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Unitarization of instanton amplitudes

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We extend an earlier numerical model of S-wave unitarization of perturbation processes to include instanton contributions in a way that bears resemblance to multi-instanton expansions. The results indicate that there is an initial growth of the unitarized cross section with the center-of-mass energy but it always remains small. Beyond a critical energy it decreases exponentially. When no perturbation processes are included, so-called half-suppression is seen; when such processes are also included, the suppression of the instanton amplitudes depends on the strength of the perturbative effects and can rise above half-suppression bounds, but not close to the unitarity limit.

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A few years ago, Ringwald and Espinosa [1,2] calculated the amplitudes for multiboson production using instanton methods in the $(B+L)$-violating sector of the standard model. Their calculations showed an exponential growth of the cross section with the center of the mass energy of the colliding fermions, a behavior that quickly violates unitarity. Since then, a huge literature has been built on the subject (for reviews, see [3,4]). The cross section is assumed to be $\sigma \sim \exp[-16\pi^2/g^2 F(E/E_0)]$ and the main effort has been focused on a more accurate determination of the behavior of the so-called "holy grail" function $F(E/E_0)$ by including refinements on the initial model, such as modifications of the incoming or outgoing states [5,6] or multi-instanton corrections [7].

It is also well known that instanton calculations are not the only ones that exhibit a violation of unitarity in multiparticle production. Tree-graph $(B+L)$-conserving calculations in perturbation theory for the ϕ^4 model result in a factorial growth [8–10] of the amplitude with the number of external particles and a violation of unitarity at the same range of number of outgoing particles and center of the mass energy. Both perturbation theory and instanton amplitudes violate unitarity at energies near the sphaleron energy ($M_W/\alpha_W \sim 10$ TeV) and multiplicities of about $N \sim 1/\alpha_W$ external particles.

Since the problem of a direct nonperturbative calculation of either $(B+L)$-violating or $(B+L)$-conserving amplitudes is at present too formidable to be solved directly, it is natural to look at an easier problem such as the $\langle N|x|0\rangle$ amplitude of the quantum-mechanical (QM) one-dimensional double-well potential. For this problem it was shown [11] that the well-to-well transition amplitude is suppressed but only by half of the 't Hooft suppression factor when the energy is at the top of the barrier. Furthermore, the same kind of behavior has been seen in $2 \to N$ cross sections using the functional Schrödinger equation, while application of the same method in the ϕ^4 model gives multiparticle amplitudes that behave like $\exp(-\beta N)$ [12,13]. Considerations of this kind, together with a combination of a variety of quantitative and qualitative arguments have led to the notion of "half-suppression." If this is correct then a "premature unitarization" takes place and the holy grail function becomes approximately $\frac{1}{2}$ even for energies near the sphaleron energy $E_{\rm sph}$ (although, of course, one cannot expect instanton calculations to be valid above the sphaleron energy).

In the following we will investigate an S-wave unitarization method first proposed by Cornwall [8]. In an earlier work [14] this method was studied for perturbative processes, where tree-level multiparticle cross sections diverge factorically. After S-wave unitarization the tree-level cross section for multiparticle production remains suppressed by a factor similar to the predictions of the quantum mechanical models, that is by

$$\sigma = \exp(-\beta N), \tag{1}$$

where β is a constant of order 1.

A natural question arises: If the above method gives the "expected" results for the tree amplitudes, is there a way to extend it to instanton calculations? And if so, what is the behavior of the cross section near the critical domain? In the following we will try to answer these questions. As we will see, instanton-only unitarization leads to a behavior that suggests half-suppression. However, the addition of the tree graphs gives an additional contribution not considered in the multi-instanton discussions. This modification leads again to unitary amplitudes but the resulting cross section depends not only on the instanton (i.e., gauge theory) coupling g, but also on the perturbation coupling λ. When $\lambda \gg g^2$ the suppression of processes with a nonzero winding number depends rather strongly on λ. With smaller λ couplings this effect becomes weaker, but it always remains substantially larger than the half-suppression predictions. In all cases, the unitarized cross sections do not approach the unitarity bound.

I. UNITARIZATION FOR TREE AMPLITUDES

It has been shown [8,9] that in the $-(\lambda/4!)\phi^4$ theory the amplitude for the decay of a particle of energy E to N particles grows with the number N of final particles as

$$T^B_{1N} \overset{N\gg 1}{\to} -c\left(\frac{E}{N}\right)^{4-N}\left(\frac{\lambda}{\alpha^2}\right)^{N/2} N!, \tag{2}$$

where $\alpha = 2.92$. Although this refers to massless particles, the same behavior is seen in the massive ϕ^4 theory, and we will continue to use this *lower bound*, since we do not want at this stage to complicate the model with an additional mass scale, which would not affect any of our final qualitative results.

