge-invariant regularization: While in theory there is nothing wrong with regulators breaking gauge invariance, we would like a gauge-invariant regularization for at least two distinct reasons. In perturbation theory a gauge-invariant regularization makes the proof of renormalizability much simpler [1,2]. For nonperturbative calculation much of the success of lattice field theory has followed directly from its manifest gauge invariance, and so we are reluctant to throw this away. Lattice regularization of a chiral theory, however, must be clever enough to evade no-go theorems [3,4] which state that it is impossible to have simultaneously (1) locality, (2) chiral invariance, and (3) the correct number of fermion species.

A good overview of the problem has been provided by Narayanan and Neuberger [5], who point out that two of the most promising recent schemes, one perturbative [1] and another on the lattice [6], both make use of a common trick. A theory which looks vectorlike is constructed by coupling right-handed particles to a mass matrix M and left-handed particles to $M^\dagger$, as in the (Euclidean) Lagrangian

$$\mathcal{L} = -\bar{\psi}\not{D}\psi + \bar{\psi}(MP^R + MP^L)\psi, \qquad (1)$$

where $P^R = \frac{1}{2}(1+\gamma_5)$, $P^L = \frac{1}{2}(1-\gamma_5)$, and $\not{D}$ is the covariant derivative. For the theory to describe a right-handed fermion we need M to have a zero mode while $M^\dagger$ has none: Thus the mass matrix M needs to have infinite dimension. The infinite number of extra fields are realized as Pauli-Villars regulators in Ref. [1]; in Ref. [6] the mass matrix is realized by a domain wall in a higher dimension, labeled by coordinate s. There is a right-handed zero mode bound to the domain wall.

It is clear from the above that in the domain wall scheme the extent of the extra dimension should be kept infinite to avoid creating fermions in pairs of opposite chirality. Indeed, if the extra dimension is made finite, we have an anti-domain-wall, to which a left-handed zero mode is bound. One can then try to restrict the gauge fields to a "waveguide" around the domain wall, with scalar fields inserted at the boundaries of the waveguide to restore gauge-invariance: This approach is still under investigation but all indications are that the theory remains vectorlike [7]. So instead we follow the rather different approach of Narayanan and Neuberger [5,8], keeping the s extent infinite. In the language of the previous paragraph this gives an infinite-dimensional mass matrix (whose explicit form is given in the next section): In "domain wall" language we eliminated the possibility of zero modes bound to the anti-domain-wall, because now there simply *is* no anti-domain-wall. There is no need to introduce new gauge fields, and so the gauge fields have the dimension of the lower-dimensional target space (they have no s dependence and are simply copied to each s slice).

That we have at least the possibility of evading the no-go theorems is evident: From the point of view of Refs. [4,3], if we imagined integrating out all the extra fermion species except the right-handed fermion at $s=0$, the action we get is no longer local [5]. Other nonlocal formulations have been tried but these all either break Lorentz invariance or dynamically generate ghost contributions which wreck the theory (for an excellent treatment of these problems see [9]). In this approach the ghosts are canceled by pure gauge terms which also come from integrating out the fermions [5].

The purpose of this paper is to carry out a test of the Kaplan-Narayanan-Neuberger (KNN) scheme in perturbation theory, using as our testing ground an exactly

*Electronic address: saoki@het.ph.tsukuba.ac.jp
†Electronic address: levien@pupgg.princeton.edu

solvable model, the Schwinger model [(1+1)-dimensional QED]. To be more precise, we examine a chiral Schwinger model that is exactly the usual Schwinger model "cut in half": That is, we have a single right-handed fermion, minimally coupled to the gauge field. In the continuum limit, we expect the model to be anomalous: The effective gauge field action induced by integrating out the fermions should be gauge variant because of this anomaly. This is of no concern for our purposes: To make an anomaly free model we have many options, one fairly routine, but the others directly relevant to the central problem in the field, that of regularizing theories such as the standard model. The routine option is to introduce a left-handed particle to make the theory vectorlike: The left-handed anomaly then has the opposite sign (we check this explicitly), so that the theory is overall anomaly free. We note that even the construction of this theory would be problematic if (as with other lattice regularizations of chiral theories) our effective action had a gauge-variant real part. A more interesting choice is to introduce two right-handed particles and a left-handed particle, with couplings to the gauge field in a ratio 3:4:5 (hence "3-4-5 model"), so that again the anomalies (which are proportional to the charge squared) cancel. In two dimensions this is the simplest "toy" standard model, with nontrivial anomaly cancellation, but clearly we could construct many other such models.

The point is that in the KNN formalism each new type of particle (by "type" we mean flavor or chirality) is completely decoupled from the others, unless we introduce an explicit mass term coupling them. So in the massless limit there are no "type-changing" vertices in perturbation theory, and we can calculate the anomalies for each type of particle separately. The essential problem is then to calculate the effective action and the anomaly for a single flavor and chirality of fermion, minimally coupled to the gauge field, knowing that for a model containing only massless fermions we can simply add the effective actions for each type of fermion together, so that in particular we can make the anomaly free theories described above.

In this paper we calculate the one-loop effective action for two external gauge fields induced by integrating out the fermions, i.e., the one-loop vacuum polarization graph. We also show that in the continuum limit fermion loops with more than two external gauge fields attached do not survive. This allows us to sum bubble graphs and get results accurate to all orders of perturbation theory, just as in the continuum theory. We can thus give exact expressions for the anomaly, the mass gap, and the chiral order parameter in the continuum limit. We then compare our results with those obtained by other regularization methods.

As was pointed out in Ref. [5], the fermion propagator has the right structure near zero momentum by construction, and so there are only two ways in which the scheme can fail this perturbative test: first, because of the new infinity in the theory, the infinite s extent necessary to create a genuinely chiral theory (a new and not so well understood problem) and, second, because of the peculiarities of momentum integration on the lattice (an old and well understood problem). Narayanan and Neuberger [5,8] have given a prescription for handling the first problem, the new infinity: They point out that it is a bulk effect in s space (this is obvious in their overlap formalism and in perturbation theory it will become clear from the fact that only the translationally invariant part of the propagator diverges) so that it is naturally cured by subtracting diagrams for which the domain wall mass term has been replaced by a constant mass. It now needs to be checked that this scheme can be implemented without introducing new singularities which might alter the continuum limit of the theory. To deal with the second problem, we need to be careful about taking the continuum limit $a \to 0$ of Feynmann integrals (where a is the lattice spacing). This is because propagators depend on the loop momenta q_μ through $q_\mu a$, which can be of order 1 since the momenta range from $-\pi/a$ to π/a. So a simple expansion of the integrand in powers of a is not valid. We follow Ref. [3] and divide the integration region into an "inside" region near the propagator pole at zero momentum and an "outside" region which is the rest of the Brioullin zone. It turns out that for fermion loops with more than two gauge fields attached, only the "inside" region (where we can replace the propagators by their continuum limits) contributes in the continuum limit. The inside region in turn vanishes because of Ward identities constraining the continuum propagators [10]. For the remaining graph with two gauge fields attached (the vacuum polarization), both inside and outside regions contribute.

In Sec. III we carry out the s subtractions for the effective action, and show that they render the initially s-divergent action s finite. In order to make the ill-defined infinite s summation well defined we first restrict the gauge interaction to a finite range $-L \leq s \leq L$, while the fermion fields propagate in infinite s space. We then take the limit $L \to \infty$ after subtractions. We emphasize strongly, however, that the limit $L \to \infty$ has to be taken, and that the fermion "lives" in an infinite s space, as we will see in Sec. V. A subtle point that needs to be addressed relates to an ambiguity in the imaginary part of the effective action, seen in the overlap formalism [8]. Because the imaginary part of the Euclidean action corresponds to the parity-violating part of the action in Minkowski space, it is the most interesting part, giving rise to the anomaly, for example. In Ref. [8] the ambiguity arose because of an ambiguous phase in the boundary states, when the effective action was rewritten as an overlap using transfer matrices in the s direction. Does such an ambiguity occur in perturbation theory? In Secs. III and IV we show that in perturbation theory the imaginary part of the effective action is finite *before* subtractions, and unaffected by the subtractions (i.e., the subtracted terms are purely real). Thus the imaginary part of the effective action is *unambiguous* in our perturbation scheme. Obviously the perturbation scheme has picked a phase for the boundary states: In Sec. V it becomes clear that in our perturbation scheme we project the boundary states onto the ground states of the free transfer matrix. In other words, the perturbation scheme makes the Brioullin-Wigner phase choice, in which the overlap of the perturbed (nonzero gauge field) state with the unperturbed (zero gauge field) state is real. Of course this is not the only way to fix the ambiguity in the effective action, and in fact it is not an adequate prescription for gauge fields with nonzero winding number (instantons) for which this overlap is zero [11], but it is a perfectly adequate prescription for ordinary perturbation theory.

