Boundary terms in the Nambu-Goto string action

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We investigate classical strings defined by the Nambu-Goto action with the boundary term added. We demonstrate that the latter term has a significant bearing on the string dynamics. It is confirmed that new action terms that depend on higher order derivatives of string coordinates cannot be considered as continuous perturbations from the starting string functional. In the case when the boundary term reduces to the Gauss-Bonnet term, a stability analysis is performed on the rotating rigid string solution. We determine the most generic solution that the fluctuations grow to. Longitudinal string excitations are found. The Regge trajectories are nonlinear.

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If some physically motivated open string model is constructed, one of the first problems we are faced with is to postulate properly the boundary conditions. In general, open string boundary conditions are dynamical equations to be held at string ends and follow from the self-interaction between the string and its ends. Their analysis can play a profound role in the understanding of open string dynamics. Referring to the context of hadronic physics, it is the question of how quarks couple with the string that approximates here an infinitely thin color flux tube. A naive way is to try to put some additional pointlike objects at the string ends, but we must remember that in the fundamental model quarks are sources of the chromoelectric field. Then, in the effective string description it is natural to expect also some reparametrization-invariant boundary string terms in the effective string action functional that describes the self-interaction between the string and its ends, without referring to any additional objects inserted on the string. In this paper, we consider the case of the Nambu-Goto string with self-interactions with string ends taken into account.

A simple noninteracting Nambu-Goto string sweeps out a timelike surface of minimal area in four-dimensional Minkowski spacetime. The minimal surface can be parametrized in such a way that nonlinear equations of motion turn into linear wave equations, and the string model becomes mathematically tractable. It was found [1] that the most general model of strings, which gives critical world sheets of minimal area, is defined by the action

$$S = -\gamma A - \frac{\alpha}{2} S_{\rm GB} - \beta S_{\rm Ch}, \tag{1}$$

where γ is string tension, A denotes world sheet area, and α and β are dimensionless parameters ($\gamma, \alpha > 0$). $S_{\rm GB}$ and $S_{\rm Ch}$ are pseudo Euclidean Gauss-Bonnet and Chern terms (related to Euler characteristics and surface self-intersections in Euclidean geometry).

String equations of motion derived from (1) are Nambu-Goto equations supplemented by some edge conditions, which depend on the action parameters. We see that additional terms in (1) do not modify bulk string equations, so that the string interacts only with its end points.

The string action (1), which depends on two arbitrary dimensionless parameters, represents a generic form allowed by reparametrization and Poincaré symmetries [1] and the requirement that the variational problem results in minimal surfaces. This statement is true as long as we do not consider additional objects that could couple to strings, such as internal fields existing on the world sheet or constant external fields in the target spacetime. Obviously, there exist also "pointlike" actions, being functionals of the trajectories of string end points. The simplest example is given by [2–6]

$$S_p = -mL_1 - mL_2, \tag{2}$$

where L_1 and L_2 are invariant lengths of the trajectories of string ends. We have here two massive pointlike particles attached to the string. Such "nonstringy" terms modify edge conditions in the variational problem for critical string world sheets, but they cannot be represented as reparametrization-invariant surface terms and do not modify local distributions of energies, momenta, and angular momenta along strings. Let us recall again the following statement [1]: for the Nambu-Goto string, if we do not introduce additional dynamical objects to couple with the string then there are only two possible invariant terms that can describe the (self-)interaction between the string and its ends. This interaction is characterized by two dimensionless couplings α and β.

The choice of the world sheet parametrization can be defined by the conditions [5]

$$(\dot{X} \pm X')^2 = 0,$$
$$(\ddot{X} \pm \dot{X}')^2 = -\frac{1}{4} q^2, \tag{3}$$