In a separate paper [14], we have dealt with the S-wave unitarization of this model. Divergences occur when we combine the amplitudes given above with the relativistic massless phase space:

$$\rho_N \sim \int \prod \frac{d^3p_i}{(2\pi)^3 2\omega_i}\delta(E-\sum \omega_i)\delta(\sum p_i)$$
$$\sim \frac{1}{(N!)^3}\left(\frac{E}{4\pi}\right)^{2N}. \tag{3}$$

This leads to a cross section

$$\sigma_{1N} = \rho_N |T^B_{1N}|^2 \sim \frac{N^{2N}}{N!}\left(\frac{\lambda}{16\pi^2\alpha^2}\right)^N, \tag{4}$$

which grows after a "critical" number of final particles:

$$N_c \simeq \frac{16\pi^2\alpha^2}{e^2\lambda}. \tag{5}$$

The cross section becomes of order 1 when the number of external legs is approximately equal to

$$N_1 \simeq eN_c. \tag{6}$$

For the S-wave unitarization of this model, we need the full T^B matrix that contains transitions with arbitrary numbers of initial and final particles. Since the factorial in Eq. (2) arises from the number of the Feynman diagrams, a reasonable expression for the amplitude of $J \to K$ transitions is [14]

$$T_{J\to K} \simeq J!K!\left(\frac{E}{N}\right)^{-N}\left(\frac{\lambda}{\alpha^2}\right)^{N/2}, \tag{7}$$

where we have dropped an unimportant constant multiplication factor. The T matrix can contain any number of external particles. For the numerical unitarization we limit ourselves to an $N_m \times N_m$ square matrix, with $N_m \geq 2N_1$. We define a matrix R as

$$R = \rho^{1/2} T^B \rho^{1/2}, \tag{8}$$

where the phase space matrix ρ is an $N_m \times N_m$ diagonal matrix with elements given by Eq. (3). The *unitarized* amplitude takes the form

$$T = \rho^{-1/2}\frac{R}{1-iR}\rho^{-1/2}, \tag{9}$$

which trivially satisfies the unitarity condition and gives a cross section that for large N behaves like $\exp(-\beta N)$, $\beta \sim 1$, in agreement with results from quantum mechanical models. In Ref. [14] we discussed the effects of the finite matrix size N_m, and we found that they are important for $N_m \leq 2N_1$ but as we get to larger sizes the behavior of the unitarized amplitude stabilizes. We also saw that the value of the coupling constant does not change the general behavior of the unitarized cross section, although it does affect the constant β, which roughly follows a scaling according to $\beta \sim a^{2/3}$, at least in the quartic oscillator [12].

II. INSTANTON-ONLY UNITARIZATION

There is a qualitative difference between the divergences leading to violation of unitarity in the tree calculations and the instanton calculations. The former is caused by the number of final particles N, while the latter is a result of the "wrong" scaling of the amplitude with the energy E. Specifically, the instanton amplitude with Higgs bosons as well as W and Z bosons in the final states is

$$T^I_{1N} = N!\left(\frac{E}{N}\right)^N\left(\frac{1}{v^2 g}\right)^N e^{-8\pi^2/g^2}, \tag{10}$$

where $E_0 = \sqrt{6}\pi m_W/\alpha_W = 2\sqrt{6}\pi^2 v/g$ is roughly the sphaleron mass. Integrating over the phase space and summing over the number N of final particles leads [5,6] to an exponential growth of the cross section with the total initial energy E:

$$\sigma \sim \exp\left\{-\frac{16\pi^2}{g^2}\left[-1+\frac{8}{9}\left(\frac{E}{E_0}\right)^{4/3}\right]\right\}, \tag{11}$$

and the above expression becomes of order 1 when $E \simeq E_0 \simeq 22$ TeV.

In the present paper, we will never sum over all the possible final states as is the common procedure in the instanton literature. This is necessary for our numerical model to work. In a sense it is also more physically relevant, since what we really want is to find out whether a $B+L$ violating scattering process of two particles to some definite number of particles can be observable in an accelerator. The trade-off is that the unitarity bounds are reached more slowly for our partial cross sections than for the "total" cross section given above.