II. MODEL, PERTURBATION THEORY, AND THE EFFECTIVE ACTION

The fermionic action in $d(=2n+1)$ dimensions proposed by Narayanan and Neuberger is [5,8]

$$S_{\rm fermion}(\bar\psi,\psi,U) = -\sum_{x,s,t}\bar\psi_s(x)[\not D + MP^R + M^\dagger P^L]_{st}\psi_t(x), \qquad (2)$$

where

$$\begin{aligned}
\not D &= \delta_{s,t}\frac{1}{2}\gamma_\mu(\nabla_\mu^\dagger + \nabla_\mu),\\
\nabla_\mu\psi_s(x) &= U(x,\mu)\psi_s(x+\hat\mu) - \psi_s(x),\\
\nabla_\mu^\dagger\psi_s(x) &= \psi_s(x) - U^{-1}(x-\hat\mu,\mu)\psi_s(x-\hat\mu),\\
M_{st} &= \delta_{s+1,t} - \delta_{s,t}a_s,\\
M_{st}^\dagger &= \delta_{s-1,t} - \delta_{s,t}a_s,\\
a_s &= 1 - m_0\,{\rm sgn}\left(s+\frac{1}{2}\right) - \frac{1}{2}\nabla_\mu^\dagger\nabla_\mu,\\
P^R &= \frac{1}{2}(1+\gamma_d),\\
P^L &= \frac{1}{2}(1-\gamma_d).
\end{aligned} \qquad (3)$$

$\psi_s(x)$ and $\bar\psi_s(x)$ are Dirac spinors, m_0 is the domain wall height, $0 < m_0 < 1$, x labels the sites on the $d-1$ dimensional "real" lattice, and s labels the infinite number of fermions, and as such can be seen as a flavor index or as the position variable in an "extra" dth dimension in which the domain wall "lives." We have set the lattice spacing a (not to be confused with a_s) equal to 1, but will restore an explicit a to expressions as we need to in taking the continuum limit ($a \to 0$) later. The γ_μ, $\mu = 1,\dots,2n$, are Euclidean gamma matrices. P^R and P^L are the usual projection operators onto right- and left-handed fermions, respectively. Note that we take the gauge fields U to be s independent; i.e., we have not introduced any extra degrees of freedom for the gauge field. The action above is explicitly invariant under both s-independent local gauge transformations and global vector transformation.

More species[1] of fermion could be incorporated into Eq. (3) by simply adding more fermion fields ψ, and changing the sign of the domain wall mass m_0 according to whether a zero mode of a fermion field is to be right handed ($m_0 > 0$) or left handed ($m_0 < 0$). That is, we make a whole new copy of the action in Eq. (2) for each new species of fermion. The resulting action then has a global "vector" invariance $U(n_+)\otimes U(n_-)$, where n_+ (n_-) is the number of fermion fields with positive (negative) m_0. This vector symmetry may be a candidate for a "chiral" symmetry $U(n_+)_R\otimes U(n_-)_L$ for the zero modes (there are n_+ right-handed zero modes and n_- left-handed zero modes by construction). For example, a model with $n_+ = n_- = n_f$ (as for QCD) the symmetry becomes $U(n_f)_L\otimes U(n_f)_R = U(1)_A\otimes U(1)_V\otimes SU(n_f)_A\otimes SU(n_f)_V$. The currents associated with this symmetry are

$$J_{\mu,a}^R = i\sum_{s=-\infty}^{\infty}\bar\psi_s^+\gamma_\mu T_a^R\psi_s^+,\quad J_{\mu,a}^L = i\sum_{s=-\infty}^{\infty}\bar\psi_s^-\gamma_\mu T_a^L\psi_s^-, \qquad (4)$$

where the T_a^R's (T_a^L's) are the generators of $U(n_+)$ [$U(n_-)$] and the indices $\pm$ represent the sign of m_0. (Here we omit the "species" index of $\psi^\pm$.) These currents are not well defined due to the infinite s summation, and so they will be redefined later in Sec. III.

Equation (3) is probably not the most familiar way of writing out the fermionic action for this model. For instance, we note that the lattice derivative $\not D$ is just the naive derivative: The Wilson terms appear in the mass matrix M. To write down the action in a simpler fashion (see Ref. [8] for instance), we would start with the free fermion action with Wilson terms in all d dimensions, gauge $d-1$ of the d dimensions, and add a domain wall mass in the dth dimension. In Eqs. (2), (3) the Wilson term for the dth dimension is obscured by the fact that the relevant γ_d matrices are hidden in P^L, P^R.

The reason for writing the action in this way is that (2) is clearly of the general form (1) first put forward in Ref. [5] as a way of understanding different schemes [6,1,5] for implementing chiral fermions. As we mentioned above, these schemes all have in common the idea that in order to create a chiral fermion (say, right handed), the mass matrix M in Eq. (2) should have a zero mode, while $M^\dagger$ should not. This cannot be achieved with a finite-dimensional M, and so we must have an infinite number of auxiliary fields. These fields may be Pauli-Villars regulators, of alternating statistics [1,12,13], or fermion fields labeled by s and coupled to a domain wall in this "internal" space, as in the scheme under investigation. Of course these two approaches do not exhaust the possibilities, but they are the only ones to have been investigated so far.

In order to do perturbation theory we need expressions for the fermion propagator and the vertices. The main complications arise from the rather messy form of the propagator, first derived in Ref. [5]. Because translational symmetry is broken in the extra dimension, we work in momentum space for the $(d-1)$-dimensional "real world" and position space for the extra dimension. Then the propagator is [note that $\bar p_\mu = \sin(p_\mu a), \hat p_\mu = 2\sin(\frac{1}{2}p_\mu a)$]

[1] We are using the word "species" rather than "flavor" simply because we have already described the s index on ψ as a "flavor" index.

$$
\begin{aligned}
S_F(p) &= [-i\gamma_\mu \bar{p}_\mu + \tfrac{1}{2}\hat{p}^2 - M(p)P^R - M^\dagger(p)P^L]^{-1} \\
&= \left[-i\gamma_\mu \bar{p}_\mu + M^\dagger(p)\right] P^R G^R + \left[-i\gamma_\mu \bar{p}_\mu + M(p)\right] P^L G^L ,
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
M_{st} &= \delta_{s+1,t} - \delta_{s,t}\tilde{a}_s(p), \\
M^\dagger_{st} &= \delta_{s-1,t} - \delta_{s,t}\tilde{a}_s(p), \\
\tilde{a}_s(p) &= \begin{cases} a_+, & s \geq 0, \\ a_-, & s < 0, \end{cases} \\
a_\pm &= 1 + \frac{\hat{p}^2}{2} \mp m_0,
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
G^L_{st}(p) = G^L_{ts}(p) &= \left(\frac{1}{\bar{p}^2 + M^\dagger M}\right)_{st} \\
&= \begin{cases} Be^{-\alpha^+|s-t|} + (A^L - B)e^{-\alpha^+(s+t)}, & s,t \geq 0, \\ A^L e^{-\alpha^+ s + \alpha^- t}, & s \geq 0, t < 0, \\ Ce^{-\alpha^-|s-t|} + (A^L - C)e^{\alpha^-(s+t)}, & s,t < 0, \end{cases} \\
G^R_{st}(p) = G^R_{ts}(p) &= \left(\frac{1}{\bar{p}^2 + MM^\dagger}\right)_{st} \\
&= \begin{cases} Be^{-\alpha^+|s-t|} + (A^R - B)e^{-\alpha^+(s+t+2)}, & s,t \geq -1, \\ A^R e^{-\alpha^+(s+1) + \alpha^-(t+1)}, & s \geq -1, t < -1, \\ Ce^{-\alpha^-|s-t|} + (A^R - C)e^{\alpha^-(s+t+2)}, & s,t < -1, \end{cases} \\
\alpha^\pm &= \operatorname{arccosh}^{-1}\left[\frac{1}{2}\left(a^\pm + \frac{1+\bar{p}^2}{a^\pm}\right)\right] \geq 0,
\end{aligned}
$$

$$
A^R = \frac{1}{a^- e^{\alpha^-} - a^+ e^{-\alpha^+}}, \qquad A^L = \frac{1}{a^+ e^{\alpha^+} - a^- e^{-\alpha^-}}, \tag{7}
$$

$$
B = \frac{1}{2a^+ \sinh \alpha^+}, \qquad C = \frac{1}{2a^- \sinh \alpha^-}. \tag{8}
$$

