Factorization and polarization in linearized gravity

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We investigate all the four-body graviton interaction processes: $gX \to \gamma X$, $gX \to gX$, and $gg \to gg$ with X as an elementary particle of spin less than 2 in the context of linearized gravity except the spin-3/2 case. We show explicitly that gravitational gauge invariance and Lorentz invariance cause every four-body graviton scattering amplitude to be factorized. We explore the implications of this factorization property by investigating polarization effects through the covariant density matrix formalism in each four-body graviton scattering process.

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I. INTRODUCTION

Among the four fundamental interactions in nature, the gravitational interaction has not yet been successfully quantized but the challenge of combining the quantum principle with the elegant theory of general relativity, based on general covariance, has been made ceaselessly. While the very small gravitational coupling constant might reduce the importance of theoretical and experimental investigation of quantum gravity, gravity becomes as strong as the other forces near the Planck scale, and it is believed to be crucial in a consistent description of the birth of the Universe according to the big bang scenario. Furthermore, the successful unification of electromagnetic and weak interactions in the standard model makes unavoidable the thought that further unifications might be realized for all other fundamental interactions. Recent developments of supergravity [1] and superstring theories [2] were inspired by the hope of constructing a consistent unified quantum theory including gravity. In all cases, any common aspect of gravity and other interactions is very much worth exploring.

It has been established by several people [3] that the Fierz-Pauli theory of a massless spin-2 particle in the Minkowski flat space-time is inconsistent when coupled to matter and the only consistent theory in the low frequency domain is Einstein's general relativity. In the light of this aspect, we use Einstein's general relativity as a correct effective gravitational theory at low energies compared to the Planck scale. Since we are interested mainly in the weak field limit, we perform the weak field expansion to get the linearized gravitational Lagrangian. After the expansion, ordinary quantum field theoretical methods are applied to the linearized gravity to obtain the graviton-graviton and graviton-matter vertices. Several graviton interaction processes have been studied previously [4–7] in this framework.

The formidable complexity in vertices with more than three gravitons might render conventional Feynman diagram techniques very much inefficient. Recently we have shown, however, that all the tree-level transition amplitudes of $ge \to \gamma e$ [8], elastic graviton-scalar, graviton-electron, graviton-photon, and graviton-graviton scattering processes [9,10], are completely factorized into a simple form composed of a kinematic factor, QED-like Compton scattering form, and other gauge invariant terms. The factorization property can be used as a powerful tool to investigate the gravitational interactions and the polarization effects. The factorization property in the linearized gravity corresponds to a well-known fact in the standard field theory [11–14] that gauge symmetry and Lorentz invariance enable all the lowest-order amplitudes of four-particle interactions with an external massless gauge boson to be always factorized into one factor depending on the charge or the internal symmetry indices, and the other depending on the spin or polarization indices. A natural question is whether or not all the four-body graviton interactions exhibit the same factorization property.

In this paper, we investigate in a more extensive way the four-body graviton interactions like $gX \to \gamma X$ and $gX \to gX$ in the context of linearized gravity, where X is any kind of particles with spin less than 2 or graviton itself. Even though we do not consider the spin-3/2 case in the present work, we considerably extend our previous works [8–10] to show the presence of the factorization property in the four-body graviton interactions including the case with a massive vector boson W for X. In addition, we investigate the polarization effects to explore the implications of the factorization property.

The paper is organized as follows. In Sec. II, we describe in detail the derivation of the gravitational Lagrangian for the graviton scattering process with matter, including graviton itself, and present its expanded form through the Gupta procedure [4] in the weak field limit. Factorization in the linearized gravity is explained in analogy with that of the standard gauge field theories in Sec. III. Section IV is devoted to investigating polarization effects in these graviton scattering processes and to exploring the implications from the factorization property to the polarization effects. A brief summary and discussion are given in Sec. V. Every Feynman rule needed in the present work is listed in the Appendix.

II. INTERACTION LAGRANGIAN

In this section, we describe a general procedure to derive the gravitational Lagrangian for a graviton scattering with a massive scalar, a massive fermion, and a massive vector boson in the presence of the electromagnetic field. Without loss of generality it can be assumed that all the massive particles have the same mass denoted by m.

The natural starting point for the derivation is the standard QED Lagrangian in the absence of gravity:

$$\begin{aligned}\mathcal{L}_{\rm QED} &= (D_\mu\phi)^*(D^\mu\phi) - m^2(\phi^*\phi) + i\bar\psi\gamma^\mu D_\mu\psi - m\bar\psi\psi\\ &\quad -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^*(D^\mu W^\nu - D^\nu W^\mu)\\ &\quad +m^2W_\mu^* W^\mu - ieW_\mu^* W_\nu F^{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad (2.1)\end{aligned}$$

where ϕ is a scalar field, ψ is a fermion field, W is a vector boson field, and A is a photon field, with which the field strength $F_{\mu\nu}$ and the covariant derivative D_μ are defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad D_\mu = \partial_\mu + ieA_\mu. \quad (2.2)$$

The gravitational Lagrangian $\mathcal{L}$ is then obtained by making the QED Lagrangian in a general covariant form. To begin with, we write down the general covariant gravitational Lagrangian without any detailed description of the derivation:

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{gs}(A) + \mathcal{L}_{gf}(A) + \mathcal{L}_{gW}(A) + \mathcal{L}_{gA}, \quad (2.3)$$

$$\mathcal{L}_g = 2\kappa^{-2}\sqrt{-g}R, \quad (2.4)$$

$$\mathcal{L}_{gs}(A) = \sqrt{-g}[g^{\mu\nu}(D_\mu\phi)^*(D_\nu\phi) - m^2\phi^*\phi], \quad (2.5)$$

$$\begin{aligned}\mathcal{L}_{gf}(A) = \sqrt{-g}\Bigg(&\frac{i}{2}[\bar\psi\gamma^\mu(\vec\nabla_\mu + ieA_\mu)\psi\\ &-\bar\psi(\overleftarrow\nabla_\mu - ieA_\mu)\gamma^\mu\psi] - m\bar\psi\psi\Bigg), \quad (2.6)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{gW}(A) = &-\frac{1}{2}\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}(D_\mu W_\alpha - D_\alpha W_\mu)^*\\ &\times(D_\nu W_\beta - D_\beta W_\nu) + \sqrt{-g}g^{\mu\nu}m^2W_\mu^*W_\nu\\ &-ie\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}W_\mu^*W_\alpha F_{\nu\beta}, \quad (2.7)\end{aligned}$$

$$\mathcal{L}_{gA} = -\frac{1}{4}\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}, \quad (2.8)$$

where $\kappa = \sqrt{32\pi G_N}$ with the Newtonian constant G_N. For the sake of discussion, the gravitational Lagrangian $\mathcal{L}$ has been separated into five parts, each of which describes an independent process under consideration. The Lagrangian $\mathcal{L}_g$ describes pure gravitational interactions. $\mathcal{L}_{gs}(A)$, $\mathcal{L}_{gf}(A)$, and $\mathcal{L}_{gW}(A)$ are for gravitational interactions of a massive scalar s, a massive fermion f, and a massive vector boson W in the presence of the electromagnetic field, respectively. The final Lagrangian $\mathcal{L}_{gA}$ is for gravitational interactions of the electromagnetic field.