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where the overdot and the prime stand for derivatives of world sheet coordinates $X_\mu(\tau,\sigma)$ with respect to internal string parameters τ and σ. The parameter q can be considered as a momentum scale unit. This parameter is freely adjustable. In the above parametrization, bulk equations of motion get linearized and their general solution reads

$$X_\mu(\tau,\sigma) = X_{L\mu}(\tau+\sigma) + X_{R\mu}(\tau-\sigma). \tag{4}$$

To solve the boundary problem at the string ends $\sigma = \pm\frac{\pi}{2}$, we make use of the correspondence between minimal surfaces X_μ parametrized according to (3) and solutions Φ of the complex Liouville equation

$$\ddot{\Phi} - \Phi'' = 2q^2 e^{\Phi}. \tag{5}$$

The real part of the Liouville field is related to the only independent component of the induced world sheet metric (conformal mode) in the orthonormal gauge, while the complex part describes external geometry, i.e., an immersion of the world sheet in four-dimensional flat spacetime. A detailed analysis of the geometric approach to four-dimensional string models together with a derivation of all formulas used throughout this paper can be found in [1].

The correspondence between solutions of Nambu-Goto string equations and solutions of the complex Liouville equation provides us a very useful framework to study minimal surfaces. To visualize this correspondence, we present general solutions both to Nambu-Goto equations together with (3) and to complex Liouville equation (5) in the common form

$$\begin{aligned} e^{\Phi} &= -\frac{4}{q^2}\frac{f_L'(\tau+\sigma)f_R'(\tau-\sigma)}{[f_L(\tau+\sigma)-f_R(\tau-\sigma)]^2}\ , \\ \dot{X}^\mu_{L,R} &= \frac{q}{4|f'_{L,R}|}(1+|f_{L,R}|^2, 2\mathrm{Re}\, f_{L,R}, 2\mathrm{Im}\, f_{L,R}, 1-|f_{L,R}|^2)\ , \end{aligned} \tag{6}$$

where f_L and f_R are arbitrary complex functions.

As the derivatives of left and right movers are lightlike vectors, we can interprete f_L and f_R as their coordinates on the complex plane, on which the stereographical projection of the sphere of null directions in four-dimensional spacetime has been performed. Let us also note that modular transformations of $f_{L,R}$ induce Lorentz transformations of world sheet coordinates while the Liouville field Φ remains unchanged.

Now, we can present edge conditions following from the string action (1) as [1]

$$\begin{aligned} e^{\Phi} &= -\frac{1}{q}\sqrt{\frac{\gamma}{\alpha}}e^{-i\theta}\ , \\ \mathrm{Im}\,\Phi' &= 0 \quad \text{for} \quad \sigma = \pm\frac{\pi}{2}\ , \end{aligned} \tag{7}$$

where the angle parameter $\theta \in [-\pi,\pi]$ is defined by

$$\tan\frac{\theta}{2} = \frac{\beta}{\alpha}\ . \tag{8}$$

In this paper we consider only the case $\theta = 0$ ($\beta = 0$). This model has been investigated earlier in papers [4, 7, 8]. Then, there exists a well known solution corresponding to a rotating rigid rod,

$$X^\mu = \frac{q}{\lambda^2}(\lambda\tau, \cos\lambda\tau\sin\lambda\sigma, \sin\lambda\tau\sin\lambda\sigma, 0), \tag{9}$$

where the angular frequency λ satisfies the relation

$$\frac{\lambda^2}{\cos^2\frac{\lambda\pi}{2}} = q\sqrt{\frac{\gamma}{\alpha}}\ . \tag{10}$$

Note that the velocity of the string ends is smaller than the velocity of light and tends to it in the limit $\alpha \to 0$ ($\lambda \to 1$).

We can compute the energy and the angular momentum of the rotating string (9) (for relevant general formulas see Ref. [1]):

$$\begin{aligned} E &\equiv P^0 = \frac{\gamma q\pi}{\lambda}\left(1+\frac{\sin\pi\lambda}{\pi\lambda}\right), \\ J &\equiv M^{12} = \frac{\gamma q^2\pi}{2\lambda^3}\left[1+2\frac{\sin\pi\lambda}{\pi\lambda}+\frac{\sin 2\pi\lambda}{2\pi\lambda}\right]. \end{aligned} \tag{11}$$

The total momentum and other components of the total angular momentum vanish.

The pertinent Regge trajectory is plotted in Fig. 1.