Note that the above form of the fermion propagator is valid only for s-space infinite.

To obtain the vertices we introduce gauge fields A_μ, defined by

$$
U(x,\mu) = e^{ieaA_\mu(x)}. \tag{9}
$$

The vertices are somewhat simpler in form than the propagator: In fact they are exactly the usual Wilson vertices (see Fig. 1), obeying the lattice Ward identity:

$$
V^{(n)}_{\mu_1\cdots\mu_n}(q,q') = \frac{a^{n-1}(-e)^n}{n!} \sum_\mu \delta_{\mu\mu_1} \cdots \delta_{\mu\mu_n} \sum_s \delta_{ss_1} \cdots \delta_{ss_n} \partial^n_\mu S_F^{-1}\left(\frac{q+q'}{2}\right), \tag{10}
$$

where we have restored the dependence on the lattice spacing a explicitly, and $\partial^n_\mu S_F^{-1}(q)$ means $\partial^n S_F^{-1}/\partial(q_\mu a)^n$. This is exactly the usual Wilson vertex, whose only momentum dependence is through the sum of the ingoing and outgoing fermion momenta, with a trivial s dependence added in. We note that there is an infinite number of "seagull" vertices, but with the addition of each photon the vertex decreases by a factor of a. We will need only $V^{(1)}_\mu$ and $V^{(2)}_{\mu,\nu}$:

$$
\begin{aligned}
V^{(1)}_\mu(q,q') &= (-e)\partial_\mu S_F^{-1}\left(\frac{q+q'}{2}\right), \\
V^{(2)}_{\mu\nu}(q,q') &= a\frac{e^2}{2}\delta_{\mu\nu}\partial^2_\mu S_F^{-1}\left(\frac{q+q'}{2}\right).
\end{aligned} \tag{11}
$$

We have left off the trivial s dependence.

The bulk of this paper is devoted to the calculation of the vacuum polarization tensor $\Pi_{\mu\nu}(p)$ (see Fig. 2) for the chiral Schwinger model ($d = 3$):

$$\Pi_{\mu\nu}(p) = \Pi^{(a)}_{\mu\nu}(p) + \Pi^{(b)}_{\mu\nu}(p), \tag{12}$$

where $\Pi^{(a)}_{\mu\nu}(p)$ is the nonseagull diagram,

$$\Pi^{(a)}_{\mu\nu}(p) = e^2 \int \frac{d^2q}{(2\pi)^2} \sum_{st} \mathrm{Tr}\left\{\partial^\mu S_F^{-1}(q)\left[S_F(q-p/2)\right]_{st} \partial^\nu S_F^{-1}(q)\left[S_F(q+p/2)\right]_{ts}\right\} a^2, \tag{13}$$

and $\Pi^{(b)}_{\mu\nu}(p)$ is the seagull diagram,

$$\Pi^{(b)}_{\mu\nu}(p) = e^2 \int \frac{d^2q}{(2\pi)^2} \sum_{st} -\delta_{st}\delta^{\mu\nu}\mathrm{Tr}\left\{(\partial^\mu)^2 S_F^{-1}(q)\left[S_F(q)\right]_{ss}\right\} a^2. \tag{14}$$

In (13) and (14) we have used the vertex factors in (11).

The one-loop effective action, to second order in the gauge fields (we note that the effective action for an odd number of gauge fields vanishes, by Furry's theorem; see the Appendix for details) is then given by

$$S^{(2)}_{\text{eff}} = \frac{1}{2}\int \frac{d^2p}{(2\pi)^2} \tilde{A}_\mu(p)\tilde{A}_\nu(-p)\Pi_{\mu\nu}(p), \tag{15}$$

where $\tilde{A}_\mu(p)$ is the Fourier transform of the gauge field:

$$A_\mu(x) = \int \frac{d^2p}{(2\pi)^2} e^{ip\cdot(x+\frac{1}{2}\hat{\mu})}\tilde{A}_\mu(p). \tag{16}$$

It is convenient for later calculations to work in a chiral basis with

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma^\dagger_\mu & 0 \end{pmatrix},$$

$$\sigma_\mu = \begin{cases} 1, & \mu = 1, \\ -i, & \mu = 2, \end{cases}$$

$$\gamma_3 = -i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{17}$$

so that $\{\gamma_1, \gamma_2, \gamma_3\}$ are just the usual three Pauli matrices.

In this basis we have

$$S_F(p) = \begin{pmatrix} M^\dagger G^R & -i\sigma\cdot\bar{p}\,G^L \\ -i\sigma^\dagger\cdot\bar{p}\,G^R & MG^L \end{pmatrix},$$

$$\partial_\mu S_F^{-1}(p) = i\gamma_\mu\tilde{p}_\mu + \bar{p}_\mu = \begin{pmatrix} \bar{p}_\mu & i\sigma_\mu\tilde{p}_\mu \\ i\sigma^\dagger_\mu\tilde{p}_\mu & \bar{p}_\mu \end{pmatrix},$$

$$\partial^2_\mu S_F^{-1}(p) = -i\gamma_\mu\bar{p}_\mu + \tilde{p}_\mu = \begin{pmatrix} \tilde{p}_\mu & -i\sigma_\mu\bar{p}_\mu \\ -i\sigma^\dagger_\mu\bar{p}_\mu & \tilde{p}_\mu \end{pmatrix}. \tag{18}$$

Here $\tilde{p}_\mu = \cos(p_\mu a)$.

III. SUBTRACTION PROCEDURE

Because of the sums over infinite s space, the effective action (15) is ill defined. In order to make the effective action well defined, we adopt the following prescription (see also Ref. [14]). We restrict the interaction between gauge fields and fermions to the range $-L \le s \le L$, though the s space itself is infinite. We immediately see that the effective action (15) diverges as $L \to \infty$. The divergence, which arises from the translationally invariant part of the fermion propagator, can be removed by a subtraction

$$S^{\text{tot}}_{\text{eff}} = S_{\text{eff}} - \frac{1}{2}(S^+_{\text{eff}} + S^-_{\text{eff}}), \tag{19}$$

where $S^\pm_{\text{eff}}$ arises from calculating (15) with a constant mass term $\pm m_0$. This is the prescription suggested in [5,8]. In the homogeneous effective actions $S^\pm_{\text{eff}}$ we also restrict the gauge fields to the finite range $-L \le s \le L$. After performing the subtraction in Eq. (19) at finite

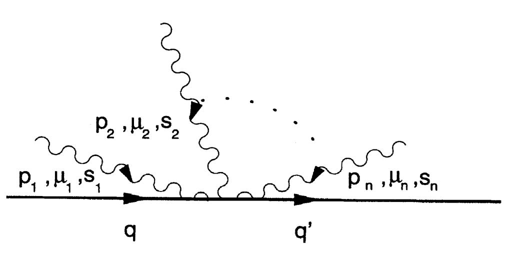

FIG. 1. An n-photon vertex.