Now let us describe in detail the derivation procedure of the gravitational Lagrangian $\mathcal{L}$ in the weak field limit and expand the Lagrangian around the flat Minkowski space to obtain the necessary interaction terms. The flat space expansion of Eq. (2.3) usually can be carried out by the Gupta procedure [4]. In the procedure one introduces a symmetric tensor field $h_{\mu\nu}$ denoting the deviation of the metric tensor $g_{\mu\nu}$ from the flat space Minkowski metric tensor $\eta_{\mu\nu}$=(+,−,−,−):

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \quad (2.9)$$

After the expansion any curved space geometrical object is expressed as an infinite series in terms of $h_{\mu\nu}$. For the present work, however, only the terms up to $O(h^3)$ are needed and therefore every expanded Lagrangian will be presented including the terms up to that order.

It is convenient to expand at first the contravariant metric tensor $g^{\mu\nu}$ and the affine connection $\Gamma^\lambda_{\mu\nu}$, whose expanded forms are given up to $O(h^3)$ by

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda}h^\nu_\lambda - \kappa^3 h^{\mu\lambda}h_{\lambda\alpha}h^{\alpha\nu}, \quad (2.10)$$

$$\begin{aligned}g \equiv \det(g_{\mu\nu}) = &-1 - \kappa h + \frac{1}{2}\kappa^2(h^\mu_\rho h^\rho_\mu - h^2)\\ &+\frac{1}{6}\kappa^3(-2h^\mu_\rho h^\rho_\gamma h^\gamma_\mu + 3hh^\mu_\rho h^\rho_\mu - h^3), \quad (2.11)\end{aligned}$$

$$\begin{aligned}\sqrt{-g} = 1 &+ \frac{\kappa}{2}h + \frac{\kappa^2}{8}(h^2 - 2h^\mu_\rho h^\rho_\mu)\\ &+\frac{\kappa^3}{48}(h^3 - 6hh^\mu_\rho h^\rho_\mu + 8h^\mu_\rho h^\rho_\gamma h^\gamma_\mu), \quad (2.12)\end{aligned}$$

$$\begin{aligned}\Gamma^\lambda{}_{\mu\nu} &\equiv \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\\ &= \frac{1}{2}\kappa(\eta^{\lambda\sigma} - \kappa h^{\lambda\sigma} + \kappa^2 h^{\lambda\alpha}h^\sigma_\alpha)\\ &\quad\times(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}), \quad (2.13)\end{aligned}$$

with the definition $h = h^\mu_\mu$.

Let us now consider the Lagrangian $\mathcal{L}_g$ for pure gravitational interactions. The scalar curvature in Eq. (2.4) is defined in terms of the affine connection $\Gamma^\lambda_{\mu\nu}$ as

$$R = g^{\mu\nu}\left[\partial_\nu\Gamma^\lambda{}_{\mu\lambda} - \partial_\lambda\Gamma^\lambda{}_{\mu\nu} + \Gamma^\tau{}_{\mu\lambda}\Gamma^\lambda{}_{\tau\nu} - \Gamma^\tau{}_{\mu\nu}\Gamma^\lambda{}_{\tau\lambda}\right]. \quad (2.14)$$

Taking the de Donder gauge $\partial_\alpha h^\alpha_\mu = \frac{1}{2}\partial_\mu h$, the Lagrangian $\mathcal{L}_g$ can be expanded [15,16] around the flat Minkowski space and then reduced to the form

$$\mathcal{L}_g = \mathcal{L}_g^0 + \kappa\mathcal{L}_g^1 + \kappa^2\mathcal{L}_g^2 + \cdots, \tag{2.15}$$

$$\mathcal{L}_g^0 = -\frac{1}{4}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu h^{\sigma\nu}\partial^\mu h_{\sigma\nu}, \tag{2.16}$$

$$\mathcal{L}_g^1 = \frac{1}{2}h^\alpha_\beta\partial^\mu h^\beta_\alpha\partial_\mu h - \frac{1}{2}h^\alpha_\beta\partial_\alpha h^\mu_\nu\partial^\beta h^\nu_\mu - h^\alpha_\beta\partial_\mu h^\nu_\alpha\partial^\mu h^\beta_\nu + \frac{1}{4}h\partial^\beta h^\mu_\nu\partial_\beta h^\nu_\mu + h^\beta_\mu\partial_\nu h^\alpha_\beta\partial^\mu h^\nu_\alpha - \frac{1}{8}h\partial^\nu h\partial_\nu h, \tag{2.17}$$