Now we can use a simplification arising from the assumption of energy equipartition in the final states. This leads to instanton cross sections similar to the tree cross sections. We write the energy as $E = N\omega$, where ω is the average energy of the outgoing particles. With this substitution, the amplitude takes a form that mimics the tree amplitude and the cross section is

$$\sigma^I_{1N} = \frac{N^{2N}}{N!}\left(\frac{\omega^2}{4\pi v^2 g}\right)^{2N} e^{-16\pi^2/g^2}. \tag{12}$$

Compared to Eq. (4) we observe that both cross sections will have a factorial growth with the number of outgoing particles. For small N the tree cross section is decreasing due to the smallness of the coupling. On the other hand, the instanton cross section grows continuously, and it is only the 't Hooft suppression factor that keeps it small.

In the numerical computations we will keep the ratio ω/v fixed and equal to some constant value. For all the results we tried different values for this constant, and we found that none of our qualitative conclusions depends on the particular value we chose. As we observed above, saturation of the cross section is achieved with much greater difficulty in this model. For example, if $\omega/v = 1$ and $g = 0.627$, then the cross section becomes of order 1 only when $N \simeq 190$, which corresponds to an energy of $E \simeq 46$ TeV.

We need to define the general $J \to K$ instanton scattering matrix. Since instantons are pointlike, there is no distinction between initial and final legs. Then the $J \to K$ amplitude is the same as the $1 \to N$ amplitude given above with N the sum of incoming and outgoing particles:

$$T^I_{JK} = N! \left(\frac{E}{N}\right)^N \left(\frac{1}{v^2 g}\right)^N e^{-8\pi^2/g^2}, \quad N = K + J. \tag{13}$$

The R matrix for the instanton-only computations has the form

$$R = \begin{pmatrix} 0 & \rho^{1/2} T^I \rho^{1/2} \\ \rho^{1/2} T^A \rho^{1/2} & 0 \end{pmatrix}, \tag{14}$$

where T^A is the anti-instanton amplitudes, and they have the same values as the instanton T^I amplitudes. The reason for choosing the above form for the R matrix will be given in the next section. For now note that when we define the unitarized amplitude as in Eq. (9) and then expand the denominator in powers of T^I, the upper-right instanton-sector of the unitarized amplitude is

$$\begin{aligned} T^I_{\text{unitarized}} &= \left(\rho^{-1/2} \frac{R}{1 - iR} \rho^{-1/2}\right)^I \\ &= T^I - T^I \rho T^A \rho T^I + \cdots, \end{aligned} \tag{15}$$

which is quite similar to the multi-instanton expansions. This similarity is demonstrated further in Fig. 1, where we have plotted the $1 \to N$ cross sections that correspond to the first and the second terms of the above equation. We used $g = 0.627$, corresponding to $\alpha_W = \frac{1}{32}$ and $\omega/v = 3$. For fixed ω/v the total energy scales with N, so Fig. 1 can also be viewed as an energy diagram for the first two terms of the expansion with $E \equiv N \times 738$ GeV. We see that the second term begins at a lower value, but after some energy it actually becomes larger than the first term, while both are still very small. This is what the premature unitarization scenario predicts as well. Of course, a matrix multiplication can contain intermediate states with all possible numbers of particles, so we have to remember to multiply only elements that are much smaller than one, when the expansion given above is still valid.

Since the diagonal elements of the R matrix are zero, we can write the following expression for the instanton sector unitarization:

$$T^I_{\text{unitarized}} = \left(\rho^{-1/2} \frac{R^I}{1 + T^A \rho T^I} \rho^{-1/2}\right)^I \tag{16}$$

where $R^I = \rho^{1/2} T^I \rho^{1/2}$. The denominator shows that any expansion will only contain pairs of instantons and anti-instantons and that the terms will alternate in sign. These properties are in agreement with the Zakharov-Maggiore-Shifman multi-instanton model.