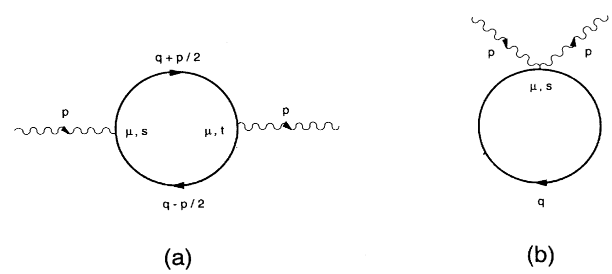

FIG. 2. Graphs contributing to the vacuum polarization. (a) The nonseagull graph. (b) The seagull graph.

L, we take the limit $L \to \infty$. Restricting the gauge fields to finite ranges does not change the form of the free fermion propagator, Eqs. (5) and (6), but it breaks the gauge-invariance of the action (2) under s-independent transformations. Of course we know that we need to break the gauge-invariance in the imaginary part of the effective action in order to recover the anomaly. Our prescription breaks gauge-invariance in both the real and imaginary parts of the effective action when L are finite, but when we take $L \to \infty$ gauge invariance is recovered in the real part and broken only in the imaginary part. This is a key advantage of the KNN scheme.

The definition of the effective action above also suggests that the currents associated with the global "vector" transformations in Sec. II should be modified as

$$J_\mu^R = i \sum_{s=-\infty}^{\infty} \bar{\psi}_s \gamma_\mu \psi_s \to J_\mu^R = i \sum_{s=-L}^{L} \bar{\psi}_s \gamma_\mu \psi_s, \tag{20}$$

so that the "vector" transformation generated by the modified currents becomes

$$\psi_s \to \psi_s' = \begin{cases} e^{i\theta}\psi_s, & -L \le s \le L, \\ \psi_s, & |s| \ge L. \end{cases} \tag{21}$$

It should be noted that the action is no longer invariant under this modified transformation due to the presence of the terms $\bar{\psi}_s(MP^R + M^\dagger P^L)_{st}\psi_t$, and, as seen in Secs. VI and VIII, this breaking of the "vector" symmetry leads to anomalies in the fermion number currents.

In this section we show explicitly that this subtraction scheme renders the real part of the effective action finite, but that the imaginary part of the action (which leads to the anomaly) is finite without subtraction. The homogeneous action $S_{\rm eff}^{\pm}$ is purely real (as we show in the next section), and so the imaginary part of the action is unaffected by the subtractions. As such it is unambiguous, in apparent contradiction with the overlap calculation of Ref. [8]. In Sec. V we address this apparent contradiction.

To see that the effective action is divergent in s is very straightforward. Looking at the expression for $\Pi_{\mu\nu}^{(a)}$ [Eq. (13)] for instance, using the chiral basis (18) for the propagators and vertices, taking the Dirac trace, and then summing over s and t gives terms of the form

$$\sum_{st} (O_1)_{st}(O_2')_{ts}, \tag{22}$$

where O, O' come from the set $\{G^L, G^R, MG^L, M^\dagger G^R\}$ and

$$\sum_{st} \equiv \sum_{s=-L}^{L}\sum_{t=-L}^{L}.$$

A subscript 1 means "evaluated at momentum $q + p/2$," while subscript 2 means "evaluated at $q - p/2$." All such terms diverge: Looking, for instance, at $\sum_{st}(G_1^L)_{st}(G_2^L)_{ts}$ we find

$$\begin{aligned}\sum_{st}(G_1^L)_{st}(G_2^L)_{ts} = B_1B_2 \sum_{s,t=0}^{L} e^{-\alpha_1^+|s-t|}e^{-\alpha_2^+|s-t|} \\ +C_1C_2\sum_{s,t=1}^{L} e^{-\alpha_1^-|s-t|}e^{-\alpha_2^-|s-t|} \\ +\text{terms finite as } L\to\infty. \end{aligned} \tag{23}$$

We note that the divergence comes only from the translationally invariant part of the propagator. As such it is natural to remove the divergence by subtracting the effective action due to a homogeneous mass term. With a constant mass term $\pm m_0$ instead of the domain wall we find that the propagators are as in Eq. (5) above, but with $G^L = G^R = G^{\pm}, M = M_{\pm}, M^\dagger = (M_\pm)^\dagger$, where

$$\begin{aligned} G_{st}^{+} &= Be^{-\alpha^+|s-t|}, \\ G_{st}^{-} &= Ce^{-\alpha^-|s-t|}, \\ (M_\pm)_{st} &= \delta_{s+1,t} - \delta_{s,t}a_\pm, \\ (M_\pm)^\dagger_{st} &= \delta_{s-1,t} - \delta_{s,t}a_\pm. \end{aligned} \tag{24}$$

From the form of the homogeneous propagator above, it is immediately obvious that (19) is the correct prescription to render the effective action finite. For instance, subtracting

$$\frac{1}{2}\sum_{s,t=-L}^{L}(G_1^+)_{st}(G_2^+)_{ts} + \frac{1}{2}\sum_{s,t=-L}^{L}(G_1^-)_{st}(G_2^-)_{ts}$$

cancels the divergence in (23).

There are some more subtle points to be made. First, we note that only the real part of the effective action is initially divergent: The imaginary part is finite *before* subtractions. To illustrate this we note that the "G^LG^L" term, for instance, is paired with a "G^RG^R" term in the following way:

$$\sum_{st}\left[(G_1^L)_{st}(G_2^L)_{ts}\zeta_{\mu\nu} + (G_1^r)_{st}(G_2^R)_{ts}\zeta_{\mu\nu}^*\right], \tag{25}$$

where

$$\zeta = \sigma\cdot\overline{(q+p/2)}\;\sigma\cdot\overline{(q-p/2)}\;\sigma_\mu\sigma_\nu\tilde{p}_\mu\tilde{p}_\nu. \tag{26}$$

We can split the above sum into its real and imaginary parts as follows:

$$\begin{aligned}\sum_{st}\Bigg[&\left[(G_1^L)_{st}(G_2^L)_{ts} + (G_1^R)_{st}(G_2^R)_{ts}\right]\left(\frac{\zeta_{\mu\nu}+\zeta_{\mu\nu}^*}{2}\right) \\ &+\left[(G_1^L)_{st}(G_2^L)_{ts} - (G_1^R)_{st}(G_2^R)_{ts}\right]\left(\frac{\zeta_{\mu\nu}-\zeta^*\mu\nu}{2}\right)\Bigg].\end{aligned} \tag{27}$$

It is easily shown that $\sum_{st}(G_1^R)_{st}(G_2^R)_{ts}$ diverges in exactly the same way as $\sum_{st}(G_1^L)_{st}(G_2^L)_{ts}$, so that the first term in (27) is infinite, but the second is finite. In the next section we show that the homogeneous actions $S_{\rm eff}^{\pm}$

are purely real, so that the imaginary part of the effective action is unaffected by the subtractions, and is hence unambiguous.

IV. REALITY OF THE HOMOGENEOUS EFFECTIVE ACTION

This can be seen by a brief but sloppy argument or a slightly longer explicit calculation. The sloppy argument first: Given that we have fermions coupled to an Abelian gauge field in three dimensions, we might expect an imaginary piece in the homogeneous effective action, of Chern-Simons form [15,16]

$$\int d^3x \epsilon_{\alpha\beta\gamma} A_\alpha(x) \partial_\beta A_\gamma(x), \tag{28}$$

where ∂_β here means $\partial/\partial x_\beta$. Now remember that our gauge field has only two components, and so the integrand in Eq. (28) above reduces to

$$A_2(x)\partial_3 A_1(x) - A_1(x)\partial_3 A_2(x), \tag{29}$$

where $\partial_3 = \partial/\partial x_3 = \partial/\partial s$. If the gauge fields are s independent as in the KNN case, then the Chern-Simons term vanishes. Of course this does not rule out other imaginary terms, and so we really should do an explicit calculation.