$$\begin{aligned}\mathcal{L}_g^2 = {}& \frac{1}{16}h^2\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} + h^\lambda_\mu h^\nu_\beta\partial_\lambda h^{\alpha\beta}\partial^\mu h_{\alpha\nu} - \frac{1}{8}h^{\mu\nu}h_{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial^\lambda h_{\alpha\beta} - 2h^{\lambda\nu}h_{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial_\alpha h^\mu_\beta + \frac{1}{2}hh^\lambda_\mu\partial_\lambda h^{\alpha\beta}\partial_\alpha h^\mu_\beta\\ &-\frac{1}{2}hh^\mu_\beta\partial_\lambda h^{\alpha\beta}\partial^\lambda h_{\alpha\mu} + h^\nu_\beta h^\mu_\nu\partial_\lambda h^{\alpha\beta}\partial^\lambda h_{\alpha\mu} - \frac{1}{2}h_{\alpha\beta}h^{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial^\lambda h_{\mu\nu} + \frac{1}{2}h^\mu_\alpha h^\nu_\beta\partial_\lambda h^{\alpha\beta}\partial^\lambda h_{\mu\nu} - \frac{1}{4}hh^\lambda_\mu\partial_\lambda h^{\alpha\beta}\partial^\mu h_{\alpha\beta}\\ &+\frac{1}{2}h^{\lambda\nu}h_{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - h^\lambda_\beta h^\nu_\mu\partial_\lambda h^{\alpha\beta}\partial^\mu h_{\alpha\nu} + \frac{1}{4}hh^\mu_\beta\partial_\lambda h\partial^\lambda h^\beta_\mu - \frac{1}{2}h^{\mu\nu}h_{\nu\beta}\partial_\lambda h\partial^\lambda h^\beta_\mu + \frac{1}{2}h^{\mu\nu}h_{\nu\beta}\partial_\lambda h\partial^\beta h^\lambda_\mu\\ &-\frac{1}{4}h^{\mu\nu}h_{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial_\alpha h^\lambda_\beta - \frac{1}{32}h^2\partial_\lambda h\partial^\lambda h + \frac{1}{8}h^{\mu\nu}h_{\mu\nu}\partial_\lambda h\partial^\lambda h.\end{aligned} \tag{2.18}$$

We emphasize that $\mathcal{L}_g^1$ and $\mathcal{L}_g^2$ have been proved to be of the most compact form by a computer program [17]. While the difference is only a total derivative, the Lagrangian (2.15) is much simpler than that of Refs. [15,16]. The gravitational Lagrangian $\mathcal{L}_{gs}(A)$ of a scalar in the presence of the electromagnetic field can be similarly expanded:

$$\mathcal{L}_{gs}(A) = \mathcal{L}_{gs}^0 + \kappa\mathcal{L}_{gs}^1 + \kappa^2\mathcal{L}_{gs}^2 + \cdots, \tag{2.19}$$

$$\mathcal{L}_{gs}^0 = (D^\mu\phi)^*(D_\mu\phi) - m^2(\phi^*\phi), \tag{2.20}$$

$$\mathcal{L}_{gs}^1 = \frac{1}{2}h\mathcal{L}_{gs}^0 - h^{\mu\nu}(D_\mu\phi)^*(D_\nu\phi), \tag{2.21}$$

$$\begin{aligned}\mathcal{L}_{gs}^2 = {}& \frac{1}{8}(h^2 - 2h^{\alpha\beta}h_{\alpha\beta})\mathcal{L}_{gs}^0\\ &+(h^\mu_\alpha h^{\alpha\nu} - \tfrac{1}{2}hh^{\mu\nu})(D_\mu\phi)^*(D_\nu\phi).\end{aligned} \tag{2.22}$$

Let us now consider the gravitational Lagrangian of a fermion. In the absence of gravity a free fermion is described by the Lagrangian

$$\mathcal{L}_f = \frac{i}{2}\left[\bar\psi\gamma^\mu\partial_\mu\psi - \partial^\mu\bar\psi\gamma_\mu\psi\right] - m\bar\psi\psi. \tag{2.23}$$

Incidentally, the fermionic Lagrangian $\mathcal{L}_f$ deserves special treatment when it is converted into a general covariant form. Mathematically, this is because the tensor representations of the GL(4) of general linear 4×4 matrices behave like tensors under the subgroup of Lorentz transformations, but there is no representation of GL(4), or even representations up to a sign, which behaves like spinor under the Lorentz subgroup. One approach to incorporate spinors into general relativity is the tetrad formalism [18], which will be briefly described below.

The formalism utilizes the fact that the equivalence principle guarantees the introduction of a locally inertial coordinate system y_P^m at each space-time point P. In the case, the metric tensor $g_{\mu\nu}$ is expressed as

$$g_{\mu\nu}(x) = \eta_{mn}e^m_\mu(x)e^n_\nu(x), \tag{2.24}$$

where the tetrad or vierbein $e^m_\mu(x)$ is defined as a coordinate derivative of y_P^m as

$$e^m_\mu(x) \equiv \left[\frac{\partial y_P^m(x)}{\partial x^\mu}\right]_{x=P}. \tag{2.25}$$

For the sake of discussion a different type of vierbein e^ν_n is introduced with the m index lowered to n with the Minkowski metric tensor η_{mn}, and also with the μ index raised upward to ν with the metric tensor $g^{\mu\nu}$:

$$e^\nu_n \equiv \eta_{mn}g^{\mu\nu}e^m_\mu(x). \tag{2.26}$$

Equation (2.24) shows that the vierbein e^μ_m is nothing but the inverse of the vierbein e^m_μ such that

$$\delta^\mu_\nu = e^\mu_m e^m_\nu, \quad \delta^m_n = e^m_\mu e^\mu_n. \tag{2.27}$$

Another requirement from the equivalence principle is that the special relativity should apply in locally inertial frames, i.e., should preserve Lorentz invariance locally. As a way to accomplish the requirement a new covariant derivative is introduced:

$$\nabla_m \equiv e^\mu_m(\partial_\mu + iw_\mu). \tag{2.28}$$

Then the locally Lorentz invariant gravitational Lagrangian of a fermion is obtained as

$$\mathcal{L}_{gf} = \frac{i}{2}\bar\psi\gamma^p e^\mu_p(\partial_\mu + iw_\mu)\psi + \text{H.c.} - m\bar\psi\psi, \tag{2.29}$$

where the field connection $w_\mu(x)$ is expressed in terms of vierbeins as

$$\begin{aligned}w_\mu(x) = {}& \frac{1}{4}\sigma^{mn}[e^\nu_m(\partial_\mu e_{n\nu} - \partial_\nu e_{n\mu})\\ &+\tfrac{1}{2}e^\rho_m e^\sigma_n(\partial_\sigma e_{l\rho} - \partial_\rho e_{l\sigma})e^l_\mu - (m\leftrightarrow n)],\end{aligned} \tag{2.30}$$

with $\sigma^{mn} = i[\gamma^m,\gamma^n]/2$ with the Dirac matrices γ^m. It now can be shown that the general covariant and $U(1)_{\rm EM}$ invariant Lagrangian $\mathcal{L}_f(A)$ of a fermion is

$$\begin{aligned}\mathcal{L}_{gf}(A) = \sqrt{-g}\Bigg[&\frac{i}{2}\{\bar\psi\gamma^\mu(\vec\nabla_\mu + ieA_\mu)\psi\\ &-\bar\psi(\overleftarrow\nabla_\mu - ieA_\mu)\gamma^\mu\psi\} - m\bar\psi\psi\Bigg],\end{aligned} \tag{2.31}$$

with the notation

$$\gamma^\mu = \gamma^p e_p^\mu, \quad \vec{\nabla}_\mu \psi = \partial_\mu \psi + i w_\mu \psi,$$
$$\bar{\psi}\overleftarrow{\nabla}_\mu = \partial_\mu \bar{\psi} - i\bar{\psi} w_\mu. \tag{2.32}$$