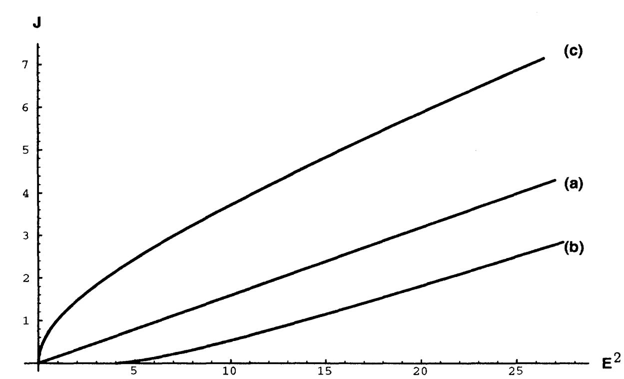

FIG. 1. Regge trajectories for various string models: (a) Nambu-Goto string, (b) Rebbi-Thorn-Chodos string with massive ends, and (c) string model with Gauss-Bonnet boundary term.

Regge trajectories represent the angular momentum J versus the squared mass E^2 relationships for given string configurations. We have compared trajectories for a rotating rigid rod obtained (a) in the standard Nambu-Goto open string model and (b) for the Nambu-Goto string with massive ends (due to the pointlike terms (2)—see Ref. [2]). Asymptotically, in the region of large masses, the trajectory can be approximated by the formula

$$J = \frac{1}{2\pi\gamma}E^2 + \frac{5}{4}\left(\frac{\alpha}{\pi^6\gamma^3}\right)^{1/4} E^{3/2} . \tag{12}$$

We see that it is slightly raised in comparison with the Nambu-Goto open string trajectory. At low masses, unlike the case (b) where the appearance of pointlike masses at string ends curves the trajectory downwards and the intercept is lowered, we find here approximately a linear dependence:

$$J = \sqrt{\frac{\alpha}{\gamma}}E. \tag{13}$$

It is interesting to note that the energy distribution along the string has also been changed. For the Nambu-Goto rotating rigid string the energy density is constant. In the modified model, the energy density (plotted in Fig. 2) is given by the formula

$$p^0 = \frac{\gamma q}{\lambda}\left[1 + \cos^4\frac{\lambda\pi}{2}\left(\frac{3}{\cos^4\lambda\sigma} - \frac{2}{\cos^2\lambda\sigma}\right)\right]. \tag{14}$$

We now turn to study small fluctuations around the solution (9). This solution is associated with a static solution of Liouville equation (5):

$$e^{\Phi_0} = -\frac{1}{q^2}\frac{\lambda^2}{\cos^2\lambda\sigma}. \tag{15}$$

A small perturbation Φ_1 from the static solution Φ_0 satisfies the linear equation

$$\ddot{\Phi}_1 - \Phi_1'' = -V(\sigma)\Phi_1, \tag{16}$$

where we have denoted

$$V(\sigma) = \frac{2\lambda^2}{\cos^2\lambda\sigma} . \tag{17}$$

The solution Φ_1 is subject to the following boundary conditions at $\sigma = \pm\frac{\pi}{2}$:

$$\Phi_1 = 0, \tag{18}$$

$$\mathrm{Im}\,\Phi_1' = 0. \tag{19}$$

One can prove that the imaginary part of Φ_1 must vanish at any world sheet point. Thus the Liouville field Φ_1 is real. We can separate variables to find a solution satisfying Eq. (16) together with the boundary conditions (18):

$$\Phi_1(\tau,\sigma) = T(\tau)\Sigma(\sigma).$$

We obtain a system of ordinary differential equations,

$$\ddot{T} + \mathcal{E}T = 0, \tag{20}$$

$$\left(-\frac{d^2}{d\sigma^2} + V(\sigma)\right)\Sigma = \mathcal{E}\Sigma, \tag{21}$$

together with the boundary conditions

$$\Sigma = 0 \quad \text{for} \quad \sigma = \pm\frac{\pi}{2}. \tag{22}$$

The solutions of the Schrödinger equation (21) that obey periodic boundary conditions (22) can exist only if $\mathcal{E} > 2\lambda^2$, where $2\lambda^2$ is the minimal value of the potential $V(\sigma)$. It implies that the separation constant $\mathcal{E}$ must be positive. For convenience we introduce a new variable ω defined as

$$\mathcal{E} = \omega^2.$$

The Schrödinger equation (21) with the potential (17) can be exactly solved. The solutions exist only for discrete values of the separation constant $\omega = \omega_n$ ($n = 1, 2, \ldots$), being roots of the equation (see Fig. 3)

$$\omega_n \tan\frac{\pi(\omega_n + n)}{2} = \lambda\tan\frac{\pi\lambda}{2} . \tag{23}$$