We used this instanton-only unitarization method for a variety of couplings and ω/v ratios ranging from 1 to 5. Representative results are shown in Fig. 2. The coupling is small, $g = 0.7$ and the ratio ω/v is equal to 3. This means that the x axis, shown as an N coordinate in the figure, also corresponds to energies $E + N \times 0.7$ TeV with the vacuum energy being $v = 246$ GeV. The S-wave ununitarized cross section shows a growth a little faster than an exponential for all energies (or numbers of particles.) The unitarized cross section also grows at the beginning but after the critical number of particles $N_c \simeq 27$ (or the corresponding critical energy $E_c = N_c\omega \simeq 20$ TeV) it starts to fall. It is always smaller than half of the full 't Hooft factor ($\simeq 140$, i.e., $\sigma \simeq e^{-140}$), and thus we do have a manifestation of half suppression. In conclusion, all the expectations from multi-instanton methods are satisfied in our model. In addition, we have the prediction that although the cross section initially grows with

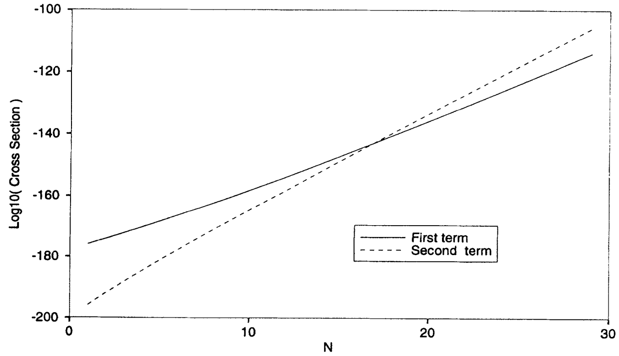

FIG. 1. Contributions to the instanton-only unitarization from: (a) One instanton and (b) two instantons and one anti-instanton. Coupling is $g = 0.627$. Energy of outgoing particles is $\omega = 3v$.

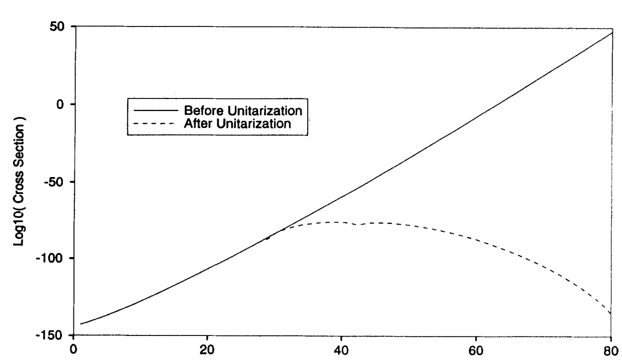

FIG. 2. The instanton cross section before and after instanton-only unitarization with $g = 0.7$, $\omega/v = 3$. The x axis is also an energy axis with values $E = N \times 0.7$ TeV.

energy, it will eventually decrease exponentially with the energy. It can never become observable.

III. INSTANTONS AND TREES

At sphaleron energies, both the perturbative tree diagram amplitudes that involve many outgoing gauge bosons and scalars and the instanton-configuration amplitudes become unsuppressed [10]. A highly energetic particle can decay either through nonzero winding number processes or by zero winding number perturbative processes. Also, corrections arising from inclusion of intermediate states can involve either combinations of multi-instanton configurations or tree and loop corrections. Our main objective is to combine amplitudes from tree level diagrams and instanton amplitudes within our numerical framework. To this end we first have to review some elementary definitions of the S matrix.

In field theory we define Hilbert spaces of a definite number of particles. If H_0 is the Hilbert space for no particles present (the vacuum) and $H_1, H_2, \ldots$ are the Hilbert spaces for one, two, ... particle states, then we define a Fock space as a direct sum of those Hilbert spaces:

$$\mathcal{F} = H_0 \oplus H_1 \oplus H_2 \oplus \cdots . \tag{17}$$

The S operator connects the Fock space of the "outgoing" states to the Fock space of the "incoming" states. The above equation leads to the representation of the S operator as a matrix, and the transition amplitude of J initial particles to K final particles can be treated as in our tree unitarization.

Now in the standard model the vacuum is "degenerate." Each sector is characterized by a winding number ν, with ν being an integer ($\nu \in Z$). Then, within the perturbation theory, we can build a different Fock space $\mathcal{F}_\nu$ in each sector in a manner similar to Eq. (17). The complete Fock space is then a direct sum of the $\mathcal{F}_\nu$ Fock spaces. Therefore, the S matrix has a $Z \times Z$ structure like

$$\begin{pmatrix} \cdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & S_0 & S_1 & S_2 & \cdots \\ \cdots & S_{-1} & S_0 & S_1 & \cdots \\ \cdots & S_{-2} & S_{-1} & S_0 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}, \tag{18}$$

where S_0 represents transitions within the same sector (the usual tree level perturbative S matrix), and S_ν represents transitions with total change of the winding number equal to ν. Since we do not consider $\nu > 1$ instanton amplitudes, it is sufficient to work with a 2×2 submatrix of the above matrix.