In order to expand the Lagrangian around the flat Minkowski space we first need to expand the vierbein e_μ^m [19], which is given by

$$e_\mu^m = \delta_\mu^m + \frac{\kappa}{2} h_\mu^m - \frac{\kappa^2}{8} h_\nu^m \delta_n^\nu h_\mu^n + O(\kappa^3). \tag{2.33}$$

The resulting Lagrangian $\mathcal{L}_{gf}(A)$ [7] is of the form

$$\mathcal{L}_{gf}(A) = \mathcal{L}_{gf}^0 - e\bar{\psi}\gamma^\mu\psi A_\mu + \kappa\mathcal{L}_{gf}^1 - \frac{1}{2}\kappa e(h\eta_{\mu\nu} - h_{\mu\nu})\bar{\psi}\gamma^\mu\psi A^\nu + \kappa^2\mathcal{L}_{gf}^2 + \cdots, \tag{2.34}$$
$$\mathcal{L}_{gf}^0 = \frac{i}{2}[\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi] - m\bar{\psi}\psi, \tag{2.35}$$
$$\mathcal{L}_{gf}^1 = \frac{1}{2}h\mathcal{L}_{gf}^0 - \frac{i}{4}h_{\mu\nu}[\bar{\psi}\gamma^\mu\partial^\nu\psi - \partial^\nu\bar{\psi}\gamma^\mu\psi], \tag{2.36}$$
$$\begin{aligned}\mathcal{L}_{gf}^2 = {} & \frac{1}{8}(h^2 - 2h_\beta^\alpha h_\alpha^\beta)\mathcal{L}_{gf}^0 + \frac{i}{16}[3h_\nu^\alpha h_{\mu\alpha} - 2hh_{\nu\mu}](\bar{\psi}\gamma^\mu\partial^\nu\psi - \partial^\nu\bar{\psi}\gamma^\mu\psi) \\ & + \frac{i}{16}[h_\alpha^\nu\partial^\alpha h_\nu^\mu - h_\nu^\mu\partial^\alpha h_\alpha^\nu](\bar{\psi}\gamma_\mu\psi) + \frac{i}{32}[h^{\tau\alpha}\partial^\mu h_\alpha^\nu - h^{\nu\alpha}\partial^\mu h_\alpha^\tau](\bar{\psi}\gamma_\mu\gamma_\nu\gamma_\tau\psi),\end{aligned} \tag{2.37}$$

where γ^μ are the ordinary Dirac matrices and from now on every Greek index refers to the flat Minkowskian space-time.

As in the scalar case, it is also possible to derive and expand the gravitational Lagrangian for a vector boson in the presence of the electromagnetic field:

$$\mathcal{L}_{gW}(A) = \mathcal{L}_{gW}^0 + \kappa\mathcal{L}_{gW}^1 + \kappa^2\mathcal{L}_{gW}^2\cdots, \tag{2.38}$$
$$\mathcal{L}_{gW}^0 = -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^*(D^\mu W^\nu - D^\nu W^\mu) + m^2 W_\mu^* W^\mu - ieW_\mu^* W_\nu F^{\mu\nu}, \tag{2.39}$$
$$\mathcal{L}_{gW}^1 = h^{\mu\nu}\left[(D_\mu W_\alpha - D_\alpha W_\mu)^*(D_\nu W^\alpha - D^\alpha W_\nu) - m^2 W_\mu^* W_\nu\right] + \frac{1}{2}h\mathcal{L}_{gW}^0 + ie[\eta^{\mu\nu}h^{\alpha\beta} + \eta^{\alpha\beta}h^{\mu\nu}]W_\mu^* W_\alpha F_{\nu\beta}, \tag{2.40}$$
$$\begin{aligned}\mathcal{L}_{gW}^2 = {} & \frac{1}{8}(h^2 - 2h^{\mu\nu}h_{\mu\nu})\mathcal{L}_{gW}^0 + \frac{1}{2}(hh^{\alpha\rho} - 2h_\lambda^\alpha h^{\lambda\rho})\left[\eta^{\beta\sigma}(D_\alpha W_\beta - D_\beta W_\alpha)^*(D_\rho W_\sigma - D_\sigma W_\rho) - m^2 W_\alpha^* W_\rho\right] \\ & - h^{\mu\nu}h^{\alpha\beta}\left[\tfrac{1}{2}(D_\mu W_\alpha - D_\alpha W_\mu)^*(D_\nu W_\beta - D_\beta W_\nu) + ieW_\mu^* W_\alpha F_{\nu\beta}\right] \\ & - ie[\eta^{\mu\nu}(h^{\alpha\lambda}h_\lambda^\beta - \tfrac{1}{2}hh^{\alpha\beta}) + \eta^{\alpha\beta}(h^{\mu\lambda}h_\lambda^\nu - \tfrac{1}{2}hh^{\mu\nu})]W_\mu^* W_\alpha F_{\nu\beta}.\end{aligned} \tag{2.41}$$

Finally, the gravitational Lagrangian of the electromagnetic field is shown to be expanded as

$$\mathcal{L}_{gA} = \mathcal{L}_{gA}^0 + \kappa\mathcal{L}_{gA}^1 + \kappa^2\mathcal{L}_{gA}^2 + \cdots, \tag{2.42}$$
$$\mathcal{L}_{gA}^0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{2.43}$$
$$\mathcal{L}_{gA}^1 = \frac{1}{2}h_\nu^\tau F^{\mu\nu}F_{\mu\tau} + \frac{1}{2}h\mathcal{L}_{gA}^0, \tag{2.44}$$
$$\begin{aligned}\mathcal{L}_{gA}^2 = {} & \frac{1}{8}(h^2 - 2h_\mu^\nu h_\nu^\mu)\mathcal{L}_{gA}^0 + \frac{1}{4}F_{\alpha\beta}F_{\rho\sigma}[hh^{\alpha\rho}\eta^{\beta\sigma} \\ & - 2h_\mu^\alpha h^{\mu\rho}\eta^{\beta\sigma} - h^{\alpha\rho}h^{\beta\sigma}].\end{aligned} \tag{2.45}$$