Looking first at the seagull contribution to $S^{\pm}_{\rm eff}$, with the two-gauge-field vertex, we find that it is proportional to the integral over q of

$$\sum_s \mathrm{Tr}\left[\begin{pmatrix} (M^\dagger G) & -i\sigma\cdot\bar{q}G \\ -i\sigma^\dagger\cdot\bar{q}G & MG \end{pmatrix}_{ss} \begin{pmatrix} \tilde{q}_\mu & -i\sigma^\dagger_\mu \bar{q}_\mu \\ -i\sigma_\mu\bar{q}_\mu & \tilde{q}_\mu \end{pmatrix}\right]$$

$$= \sum_s \left[(M^\dagger G)_{ss}\tilde{q}_\mu + (MG)_{ss}\tilde{q}_\mu - G_{ss}\sigma_\mu\sigma\cdot\bar{q}\bar{q}_\mu - G_{ss}\sigma^\dagger_\mu\sigma^\dagger\cdot\bar{q}\bar{q}_\mu\right]. \tag{30}$$

For simplicity we have left off the $\pm$ subscripts and superscripts. This sum is divergent, but real: We get no contribution to the imaginary part of $S^{\pm}_{\rm eff}$.

To see that the nonseagull term is real is a bit (but not much) more subtle. Writing out the vertices and propagators in the chiral basis as in Eq. (30) above we obtain

$$\Pi^{(a)}_{\mu\nu}(p) = e^2 \int \frac{d^2q}{(2\pi)^2} \sum_{st} \left[G_1 (M^\dagger G)_2 (\zeta_{\mu\nu})_1^* + G_1(MG)_2(\zeta_{\mu\nu})_1 \right.$$
$$\left. + (M^\dagger G)_1 G_2 (\zeta_{\mu\nu})_2 + (MG)_1 G_2 (\zeta_{\mu\nu})_2^* + \text{other terms}\right]_{st}, \tag{31}$$

where

$$(\zeta_{\mu\nu})_1 = \sigma^\dagger\cdot\overline{(q+p/2)}\,\sigma^\dagger_\mu \tilde{q}_\mu \bar{q}_\mu,$$
$$(\zeta_{\mu\nu})_2 = \sigma^\dagger\cdot\overline{(q-p/2)}\,\sigma^\dagger_\mu \tilde{q}_\mu \bar{q}_\mu. \tag{32}$$

The subscripts 1 and 2 on G, MG, or $M^\dagger G$ again mean "evaluated at momentum $q+p/2$" or "evaluated at momentum $q-p/2$," respectively. The imaginary part of the expression in (31) is

$$e^2 \int \frac{d^2q}{(2\pi)^2} \sum_{st} \left[\left[G_1(M^\dagger G)_2 - G_1(MG)_2\right]\left(\frac{(\zeta_{\mu\nu})_1^* - (\zeta_{\mu\nu})_1}{2i}\right)\right.$$
$$\left. - \left[G_2(M^\dagger G)_1 - G_2(MG)_1\right]\left(\frac{(\zeta_{\mu\nu})_2^* - (\zeta_{\mu\nu})_2}{2i}\right) + \text{other terms}\right]_{st}. \tag{33}$$

This sum can be easily shown to converge (the subtractions ensure this). Thus we are justified in doing the q integral *inside* the sum. Putting $q \to -q$ in the second term in the square brackets above turns all the subscript 1's into 2's and vice versa: For G, MG, and $M^\dagger G$ this is obvious since these are only functions of q through $|q \pm p/2|$. The ζ terms are also easily seen to interchange the 1 and 2 subscripts. Then the first term in Eq. (33) cancels with the second and we get zero for the imaginary part once more.

In Eq. (31) we only listed 4 of 16 terms, but the argument goes through in a similar fashion for all the others.

The last two sections have shown that the imaginary part of the effective action is finite and unambiguous, in apparent contradiction with the calculation of Ref. [8]. In the following section we look at this apparent contradiction.

V. PERTURBATION THEORY AND THE PHASE AMBIGUITY

The overlap formula [8] for the effective action in our scheme is of the form

$$\begin{aligned}\exp[S_{\rm eff}(A_\mu)] &= \lim_{L\to\infty}\{\lim_{s_0\to\infty}\langle b-|[\hat{T}_-(0)]^{s_0}[\hat{T}_-(A_\mu)]^L[\hat{T}_+(A_\mu)]^L[\hat{T}_+(0)]^{s_0}|b+\rangle\}\\ &= \langle b-|0-\rangle\langle 0+|b+\rangle \lim_{L\to\infty}\langle 0-|[\hat{T}_-(A_\mu)]^L[\hat{T}_+(A_\mu)]^L|0+\rangle,\end{aligned} \tag{34}$$

where $\hat{T}_\pm(A_\mu)$ are the transfer matrices and $|0\pm\rangle$ are the ground states of the free transfer matrices. The point is that when we take $s_0\to\infty$ we project the boundary states $|b\pm\rangle$ onto the ground states of the free transfer matrix. Since $|b\pm\rangle$ is naturally taken to be independent of A_μ, $\langle b-|0-\rangle\langle 0+|b+\rangle$ does not depend on A_μ at all, and as such it affects only the irrelevant *constant* part of the effective action. We can see also that the boundary conditions for the fermions do not affect the final result. Even periodic boundary conditions for fermions give the same result as long as the limit $s_0\to\infty$ is taken *before* taking the limit $L\to\infty$. (To get a nonzero result the condition $\langle 0+|0-\rangle\neq 0$ is also needed.) Aside from the irrelevant constant $\langle b-|0-\rangle\langle 0+|b+\rangle$ the final answer is the same answer we would have gotten if we had taken $|b\pm\rangle=|0\pm\rangle$ in the first place, which is the Wigner-Brioullin phase choice [17,8], where the overlap of the perturbed state $\lim_{L\to\infty}\{\lim_{s_0\to\infty}[\hat{T}_\pm(A_\mu)]^L[\hat{T}_\pm(0)]^{s_0}\}|b\pm\rangle$ with the unperturbed state $\lim_{s_0,L\to\infty}[\hat{T}_\pm(0)]^{s_0+L}|b\pm\rangle$ is real.

VI. EFFECTIVE ACTION IN THE CONTINUUM LIMIT

In Sec. III we showed that the subtractions render the s,t sums in the effective action in (15) finite. But we still have to integrate over p and q. It can easily be checked that after the subtractions the integrand has no singularities in p or q when we take the continuum limit $a\to 0$, other than the singularity noted in Ref. [5]:

$$\lim_{a\to 0} A^L(p)=\frac{m_0(4-m_0^2)}{4p^2a^2}. \tag{35}$$

This part of the fermionic propagator corresponds to the zero mode bound to the domain wall. The zero mode is absent in the homogeneous propagators (there is no domain wall for it to be bound to), and in fact the homogeneous action will give no contribution in our continuum calculation. We will start by just leaving the homogeneous terms out entirely, but justify our rashness explicitly as we go along. The homogeneous action has of course already fulfilled its role to tame the s divergence of the integrand so that meaningful statements can be made about its continuum limit. We note that the method used in this section is basically identical to that used in Ref. [18] on Kaplan and Shamir fermions. We explicitly use the Karsten-Smit approach [3] to momentum integration on the lattice.

We wish to evaluate (15) in the continuum limit $a\to 0$. We will see that because of the divergence of the propagators at small momentum it is natural to divide the region of q integration, $A=\{(q_1,q_2):|q_1|<\pi/a,|q_2|<\pi/a\}$ into an "inside," $A_1=\{(q_1,q_2):|q_1|<\epsilon/a,|q_2|<\epsilon/a\}$, and an "outside," $A_2=A-A_1$ (see Ref. [3], pp. 121–122). ϵ is a small positive number which we take to zero only *after* we take $a\to 0$. In the outside region A_2 we can rescale $q\to q'=qa$ and take $a\to 0$ in the integrand.