Thus, the correct way to write the total R matrix in our method is

$$R = \begin{pmatrix} R^B & R^I \\ R^A & R^B \end{pmatrix} \tag{19}$$

with R^B, R^I, R^A the tree, instanton, and anti-instanton R matrices defined as before.

If we ignore the effects of the perturbative tree diagrams, then the diagonal of the above matrix is zero and the result is the instanton-only unitarization scheme of the last section. If the tree level amplitudes lead themselves to divergent cross sections, we cannot justify their omission. Since the essence of this method is to include all the possible intermediate states, we need both trees and instantons.

Expanding the denominator of the instanton sector of the unitarized T matrix, we see that the instanton cross section arising from Eq. (19) contains

$$\left| \left(\frac{R}{1 - iR} \right)^I_{1N} \right|^2 = [R^I - (R^3)^I + \cdots]^2_{1N} + [(R^2)^I - (R^4)^I + \cdots]^2_{1N}. \tag{20}$$

The first term of the right-hand side depends only on odd powers of R and it is the same as in Eq. (15). However, the second term with the odd powers of R is zero when there are no tree contributions present and positive otherwise. Then we expect that with the addition of the trees in the model we will get cross sections larger than those from the instanton-only unitarization.

We are ready to proceed with the numerical unitarization of this model. There are three parameters that have to be specified: the two couplings and the ratio ω/v. The value for the weak coupling in the standard model is $\alpha_W = g^2/4\pi \simeq \frac{1}{32}$. This leads to an exponential suppression factor of the order of 10^{-170} for the cross section. As we explained, in our formulation of the problem we do not have a summation over the final states, and, as a result, we need very large numbers of outgoing particles, $N \geq 190$, if we are going to see any explicit effects of the unitarization.

This would mean that we have to compute the inverse of a 200×200 matrix or of an even larger matrix if we want to avoid spurious effects from the finite size of the matrix. And this might be possible if we did not have the additional problem that the matrix contains complex numbers spread in a range of more than a hundred orders of magnitude. This last problem can be successfully solved, as we explained in [14], by using the MAPLE software package with a setting of high precision (300 digits for the tree-only unitarization). However, computer memory availability and processor power forbid us from working with extremely large matrices. Using the time-honored trial-and-error method, we found out that the computational errors become negligible (or order 10^{-200}) when we use Eq. (19) with 60 elements for each of the tree and instanton matrices (i.e., a 120×120 matrix) and precision set to 400 digits. The completion of the computation of the inverse requires more than 10^6 sec of processor time in a Sparc10 workstation.

For the unitarization to be apparent in the results, we must have $N_1 < N_m \simeq 60$. For the instanton sector we can do this by increasing either the value of the coupling or the ratio ω/v or both. We tried a variety of values for both constants, and we found that their fluctuations have little effect on the general behavior of the cross section.

For the tree sector, if the tree coupling λ is taken to be

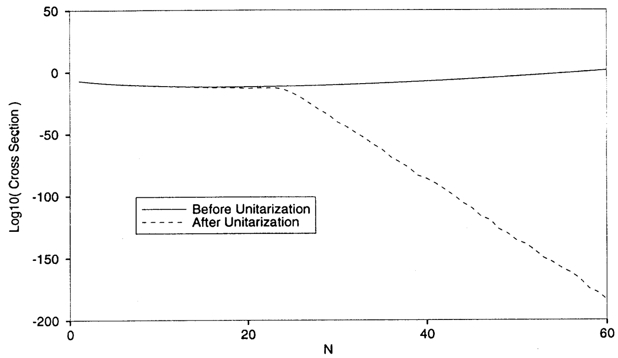

FIG. 3. The tree sector cross section before and after unitarization. The perturbative coupling is large, $\lambda = 24g^2 = 9.7$.

the coupling of the Higgs doublet in the standard model, then its value is unknown. For want of a better model, we assume $\lambda \sim g^2$, so that the Higgs sector is weakly coupled in general. We tried different values (from $\lambda = g^2$ up to $\lambda = 24g^2$). The most interesting situation is when both the tree and the instanton cross sections become of order 1 within the limits of the matrices we are using. Of course, when $\lambda = 24g^2$ the Higgs sector is actually strongly coupled, with consequences easily seen in the calculations.