To summarize, we have described in detail how to derive the general covariant Lagrangian for gravitational interactions with a scalar, a fermion, and a vector boson in the presence of the electromagnetic field. Then we have expanded the Lagrangian around the flat Minkowski space through the Gupta procedure. The expanded Lagrangian is not only Lorentz invariant at any order of h but also invariant under the transformation:

$$h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu X^\nu + \partial^\nu X^\mu, \tag{2.46}$$

with an arbitrary nonsingular function X^μ. The latter invariance will be called gravitational gauge invariance in the present work. It is now rather straightforward to obtain all the Feynman rules of propagators and vertices up to $O(\kappa^2)$. We present all the Feynman rules needed in the present work in the Appendix.

III. FACTORIZATION

In the standard gauge theory every four-body Born amplitude with a massless gauge boson as an external particle has been well known to be factorizable [11–14] into one factor which depends only on charge or other internal-symmetry indices and the other factor which depends on spin or polarization indices.

In this section we show that gravitational gauge invariance and Lorentz invariance in the linearized gravity force all the transition amplitudes of four-body graviton interactions to be factorized as well.

First of all let us explain factorization in a (non-)Abelian gauge theory following the procedure by Ref. [12]. The crucial point is that any amplitude with an incoming gauge boson is always arranged as a sum of terms of which each one consists of three distinctive parts—a charge factor A_i, a polarization-dependent part B_i, and a propagator C_i,

$$\mathcal{M} = \sum_{i=1}^{N} \frac{A_i B_i}{C_i}. \tag{3.1}$$

Then each group factor is summed up to vanish,

$$\sum_{i=1}^{N} A_i = \sum_{i=1}^{N} B_i = \sum_{i=1}^{N} C_i = 0, \quad (3.2)$$

due to charge conservation (gauge invariance), energy-momentum conservation (Lorentz invariance)

$$\sum_{i=1}^{N} \delta_i p_i = k, \quad (3.3)$$

and transversality ($k \cdot \epsilon = 0$), where ϵ^μ and k^μ are the polarization vector and four-momentum of the massless gauge boson, respectively, and $\delta_i = 1(-1)$ is for an outgoing (incoming) particle. Every amplitude $\mathcal{M}$ for $N = 3$ is then written in a factorized form as

$$\mathcal{M} = -\frac{C_1 C_2}{C_3}\left(\frac{A_1}{C_1} - \frac{A_2}{C_2}\right)\left(\frac{B_1}{C_1} - \frac{B_2}{C_2}\right), \quad (3.4)$$

or in equivalent forms with the indices (1,2,3) permuted. It is now clear that the expression (3.4) exhibits factorization of the transition amplitude into a charge-dependent part and a polarization-dependent part.

As an example, let us consider the gluon-gluon elastic scattering process $G^a G^c \to G^b G^d$ where the superscripts denote color indices. The factorization theorem enables the transition amplitude to be factorized as [12,13]

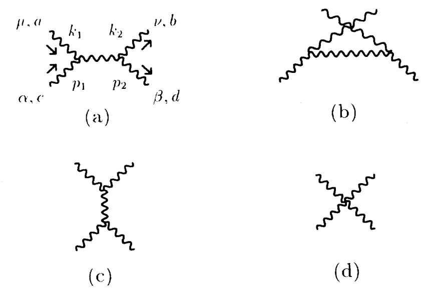

FIG. 1. Feynman diagrams for the gluon-gluon scattering process $G^a G^c \to G^b G^d$. The wavy line represents a gluon but not a photon.

$$\mathcal{M}\left(G^a G^c \to G^b G^d\right) = \left(\frac{\alpha_s}{\alpha}\right)\frac{(p_1\cdot k_1)(p_1\cdot k_2)}{k_1\cdot k_2}\left[\frac{f^{ace}f^{bde}}{p_1\cdot k_1} - \frac{f^{ade}f^{bce}}{p_1\cdot k_2}\right]\mathcal{M}_{\gamma v}, \quad (3.5)$$

where $k_1(p_1)$ are four-momenta of the incident gluon $G^a(G^c)$, $k_2(p_2)$ are four-momenta of the final gluon $G^b(G^d)$, and v stands for a massless vector boson with a positive electric charge (see Fig. 1). Here, f^{abc} are the structure constants of the SU(3) color-gauge group. The amplitude $\mathcal{M}_{\gamma v}$, which is of the same form as the Compton scattering amplitude of a charged massless vector boson, is given by

$$\mathcal{M}_{\gamma v} = e^2\left[\frac{B_1}{C_1} - \frac{B_2}{C_2}\right], \quad (3.6)$$

where

$$B_1 = -\epsilon_1^\mu \epsilon_2^{*\nu} \varepsilon_1^\alpha(p_1)\varepsilon_2^{*\beta}(p_2)[C_{\mu\alpha\delta}(k_1, p_1, -q_1)C_{\nu\beta}{}^{\delta}(-k_2, -p_2, q_1) + 2p_1\cdot k_1(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\beta}\eta_{\nu\alpha})],$$
$$B_2 = \epsilon_1^\mu \epsilon_2^{*\nu} \varepsilon_1^\alpha(p_1)\varepsilon_2^{*\beta}(p_2)[C_{\mu\beta\delta}(k_1, -p_2, q_2)C_{\nu\alpha}{}^{\delta}(-k_2, p_1, -q_2) - 2p_1\cdot k_2(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta})], \quad (3.7)$$
$$B_3 = \epsilon_1^\mu \epsilon_2^{*\nu} \varepsilon_1^\alpha(p_1)\varepsilon_2^{*\beta}(p_2)[C_{\mu\nu\delta}(k_1, -k_2, -q_3)C_{\alpha\beta}{}^{\delta}(p_1, -p_2, q_3) - 2k_1\cdot k_2(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha})],$$
$$C_1 = 2(p_1\cdot k_1), \quad C_2 = -2(p_1\cdot k_2), \quad C_3 = -2(k_1\cdot k_2), \quad (3.8)$$

with the definition

$$C_{\lambda\mu\nu}(p,q,r) = (p-q)_\nu \eta_{\lambda\mu} + (q-r)_\lambda \eta_{\mu\nu} + (r-p)_\mu \eta_{\nu\lambda}. \quad (3.9)$$