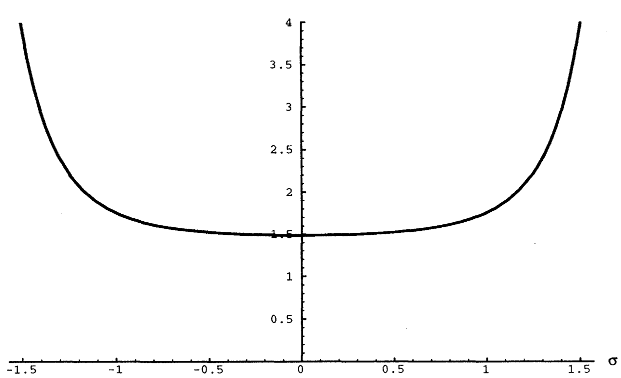

FIG. 2. The shape of energy density distribution along the string.

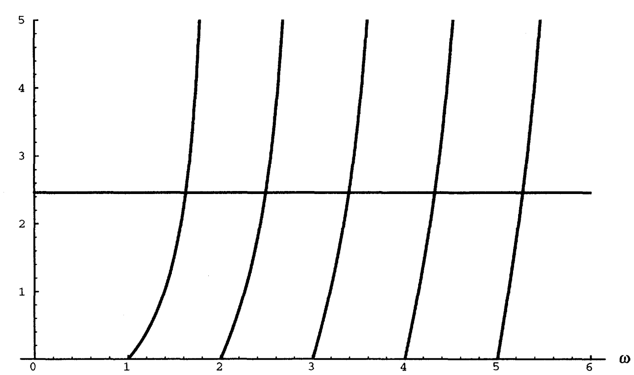

FIG. 3. Graphical solution of the equation $\lambda\tan\left(\frac{\pi\lambda}{2}\right) = \omega\tan\left[\frac{\pi(\omega+n)}{2}\right]$ for eigenfrequencies ω of fluctuation modes. The parameter λ is a fixed value of angular frequency the rigid rod rotates with.

The general solution of Eq. (16) satisfying boundary conditions (18) and (19) reads

$$\Phi_1 = 2\sum_{n=1}^{\infty} D_n \sin(\omega_n\tau+\varphi_n)\left[\tan\lambda\sigma\cos\left(\omega_n\sigma+\frac{n\pi}{2}\right) -\frac{\omega_n}{\lambda}\sin\left(\omega_n\sigma+\frac{n\pi}{2}\right)\right], \qquad (24)$$

where D_n and φ_n are arbitrary real constants.

To visualize string world sheets that correspond to Liouville fields $\Phi = \Phi_0 + \Phi_1$ we must employ the relations (6). Taking into account that e^{Φ} is real, we can make functions f_L and f_R unimodular (by some modular transformation—it is equivalent to specifying some reference frame in Minkowski spacetime). Then, it is convenient to introduce new real fields F_L and F_R,

$$f_{L,R} = e^{iF_{L,R}},$$

and the relations (6) go over into

$$e^{\Phi} = -\frac{1}{q^2}\frac{F'_L F'_R}{\sin^2\frac{F_L-F_R}{2}}, \qquad (25)$$

$$\dot{X}^{\mu}_{L,R} = \frac{q}{2F'_{L,R}}(1,\cos F_{L,R},\sin F_{L,R},0). \qquad (26)$$

The static field Φ_0 corresponds to

$$F_L^{(0)} = \lambda(\tau+\sigma), \quad F_R^{(0)} = \lambda(\tau-\sigma)+\pi, \qquad (27)$$

while the first order fluctuations Φ_1 are associated with the corrections

$$F_{L,R}^{(1)} = \pm\sum_{n=1}^{\infty} D_n \sin\left[\omega_n(\tau\pm\sigma)+\varphi_n\pm\frac{n\pi}{2}\right], \qquad (28)$$

where plus and minus signs correspond to left and right movers, respectively.