However, in the inside region A_1 we cannot do this asymmetric rescaling, since we do not have a guarantee that $q'>\epsilon\gg pa$. In this region we must take the $a\to 0$ limit of the integrand symmetrically. We get the following contribution to $\Pi^{(a)}_{\mu\nu}(p)$:

$$e^2\int_{A_1}\frac{d^2q}{(2\pi)^2}\sum_{st}\mathrm{Tr}\{i\gamma^\mu[-i\gamma^\alpha(q-p/2)_\alpha a]\,G_0^L(q-p/2)P^L i\gamma^\nu[-i\gamma^\beta(q+p/2)_\beta a]G_0^L(q+p/2)P^L\}a^2, \tag{36}$$

where

$$G_0^L(q)_{st}=\lim_{a\to 0}G^L(q)_{st}=\frac{1}{q^2a^2}F^L(s,t) \tag{37}$$

and

$$F^L(s,t)=F^L(t,s)=\frac{m_0(4-m_0^2)}{4}\times\begin{cases}(1-m_0)^{s+t}, & s,t\geq 0,\\ (1-m_0)^s(1+m_0)^t, & s\geq 0, t<0,\\ (1+m_0)^{s+t}, & s,t<0.\end{cases} \tag{38}$$

The M terms in the propagators give zero because of the Dirac trace. We have taken the $a \to 0$ limit of the integrand. Note that we have interchanged the order of the limit and the sum, which is valid only because we have explicitly shown that the s, t sums are finite. The continuum limit sum has been done in Ref. [18] and is a very simple result:

$$\sum_{st} F^L(s,t)^2 = \sum_{s} F^L(s,s) = 1. \tag{39}$$

The contribution of the seagull graph $\Pi^{(b)}_{\mu\nu}$ to the inner region is of order a^2, as are the contributions from the homogeneous effective actions. In the latter case, though, we note (at the risk of being suffocatingly pedantic) that the subtractions *were* needed to make the s, t sums finite first, so that the continuum limit made sense.

So now all that remains to be done is the q integration:

$$\int \frac{d^2q}{(2\pi)^2}\, \mathrm{Tr}(P^L\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta)\frac{(q-p/2)_\alpha(q+p/2)_\beta}{(q-p/2)^2(q+p/2)^2}. \tag{40}$$

The two-dimensional Dirac trace is

$$\mathrm{Tr}(P^L\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta) \tag{41}$$

$$= g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\nu} - i\delta^{\mu\alpha}\epsilon^{\nu\beta} - i\delta^{\nu\beta}\epsilon^{\mu\alpha}. \tag{42}$$

Note that if we replaced P^L by P^R in the above equation (as we would do to make a left-handed fermion), the imaginary part of the trace would reverse sign: This will also lead to an anomaly with the opposite sign. The integral at first sight looks logarithmically divergent, but in fact it converges, and so we can let the integration region run from $-\infty$ to ∞ (remember ϵ is to be taken to zero only after $a \to 0$) and the integral gives (using either Feynmann parameters or exponentiation of the denominators)

$$\int \frac{d^2q}{(2\pi)^2}\, \frac{(q-p/2)_\alpha(q+p/2)_\beta}{(q-p/2)^2(q+p/2)^2} = -\frac{1}{4\pi}\frac{p_\alpha p_\beta}{p^2}, \tag{43}$$

and so the final contribution to $\Pi_{\mu\nu}(p)$ from the inner region is

$$\frac{e^2}{4\pi}\frac{1}{p^2}\left[i\epsilon^{\mu\alpha}p_\alpha p_\nu + i\epsilon^{\nu\alpha}p_\alpha p_\mu + 2(\delta_{\mu\nu}p^2 - p_\mu p_\nu) - \delta_{\mu\nu}p^2\right]. \tag{44}$$

The very last term in (44) makes us feel slightly uncomfortable because it breaks gauge-invariance (in the two-dimensional sense) but is not the usual anomaly because it is real [the anomaly terms are in fact the first two terms in (44)]. Such a "longitudinal" term was found for models with s-dependent gauge fields [18]. Thankfully the contribution from the outside region exactly cancels this term.

The contribution to $\Pi_{\mu\nu}$ from the outside region of integration A_2 is

$$e^2\int_{A_2'}\frac{d^2q}{(2\pi)^2}\sum_s\left[\mathrm{Tr}\{\partial_\mu S_F^{-1}(q)\partial_\nu S_F(q)\}_{ss} + \delta^{\mu\nu}\mathrm{Tr}\{\partial_\mu^2 S_F^{-1}\cdot S_F\}_{ss}\right], \tag{45}$$

where in the first term we have used $\partial_\nu S_F^{-1} = -S_F^{-1}\partial_\nu S_F \cdot S_F^{-1}$. We have done the rescaling $q \to q' = qa$ and dropped the primes. The integration region A_2' is given by $A_2^{(a)} \cup A_2^{(b)} \cup A_2^{(c)} \cup A_2^{(d)}$, where

$$\begin{aligned} A_2^{(a)} &= \{(q_1,q_2) : q_1 \in (\epsilon,\pi), q_2 \in (-\pi,\pi)\},\\ A_2^{(b)} &= \{(q_1,q_2) : q_1 \in (-\pi,-\epsilon), q_2 \in (-\pi,\pi)\},\\ A_2^{(c)} &= \{(q_1,q_2) : q_1 \in (-\epsilon,\epsilon), q_2 \in (\epsilon,\pi)\},\\ A_2^{(d)} &= \{(q_1,q_2) : q_1 \in (-\epsilon,\epsilon), q_2 \in (-\pi,-\epsilon)\}. \end{aligned} \tag{46}$$

Let us first look at the case $\mu = \nu$. Then we note that

$$\partial_\mu S_F^{-1}\partial_\mu S_F + \partial_\mu^2 S_F^{-1}\cdot S_F = \partial_\mu(\partial_\mu S_F^{-1}\cdot S_F), \tag{47}$$

and putting $\mu = 2$ for definiteness, we have the following integral to evaluate:

$$\int \frac{dq_1}{(2\pi)^2}\sum_s\left[\mathrm{Tr}\left\{\frac{\partial}{\partial q_2}S_F^{-1}\cdot S_F\right\}_{ss}\right]_{q_2=(q_2)_{\min}}^{q_2=(q_2)_{\max}}. \tag{48}$$

We note that the integration region A_2' involves large momenta (our rescaled q of order 1), and so at this point we have no justification to replace the propagators in (48) with their zero mode piece. However, noting that the terms in square brackets in (48) must be odd in q_2 to contribute, we find that all such terms have a factor of $\bar{q}_2 = \sin q_2$ (remember we have rescaled q, and so it is as if $a = 1$), and as such give zero at the integration limits in (46) where $p_2 = \pi$ or $-\pi$. In particular, regions $A_2^{(a)}$ and $A_2^{(b)}$ give zero, and $A_2^{(c)}$ and $A_2^{(d)}$ combine to give

$$-\int_{-\epsilon}^{\epsilon}\frac{dq_1}{(2\pi)^2}\sum_s\left[\mathrm{Tr}\left\{\gamma_2^2\tilde{q}_2\bar{q}_2(P^RG^R + P^LG^L)_{ss} + \bar{q}_2(M^\dagger G^RP^R + MG^LP^L)_{ss}\right\}\right]_{q_2=-\epsilon}^{q_2=\epsilon}. \tag{49}$$

Now that all momenta are small ($< \epsilon$) we see that only the zero mode part of the propagator in the G^L term contributes (the other terms are down by a factor of ϵ, as indeed are the terms that would have come from the homogeneous effective action), and we get [noting that $\sum_s F^L(s,s) = 1$ [18]]

$$2\mathrm{Tr}(\gamma_2^2P^L)\int_{-\epsilon}^{\epsilon}\frac{dq_1}{(2\pi)^2}\,\frac{\epsilon}{q_1^2+\epsilon^2} = \frac{1}{2\pi^2}\left[\arctan\frac{q_1}{\epsilon}\right]_{q_1=-\epsilon}^{q_1=\epsilon} = \frac{1}{4\pi}. \tag{50}$$

Applying similar reasoning to the cases $\mu = 1$ and to $\mu \neq \nu$, we finally obtain, for the integral in (45),

$$\frac{e^2}{4\pi}(\delta_{\mu\nu} + i\epsilon_{\mu\nu}). \tag{51}$$

The first term is exactly what was needed to cancel the last term in (44), as advertised; the second term would give rise to a Chern-Simons interaction in the effective action (15) if the gauge fields were s dependent, but with s-independent gauge fields gives no contribution to the effective action. We omit this term from now on.