The transferred momenta q_1, q_2, and q_3 are given in terms of external momenta by

$$q_1 = p_1 + k_1, \quad g_2 = p_1 - k_2, \quad q_3 = k_1 - k_2. \quad (3.10)$$

It is a simple matter to determine the A_i factors for the process $G^a G^c \to G^b G^d$ by the use of the Jacobi identity of the structure functions:

$$A_1 = f^{ace}f^{dbe}, \quad A_2 = f^{ade}f^{bce}, \quad A_3 = f^{abe}f^{cde}. \quad (3.11)$$

We emphasize that the factorization in an ordinary gauge field theory stems from gauge invariance and Lorentz invariance of the theory. Since the linearized gravity is gravitational gauge invariant as well as Lorentz invariant, it is expected to have a similar factorization property in the linearized gravity. In the present work we show explicitly that every amplitude of a graviton scattering with a scalar, a fermion, a vector boson, or a graviton itself indeed exhibits such a remarkable factorization.

In order to prove factorization in the linearized gravity, we first note that gravitational gauge invariance guarantees the decomposition [20] of a graviton wave tensor $\epsilon^{\mu\nu}(2\lambda)$ into a multiplication of two spin-1 wave vectors,

$$\epsilon^{\mu\nu}(2\lambda) = \epsilon^\mu(\lambda)\epsilon^\nu(\lambda), \quad (3.12)$$

where the wave vector $\epsilon^\mu(\lambda)$ satisfies

$$k\cdot\epsilon(\lambda) = 0, \quad \epsilon(\lambda)\cdot\epsilon(\lambda') = -\delta_{\lambda,-\lambda'}, \quad (3.13)$$

so that the wave tensor $\epsilon^{\mu\nu}(2\lambda)$ satisfies

$$k_\mu \epsilon^{\mu\nu}(2\lambda) = \epsilon^{\mu\nu}(2\lambda)k_\nu = 0, \quad \epsilon^\mu{}_\mu(2\lambda) = 0, \tag{3.14}$$

with the graviton four-momentum k.

In order to show clearly the common features of four-body graviton processes, we organize the presentation of our results in the following. First, we introduce X as a generic notation for a scalar s, a fermion f, or a vector boson W. Second, the k_i ($i = 1, 2$) are for the four-momenta of the incident g and the final $g(\gamma)$ in the process $gX \to g(\gamma)X$, and p_i ($i = 1, 2$) for the four-momenta of the initial X and the final X, respectively. We note that there are four Feynman diagrams for every process (see Figs. 2 and 3). The last diagram in each figure set is a so-called contact term, which can be always absorbed into the other parts. We present only the results after absorbing the contact term and rearranging the amplitude according to the factorization theorem. Finally, we mention that all the processes under consideration have the same set of the kinematical factors C_i, denoting the s, t, and u channel momentum transfers,

$$C_1 = 2(p_1 \cdot k_1), \quad C_2 = -2(p_1 \cdot k_2), \quad C_3 = -2(k_1 \cdot k_2), \tag{3.15}$$

which are the same as Eq. (3.8). For simplicity, we will no longer write down this kinematical set in the following.

A. Graviton conversion into a photon

In this subsection, we consider the process of a graviton scattering off a particle X for the photon production, where X can be a scalar s, a fermion f, or a vector boson W. The graviton conversion into a photon can be considered as a means [21] to detect a gravitational wave. As mentioned before, k_1 and ϵ_1 are the incident graviton momentum and a wave vector for the graviton wave tensor, while k_2 and ϵ_2 are the final photon momentum and the photon wave vector, respectively. $p_1(p_2)$ denote the four-momentum of the incident(final) X particle.

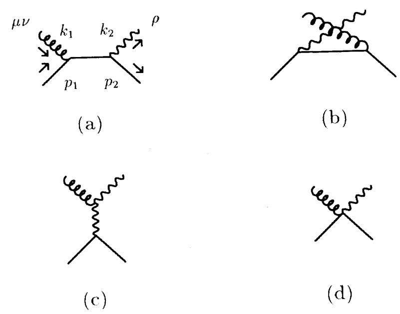

FIG. 2. Feynman diagrams for the process $gX \to \gamma X$. The curly line is for a graviton and the wavy line for a photon. Here X, represented by a solid line, can be a scalar s, a fermion f, or a vector boson W.

1. $gs \to \gamma s$

The simplest nontrivial process is the graviton scattering off a scalar particle for a photon production $gs \to \gamma s$. The process is of order e and κ in both the gravitational and the electromagnetic interactions and therefore the relevant Lagrangian consists of four parts as

$$\mathcal{L}_I^{\gamma s} = \mathcal{L}^0_{gs}(A) + \kappa\mathcal{L}^1_{gs}(A) + \mathcal{L}^0_{gA} + \kappa\mathcal{L}^1_{gA}. \tag{3.16}$$

The Feynman diagrams are shown in Fig. 2, where the solid line is for the scalar particle. After absorbing the contact term denoted by the last diagram, we obtain the resulting transition amplitude for the process $gs \to \gamma s$ divided into three parts:

$$\mathcal{M}_{gs\to\gamma s} = \mathcal{M}_a^{\gamma s} + \mathcal{M}_b^{\gamma s} + \mathcal{M}_c^{\gamma s}, \tag{3.17}$$

$$\mathcal{M}_a^{\gamma s} = e\kappa\frac{(p_1\cdot\epsilon_1)}{2(q_1^2-m^2)}\left[(q_1^2-m^2)(\epsilon_1\cdot\epsilon_2^*) - 4(p_2\cdot\epsilon_2^*)(p_1\cdot\epsilon_1)\right], \tag{3.18}$$

$$\mathcal{M}_b^{\gamma s} = e\kappa\frac{(p_2\cdot\epsilon_1)}{2(q_2^2-m^2)}\left[(q_2^2-m^2)(\epsilon_1\cdot\epsilon_2^*) - 4(p_2\cdot\epsilon_1)(p_1\cdot\epsilon_2^*)\right], \tag{3.19}$$

$$\mathcal{M}_c^{\gamma s} = e\kappa\frac{(k_2\cdot\epsilon_1)}{2q_3^2}\left[(q_1^2-q_2^2)(\epsilon_1\cdot\epsilon_2^*) + 4(p_2\cdot\epsilon_1)(p_1\cdot\epsilon_2^*) - 4(p_2\cdot\epsilon_2^*)(p_1\cdot\epsilon_1)\right]. \tag{3.20}$$