In contrast with the Nambu-Goto case, there appear longitudinal excitations of the string. Moreover, only such kind of fluctuations come out at the first order. With the help of the formulas above, the total string length L can be evaluated at some fixed time X^0:

$$\frac{\lambda^2 L}{2q} = \sin\frac{\pi\lambda}{2} - \sum_{n=1}^{\infty}\frac{D_n}{\omega_n^2-\lambda^2}\sin\left(\frac{\lambda\omega_n}{q}X^0+\varphi_n\right) \times\left[\omega_n^2+\lambda^2+2\lambda^2\tan^2\left(\frac{\pi\lambda}{2}\right)\right] \times\cos\left[\frac{\pi(\omega_n+n)}{2}\right]\cos\left(\frac{\pi\lambda}{2}\right). \qquad (29)$$

Let us now calculate the contribution to the energy coming from fluctuations. The general formula for the total string energy reads

$$P^0 = \frac{\gamma q}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\sigma\left(\frac{1}{F'_L}+\frac{1}{F'_R}\right) -\frac{\gamma q}{2}\left[\frac{\sin(F_L-F_R)}{F'_L F'_R}\right]_{\sigma=-\frac{\pi}{2}}^{\sigma=+\frac{\pi}{2}}. \qquad (30)$$

Straightforward calculations lead to the result

$$P^0 = \frac{\gamma q\pi}{\lambda}\left(1+\frac{\sin\pi\lambda}{\pi\lambda}\right) +\frac{\gamma q\pi}{2\lambda^3}\sum_{n=1}^{\infty}D_n^2\omega_n^2\left[1+\frac{\sin\pi\lambda}{\pi\lambda} +2(-1)^{n+1}\cos\pi\lambda\frac{\sin\pi\omega_n}{\pi\omega_n}\right]. \qquad (31)$$

One can easily check that the energy of fluctuations is always positive. It means that the solution (9) is stable against small perturbations. In fact, to calculate the total string energy (31) up to the second order we need also to consider second order corrections to the zero order solution. It can be proved by straightforward calculations that they do not contribute to the energy at the second order.

Finally, we want to summarize our results. We examined a classical string model in four-dimensional Minkowski spacetime defined by the Nambu-Goto action with some boundary term added. It warrants that critical world sheets are minimal surfaces, but some nonlinear equations that are third order in time derivatives hold at the string ends. It is evident from this paper that additional terms to the action functional depending on the second order derivatives of string coordinates cannot be regarded as higher order corrections to the starting Nambu-Goto action. In the limit of vanishing coupling constants ($\alpha, \beta \to 0$) our model does not revert to the original Nambu-Goto string model. There are still higher order Eqs. (7) to be satisfied. This is an unavoidable consequence of employing the theoretical framework for string actions with second order derivatives. The number of boundary conditions for dynamical equations of motion is doubled. The same is true for the number of initial data necessary to formulate properly the Cauchy problem. Roughly speaking, passing to dynamical models that are governed by the variational principle with actions depending on second order derivatives of dynamical variables doubles the number of independent degrees of freedom.

A generic minimal world sheet model (1) has been investigated for $\beta = 0$. We have found a classical ground state solution that corresponds to a rotating rigid rod. Unlike the case for the analogous Nambu-Goto configuration, the string ends move with a velocity smaller than the velocity of light and the nonrelativistic limit can be defined. It has been shown that the mass distribution along the string has been changed. Regge trajectories in this model are nonlinear. The ground state solution is stable against small perturbations. Eigenfrequencies for each fluctuation mode are found to be solutions of some simple transcendental equation. The excitations give a positive contribution to the total energy of the string. Another interesting property is that perturbations do not disturb the string from the planar motion; the shape of the string lies in a plane. But its total length measured in the laboratory frame oscillates, in contrast with other classical string models [9].

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