The final result for the continuum limit of the vacuum polarization is then

$$\Pi_{\mu\nu}(p) = \frac{e^2}{4\pi}\frac{1}{p^2}[i\epsilon^{\mu\alpha}p_\alpha p_\nu + i\epsilon^{\nu\alpha}p_\alpha p_\mu + 2(\delta_{\mu\nu}p^2 - p_\mu p_\nu)]. \tag{52}$$

The one-loop effective action is then given by (15). The consistent anomaly $\mathcal{A}(x)$ is defined as the variation of the effective action under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$:

$$\mathcal{A}(x) = -\frac{\delta S_{\text{eff}}}{\delta \Lambda} = e\partial_\mu J_\mu(x), \tag{53}$$

where $J_\mu(x)$ is the fermion number current,

$$J_\mu(x) = \frac{1}{e}\frac{\delta S_{\text{eff}}}{\delta A_\mu(x)}. \tag{54}$$

In momentum space we have

$$\mathcal{A}_p = p_\mu \tilde{J}_\mu(p) = i\frac{e^2}{4\pi}p_\mu \epsilon^{\mu\nu}\tilde{A}_\nu(p). \tag{55}$$

We make note of two points. First, we have calculated the consistent form of the anomaly; the covariant form for the Abelian theory differs only by a factor of 2 (i.e., the factor 4π in the denominator is replaced by 2π). Second, we should discuss how to make an anomaly-free theory. We have already checked that for a fermion of opposite chirality, the anomaly reverses sign, and so the vector theory with a right-handed and a left-handed fermion is anomaly free. We could also implement the 3-4-5 model, with, say, two right-handed particles with charges $e_1^R = 3e$, $e_2^R = 4e$, and one left-handed particle with charge $e_5^L = 5e$: It is clear from the above arguments that the anomaly vanishes in the continuum and we can also implement this model on the lattice.

VII. GRAPHS WITH MORE THAN TWO EXTERNAL PHOTONS

In this section we show that graphs having fermion loops with more than two photons attached vanish in the continuum limit (as they do in the continuum theory, this fact making the model easily solvable). This will allow us to give exact results for the mass gap and the chiral order parameter in the continuum limit, for the vectorlike theory (the usual Schwinger model). We must first show that the higher-order graphs are s finite and then look at the momentum integrals in the continuum limit, using the same division into "inside" and "outside" regions that we used in Sec. VI.

A. s-subtractions

The s subtractions render graphs with more than two external photon lines s finite because, as for the vacuum polarization graph, the s divergence comes from the translationally invariant parts of the propagators, and is exactly canceled by the homogeneous subtractions. Let us look for example at the nth-order graph in Fig. 3. It is clear that the *most* s-divergent terms are of the form

$$\sum_{s_1,s_2,\ldots,s_n} e^{-\alpha_1|s_1-s_2|}e^{-\alpha_2|s_2-s_3|}\cdots e^{-\alpha_n|s_n-s_1|}. \tag{56}$$

This divergence is exactly the one that will be canceled by the corresponding homogeneous terms. The only potential problem occurs if we take one of the factors $\exp\{-\alpha_i|s_i - s_{i+1}|\}$ and replace it with a *less* dangerous part of the propagator of the form $\exp\{-\beta_i s_i - \beta_{i+1}s_{i+1}\}$ (for $s_i, s_{i+1} \geq 0$). There is no corresponding term from the homogeneous effective actions to cancel this term, and so it must be finite by itself. It is not immediately obvious that this is so, given the $n-1$ dangerous-looking factors that are left. However, it is easy to see that in fact with this modification the sum in Eq. (56) is finite. Consider making exactly the replacement described above, and summing over s_i. It is easily shown that

$$\sum_{s_i=0}^{L} e^{-\alpha_{i-1}|s_{i-1}-s_i|}e^{-\beta_i s_i-\beta_{i+1}s_{i+1}} \tag{57}$$

gives factors of the form $\exp\{-\beta_{i-1}s_{i-1} - \beta_{i+1}s_{i+1}\}$ or $\exp\{-\beta'_{i-1}L - \beta_{i+1}s_{i+1}\}$. But these factors are exactly what are needed to make the sums over s_{i-1} and s_{i+1} converge. Carrying out the sums over $s_1, \ldots, s_n$, we arrive at a finite answer. Replacing more than one of the factors in (56) in this way just makes the sum even more convergent, and so we are done.

We note that the above argument still holds if we replace some of the vertices in Fig. 3 with seagull vertices,

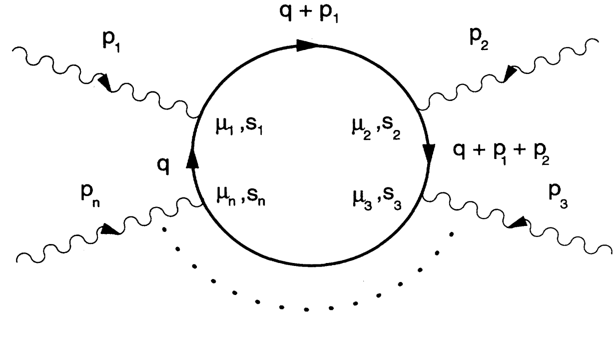

FIG. 3. A fermion loop with n photons attached.

because the s divergences come from the propagators: the s dependence of the vertices [Eq. (10)] is trivial.

B. Integration over the "inside" region

Having shown s convergence, we can now worry about doing the momentum integration in the continuum limit. In the "inside" region defined in Sec. VI, we can replace the propagators and vertices by their $a \to 0$ limits. We find immediately that (as for the vacuum polarization) any graph with a seagull vertex is down by factors of a and gives a vanishing contribution to the "inside" integration. For the nonseagull graphs (see Fig. 3) we replace the propagators with their $a \to 0$ limit:

$$\lim_{a\to 0}[S_F(p)]_{st} = -i\gamma^\mu p_\mu a G_0^L(p)P^L, \tag{58}$$

where $G_0^L(p)_{st}$ is given by Eq. (37). We have shown that the s sum associated with Fig. 3 is finite but it still must be done: However, it is easily shown that

$$\sum_{s_2=-\infty}^{\infty} F^L(s_1,s_2)F^L(s_2,s_3) = F^L(s_1,s_3), \tag{59}$$

so that

$$\begin{aligned}\sum_{s_1,s_2,\ldots,s_n} & F^L(s_1,s_2)F^L(s_2,s_3)\cdots F^L(s_n,s_1)\\ &= \sum_{s_n} F^L(s_n,s_n)\\ &= 1,\end{aligned} \tag{60}$$

where the first equality follows from repeated application of (59) and the second from (39).

We note that because the s sums just give a factor of unity, the propagators and vertices may be replaced by the following s-independent forms and the s sums ignored:

$$\begin{aligned}S_F(p) &= \frac{1}{\not p}P^L,\\ V_\mu &= (-e)\gamma_\mu.\end{aligned} \tag{61}$$

We can now follow the methods of Ref. [10] to show that for $n > 2$ the graphs of Fig. 3 vanish in the inner region. The trick is to make use of the following vector and axial-vector Ward identities, which hold for the continuum forms of the propagator and vertex in Eq. (61):

$$\begin{aligned}S_F(p+l)l_\mu V_\mu S_F(p) &= (-e)[S_F(p) - S_F(p+l)],\\ S_F(p+l)l_\mu V_\mu^5 S_F(p) &= (-e)[\gamma_5 S_F(p) - \gamma_5 S_F(p+l)]\end{aligned} \tag{62}$$

[note that $V_\mu^5 = (-e)\gamma_\mu\gamma_5$]. In two dimensions we have the relationship $\gamma_\mu\gamma_5 = -i\epsilon^{\mu\nu}\gamma_\nu$, so that we can relate the graph in Fig. 3 to the graph with the vertex factors γ_{μ_i} replaced by $\gamma_{\mu_i}\gamma_5$. To carry out the proof (whose details we do not repeat), one considers the subset of graphs obtained by leaving the order of vertices $2,\ldots,n$ fixed but attaching photon 1 in any position relative to the other vertices. Using (62) one can show that the contraction of the sum of this subset of graphs with the external momentum l_1 of photon 1 vanishes. This means that the sum of this subset of graphs has zero divergence. Carrying out the same procedure for the axial-vector-graphs shows that the sum has zero curl as well, meaning that it must be identically zero.