After extracting the kinematical factors C_i, it is straightforward to determine $A_i^{\gamma s}$ and $B_i^{\gamma s}$ ($i = 1, 2, 3$) as

$$A_1^{\gamma s} = e\kappa(\epsilon_1\cdot p_1), \quad A_2^{\gamma s} = -e\kappa(\epsilon_1\cdot p_2), \quad A_3^{\gamma s} = -e\kappa(\epsilon_1\cdot k_2), \tag{3.21}$$

$$\begin{aligned}
B_1^{\gamma s} &= (p_1\cdot k_1)(\epsilon_1\cdot\epsilon_2^*) - 2(p_2\cdot\epsilon_2^*)(p_1\cdot\epsilon_1),\\
B_2^{\gamma s} &= (p_2\cdot k_1)(\epsilon_1\cdot\epsilon_2^*) + 2(p_2\cdot\epsilon_1)(p_1\cdot\epsilon_2^*),\\
B_3^{\gamma s} &= -[(p_2+p_1)\cdot k_1](\epsilon_1\cdot\epsilon_2^*) - 2(p_2\cdot\epsilon_1)(p_1\cdot\epsilon_2^*) + 2(p_2\cdot\epsilon_2^*)(p_1\cdot\epsilon_1).
\end{aligned} \tag{3.22}$$

The transition amplitude is reduced to the factorized form

$$\mathcal{M}_{gs\to\gamma s} = -e\kappa F\left[\frac{\epsilon_1\cdot p_1}{p_1\cdot k_1} - \frac{\epsilon_1\cdot p_2}{p_1\cdot k_2}\right]\left[(\epsilon_1\cdot\epsilon_2^*) - \frac{(\epsilon_1\cdot p_1)(\epsilon_2^*\cdot p_2)}{p_1\cdot k_1} + \frac{(\epsilon_1\cdot p_2)(\epsilon_2^*\cdot p_1)}{p_1\cdot k_2}\right], \tag{3.23}$$

where ϵ_1^μ and ϵ_2^μ are two wave vectors for a graviton and a photon, respectively. Here, the overall kinematical factor F is

$$F = \frac{(p_1 \cdot k_1)(p_1 \cdot k_2)}{(k_1 \cdot k_2)}. \tag{3.24}$$

We note that the last factor is of the same form as the scalar-Compton scattering amplitude.

Before proceeding further, let us note here that the introduction of a manifestly gauge invariant four-vector $\tilde{\epsilon}_i$ $(i = 1, 2)$:

$$\tilde{\epsilon}_i = \epsilon_i - \frac{(p_1 \cdot \epsilon_i)}{(p_1 \cdot k_i)} k_i, \tag{3.25}$$

renders the expression of the scalar Compton scattering amplitude $\mathcal{M}_{\gamma s}$ greatly simplified:

$$\mathcal{M}_{\gamma s} = 2e^2(\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2^*). \tag{3.26}$$

Along with this simplification, gravitational gauge invariance and graviton transversality render the transition amplitude $\mathcal{M}_{gs\to\gamma s}$ completely factorized into a quite simple form:

$$\mathcal{M}_{gs\to\gamma s} = \frac{\kappa}{2e} F \left(\frac{p_2 \cdot \tilde{\epsilon}_1}{p_2 \cdot k_1}\right) [\mathcal{M}_{\gamma s}]. \tag{3.27}$$

2. $gf \to \gamma f$

The next simplest case is the process $gf \to \gamma f$ with a graviton in place of the incident photon in the ordinary fermion-Compton scattering process. The relevant Lagrangian for the process $gf \to \gamma f$ is composed of four parts:

$$\mathcal{L}_I^{\gamma f} = \mathcal{L}_{gf}^0(A) + \kappa \mathcal{L}_{gf}^1(A) + \mathcal{L}_{gA}^0 + \kappa \mathcal{L}_{gA}^1. \tag{3.28}$$

The Feynman diagrams for the process can be drawn in the same way as for the process $gs \to \gamma s$. The only difference is that in the present case the solid line in Fig. 2 is for a fermion. After absorbing the contact term denoted by the last diagram in Fig. 2, we obtain the transition amplitude $\mathcal{M}_{gf\to\gamma f}$:

$$\mathcal{M}_{gf\to\gamma f} = \mathcal{M}_a^{\gamma f} + \mathcal{M}_b^{\gamma f} + \mathcal{M}_c^{\gamma f}, \tag{3.29}$$

$$\mathcal{M}_a^{\gamma f} = -\frac{e\kappa}{2}\frac{(\epsilon_1 \cdot p_1)}{(q_1^2 - m^2)} \bar{u}(p_2, s_2) [\not\epsilon_2^*(\not q_1 + m) \not\epsilon_1] u(p_1, s_1), \tag{3.30}$$

$$\mathcal{M}_b^{\gamma f} = -\frac{e\kappa}{2}\frac{(\epsilon_1 \cdot p_2)}{(q_2^2 - m^2)} \bar{u}(p_2, s_2) [\not\epsilon_1(\not q_2 + m) \not\epsilon_2^*] u(p_1, s_1), \tag{3.31}$$

$$\mathcal{M}_c^{\gamma f} = \frac{e\kappa}{q_3^2}(k_2 \cdot \epsilon_1)\bar{u}(p_2, s_2)\left[(\epsilon_1 \cdot \epsilon_2^*) \not k_2 - (\epsilon_1 \cdot k_2) \not\epsilon_2^* - (\epsilon_2^* \cdot k_1) \not\epsilon_1\right]u(p_1, s_1). \tag{3.32}$$