Figures 3 and 4 show the unitarization results when $g = 0.627$, $\omega/v = 5$, and $\lambda = 9.7 \simeq 24g^2$. The tree coupling is taken so large because we want to compare tree and instanton unitarizations simultaneously. Figure 3 refers to the tree sector. The cross section becomes of order 1 at the edge of the matrix, $N_1 \simeq 60$. The critical number of particles is $N_c \simeq 22$. The behavior of the unitarized cross section is similar to what we got in the tree-only computations of Ref. [14], which is expected. In terms of denominator expansions this means that the $T^B\rho T^B$ tree-only contributions is much larger than the instanton $T^A\rho T^I$ contributions.

Figure 4 contains the instanton sector. The coupling here is equal to the standard model coupling, and the ununitarized cross section is suppressed at the beginning with the full 't Hooft factor. It grows quickly to become of order 1 at $N_1 \simeq 54$. For all N values we have a growth due to the chosen ratio ω^2/gv^2 [see Eq. (12)]. One might believe that if we were able to work with a small ω/v ratio we would have an initial decrease of the cross section, similar to the one seen in the tree sector. However, if we want to claim any relevance to the standard model, then we are forced to consider outgoing energies larger than the W mass. Then the smallness of the coupling guarantees a continuous growth with N of the cross section of Eq. (12). The unitarized cross section is quite different from the instanton-only unitarization results. The reason is the tree contributions. However, it is not true that the plot suggests that the cross section becomes observable in a weakly coupled theory; the relatively large cross section appears only because the tree coupling is large. What we can certainly infer is that the cross section is larger than half-suppression limits but a few orders of magnitude smaller than tree cross sections.

Figure 5 shows what happens for weak coupling. Both couplings are small, with $g = 0.627$, $\lambda = g^2$, and $\omega/v = 5$. Here the tree diagrams require unitarization only for very large numbers of external particles. From Eq. (5) we see that $N_c = 460$, outside the bounds of our matrix. In this case the unitarized instanton cross section is more similar to the instanton-only result although it remains larger than half-suppression predictions.

In conclusion, half suppression is a lower bound only as long as perturbation effects can be ignored altogether. When perturbative effects are taken into account, the result depends on the relation between the gauge and the tree couplings. The limit $\lambda < g^2 \ll 1$ is equivalent to assuming that all large-order perturbative diagrams are always small. In this case the results are not too different from pure instanton unitarization. However, if we accept the growth of both the perturbative and the instanton

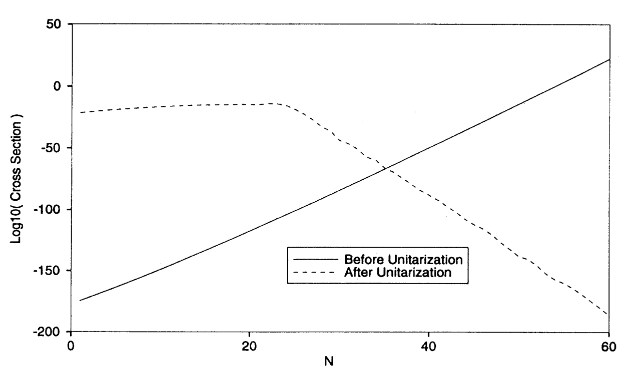

FIG. 4. The instanton sector cross section, before and after unitarization. Gauge coupling is small, $g = 0.627$ and $\omega/v = 5$. Tree coupling is large, $\lambda = 24g^2 = 9.7$. The x axis is also an energy axis with values $E = N \times 1.2$ TeV.

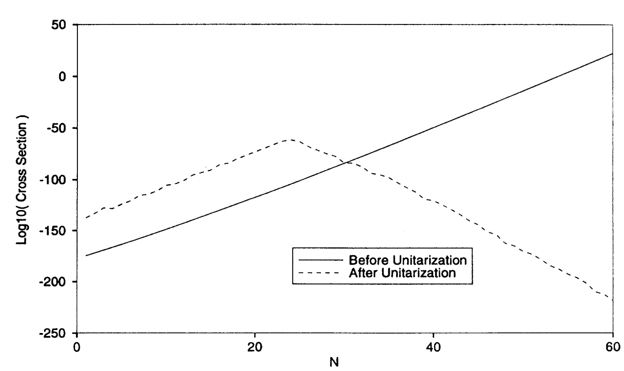

FIG. 5. The instanton sector cross section when all couplings are small: $g = 0.627$, $\lambda = g^2$, and $\omega/v = 5$. The x axis corresponds to energies $E = N \times 1.2$ TeV.

cross sections with divergences occurring at similar energies and numbers of external particles, then half suppression can be overcome within the unitarity bounds. In our numerical model, the maximum value of the cross section depends on the exact relation of the two couplings.

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