But what about $n = 2$, which we have already seen to give a finite answer? Well of course we have been a bit sloppy: The above argument only holds for $n > 2$, because for $n = 2$, the contraction of the graph with an external momentum gives an expression which is linearly divergent, and we are not allowed to use the Ward identities in (62).

C. Integration over the "outside" region

The integrals over the outside region vanish for $n > 2$, by a simple power counting argument. In Ref. [10] it is shown that for $n > 2$ any graph with n photons attached to a fermion loop (i.e., the graph in Fig. 3 or any variation with seagull vertices) vanishes as $a \to 0$, unless the propagator has a pole. Since in the outside region we have excluded the only pole in the propagator by cutting out the region $|q_\mu| < \epsilon$, where ϵ was to be taken to zero only *after* $a \to 0$, we are done.

VIII. COMPARISON WITH OTHER REGULARIZATIONS

The vacuum polarization in the chiral Schwinger model has been calculated by several authors, using both continuum [19,20] and lattice regularizations [21–23]. Their results may be summarized by

$$\Pi^{\rm others}_{\mu nu}(p) = \Pi^{\rm our}_{\mu\nu}(p) + \frac{e^2}{4\pi}C\delta_{\mu\nu}, \tag{63}$$

where $\Pi^{\rm our}_{\mu\nu}(p)$ is given by Eq. (52). Here C is a constant called the regularization constant, which was allowed but undetermined in the continuum calculations [19,20], and explicitly given in the lattice calculations [21–23]. In these previous lattice regularizations, C was nonzero and real, and depended on the Wilson parameter r. It emerges as a necessary consequence of the Wilson formulation for removing the doubler modes. The problem with a nonzero C in Eq. (63) is that this term breaks gauge-invariance in the *real* part of the effective action, meaning that the effective action for a left-handed fermion is *not* the complex conjugate of that for a right-handed fermion. In other words, to restore gauge-invariance by, say, making a vector theory with a left- and a right-handed particle, we have to give up the property that chiral determinants for left- and right-handed particles are complex conjugates. A further peculiarity relating to nonzero C is that the chiral Schwinger model develops a boson excitation of mass $e^2(C+1)^2/4\pi C$. So perhaps our most important result is that our C is zero. The gauge boson for the chiral Schwinger model then becomes infinitely heavy and decouples from the theory. A gauge-invariant vector theory is easily constructed by simply adding the effective actions for a left-handed fermion and a right-handed fermion. The vacuum polarization in the vector theory then changes from the expression (52) to

$$\Pi_{\mu\nu}^{(L+R)}(p) = \frac{e^2}{\pi}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right). \tag{64}$$

Because graphs with fermion loops having more than two photons attached vanish, we can obtain the exact current-current correlation function (and hence the mass gap) by just summing bubble graphs, exactly as in the continuum theory. We simply quote the result from Ref. [10], noting that it is exactly the result of the continuum theory:

$$\mu = \frac{e}{\sqrt{\pi}}. \tag{65}$$

We can also get a result for the chiral order parameter $\langle\bar{\psi}\psi\rangle$. This is zero in the perturbative vacuum, but may still be calculated perturbatively from the four-point function. Once again we simply quote the result, referring the interested reader to Ref. [10] and the references therein:

$$\langle\bar{\psi}\psi\rangle = \frac{\mu^2}{4\pi^2}e^{2\gamma_E}, \tag{66}$$

where γ_E is Euler's constant. The actual results are not terribly important for our purposes: Our main purpose in quoting them is to emphasize that we have obtained the correct continuum limit at all orders in perturbation theory. Of course the main result we needed was that, as for the continuum theory, fermion loops with more than two photons attached vanish, so that perturbation theory is rather easily summed.

We could also consider the 3-4-5 model. The vacuum polarization for this model becomes

$$\Pi_{\mu\nu}^{(3\text{-}4\text{-}5)}(p) = \frac{e^2}{\pi}\frac{(3^2+4^2+5^2)}{2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right), \tag{67}$$

and hence a mass gap $\mu^{(3\text{-}4\text{-}5)} = 5\frac{e}{\sqrt{\pi}}$. It is easy to see that the fermion number current of the model, defined by

$$J_\mu^F = J_\mu^{R,3} + J_\mu^{R,4} + J_\mu^{L,5}, \tag{68}$$

is anomalous:

$$\partial^\mu J_\mu^F(x) = i(3+4-5)\frac{e}{4\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu(x). \tag{69}$$

This result agrees with a previous calculation in Ref. [18].

IX. DISCUSSION

We have shown that the KNN scheme for implementing chiral fermions passes a simple perturbative test in 2+1 dimensions. Our perturbative scheme renders the effective action finite after the subtraction of effective actions with homogeneous mass terms. The gauge-variant term in the effective action corresponds exactly to the consistent anomaly: In contrast with other regularization schemes, the real part of the effective action *is* gauge-invariant. To obtain this result, we made the infinite s summation well defined by restricting the range of the gauge interaction. This restriction of course breaks gauge-invariance, in both the real and the imaginary parts of the effective action, but when the range of the gauge interaction is taken to infinity gauge invariance is restored in the real part and broken only in the imaginary part, giving the correct anomaly.

A rather more difficult test of the KNN scheme is a perturbative calculation in $4+1$ dimensions. We expect the scheme to work just as well in $4+1$ dimensions as in $2+1$, but of course an explicit demonstration is necessary. If this test is passed, we would expect the KNN regularization of more complicated anomaly free chiral gauge theories such as the standard model to also have the correct continuum limit.

The infinite extra dimension that is needed to make a truly chiral fermion has been shown here to be tamable in perturbation theory: Narayanan and Neuberger have also given a finite and hence computable nonperturbative effective action, in the form of an overlap. One obvious research goal is to produce a version of the nonperturbative overlap formula [8] that can be used in practical Monte Carlo-type calculations.

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APPENDIX: FURRY'S THEOREM

The effective action for an odd number of gauge fields vanishes, by Furry's theorem. This holds not only in the continuum [24] but also on the lattice, where we have to worry about graphs with seagull vertices. The point is that the charge conjugation matrix C, defined by $C\gamma_\mu C^{-1} = -\gamma_\mu^T$, transforms the propagator like $CS(q)C^{-1} = S^T(-p)$ and the m-gauge field vertex factor like $C\partial_\mu^m S^{-1}(p)C^{-1} = (-1)^m\partial_\mu^m[S^{-1}(-p)]^T$, where T denotes the transpose in spinor space. So regardless of the number of seagull vertices, it is easy to show (by insertion of factors CC^{-1}) that a given graph with a fermion loop and n attached gauge fields is equal to the same graph with the reverse orientation, up to a sign $(-1)^n$. Thus for odd n the two orientations cancel. The only complication we have blithely skipped over is the action of the charge conjugation matrix on the chiral "mass" term $M(p)P^R + M^\dagger(p)P^L$ in the propagator. For a lower-dimensional "target" space of dimension

$d-1=2,6,10,\ldots$, we find that $C\gamma_5C^{-1}=-\gamma_5^T$. We still get $CS(q)C^{-1}=S^T(-p)$, but now T denotes transposition in s space *as well as* Dirac space. Of course this is exactly what we need to transform the graph (say, the one in Fig. 3, with n odd) into the graph with reverse orientation, up to a sign $(-1)^n$. For target dimension $d-1=4,8,12,\ldots$, we find that $C\gamma_5C^{-1}=\gamma_5^T$, and so the transformed graph is equal to the graph with reverse orientation up to a sign $(-1)^n$ *and* a transposition of each propagator in s space. But since we trace over s, the graph is invariant under an s transposition of each propagator and we are done.

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