Now the factors A_i and B_i $(i = 1, 2, 3)$ for the process $gf \to \gamma f$ are

$$A_i^{\gamma f} = A_i^{\gamma s} \quad (i = 1, 2, 3), \tag{3.33}$$

$$\begin{aligned} B_1^{\gamma f} &= -\frac{1}{2}\bar{u}(p_2, s_2) [\not\epsilon_2(\not q_1 + m) \not\epsilon_1] u(p_1, s_1), \\ B_2^{\gamma f} &= \frac{1}{2}\bar{u}(p_2, s_2) [\not\epsilon_1(\not q_2 + m) \not\epsilon_2^*] u(p_1, s_1), \\ B_3^{\gamma f} &= \bar{u}(p_2, s_2)[\not\epsilon_1(\epsilon_2^* \cdot k_1) + \not\epsilon_2^*(\epsilon_1 \cdot k_2) \\ &\quad -(\epsilon_1 \cdot \epsilon_2) \not k_2]u(p_1, s_1). \end{aligned} \tag{3.34}$$

As mentioned before the C_i factors are the same as those for the process $gs \to \gamma s$. Consequently we are led to the factorized transition amplitude

$$\mathcal{M}_{gf\to\gamma f} = \frac{\kappa}{2e} F \left(\frac{p_2 \cdot \tilde{\epsilon}_1}{p_2 \cdot k_1}\right) [\mathcal{M}_{\gamma f}], \tag{3.35}$$

where the transition amplitude $\mathcal{M}_{\gamma f}$ is of the same form as the standard Compton scattering amplitude:

$$\mathcal{M}_{\gamma f} = -e^2\bar{u}(p_2, s_2) \left[\not\epsilon_2^* \frac{1}{\not q_1 - m} \not\epsilon_1 + \not\epsilon_1 \frac{1}{\not q_2 - m} \not\epsilon_2^*\right] \times u(p_1, s_1). \tag{3.36}$$

3. $gW \to \gamma W$

Since the process $gW \to \gamma W$ involves three vector particles and one graviton, Feynman rules are complicated and as a result the expression of the amplitude is complicated as well. Without any new insight of the amplitude structure, any conventional method will require a lot of time to calculate the cross section. This formidable algebra can be avoided by a simple reorganization of the amplitude due to the factorization as described below.

First we write down the relevant Lagrangian for the process $gW \to \gamma W$, which consist of four parts:

$$\mathcal{L}_I^{\gamma W} = \mathcal{L}_{gW}^0(A) + \kappa \mathcal{L}_{gW}^1(A) + \mathcal{L}_{gA}^0 + \kappa \mathcal{L}_{gA}^1. \tag{3.37}$$

Through the same procedure as in the process $gf \to \gamma f$ we find

$$A_i^{\gamma W} = A_i^{\gamma s} \quad (i = 1, 2, 3), \tag{3.38}$$

$$\begin{aligned} B_1^{\gamma W} &= \frac{1}{2}\left[B_1 - m^2(\epsilon_1 \cdot \varepsilon_1)(\epsilon_2^* \cdot \varepsilon_2^*)\right], \\ B_2^{\gamma W} &= \frac{1}{2}\left[B_2 + m^2(\epsilon_1 \cdot \varepsilon_2^*)(\epsilon_2^* \cdot \varepsilon_1)\right], \quad B_3^{\gamma W} = \frac{B_3}{2}, \end{aligned} \tag{3.39}$$

where B_i $(i = 1, 2, 3)$ are given in Eq. (3.7). Then the amplitude reduces to the factorized form as

$$\mathcal{M}_{gW\to\gamma W} = -e\kappa F\left(\frac{p_2\cdot\tilde{\epsilon}_1}{p_2\cdot k_1}\right)\Bigg[(\tilde{\epsilon_1}\cdot\tilde{\epsilon_2^*})(\tilde{\epsilon_1}\cdot\tilde{\varepsilon_2^*}) - (k_1\cdot k_2)\left\{\frac{(\tilde{\epsilon_1}\cdot\tilde{\varepsilon_1})(\tilde{\epsilon_2^*}\cdot\tilde{\varepsilon_2^*})}{p_1\cdot k_1} - \frac{(\tilde{\epsilon_1}\cdot\tilde{\varepsilon_2^*})(\tilde{\epsilon_2^*}\cdot\tilde{\varepsilon_1})}{p_1\cdot k_2}\right\}$$
$$-\frac{1}{p_1\cdot k_2}\{(\tilde{\epsilon_2^*}\cdot\tilde{\varepsilon_1})(p_2\cdot\tilde{\epsilon_1})(k_2\cdot\tilde{\varepsilon_2^*}) + (\tilde{\epsilon_1}\cdot\tilde{\varepsilon_2^*})(p_2\cdot\tilde{\epsilon_2^*})(k_2\cdot\tilde{\varepsilon_1})$$
$$-(\tilde{\epsilon_2^*}\cdot\tilde{\varepsilon_2^*})(p_2\cdot\tilde{\epsilon_1})(k_2\cdot\tilde{\varepsilon_1})\} + \frac{1}{p_1\cdot k_1}(\tilde{\epsilon_1}\cdot\tilde{\varepsilon_1})(p_2\cdot\tilde{\epsilon_2^*})(k_2\cdot\tilde{\varepsilon_2^*})\Bigg], \quad (3.40)$$

where $p_1(p_2)$ and $\varepsilon_1(\varepsilon_2)$ are the four-momentum and wave vector of the initial(final) vector boson, respectively. The $\tilde{\varepsilon_i}$ $(i = 1, 2)$ is a polarization vector defined in a manifestly gauge invariant form as

$$\tilde{\varepsilon_i} = \varepsilon_i - \frac{(k_1\cdot\varepsilon_i)}{(k_1\cdot p_i)}p_i. \quad (3.41)$$

The introduction of the gauge invariant wave vector considerably simplifies the amplitude form and directly justifies gauge invariance. The second bracketed term of Eq. (3.40) can be shown to be the same as the Compton scattering amplitude of a charged vector boson given by

$$\mathcal{M}_{\gamma W} = -2e^2\Bigg[(\tilde{\epsilon_1}\cdot\tilde{\epsilon_2^*})(\tilde{\varepsilon_1}\cdot\tilde{\varepsilon_2^*}) - (k_1\cdot k_2)\left\{\frac{(\tilde{\epsilon_1}\cdot\tilde{\varepsilon_1})(\tilde{\epsilon_2^*}\cdot\tilde{\varepsilon_2})}{p_1\cdot k_1} - \frac{(\tilde{\epsilon_1}\cdot\tilde{\varepsilon_2})(\tilde{\epsilon_2^*}\cdot\tilde{\varepsilon_1})}{p_1\cdot k_2}\right\}$$
$$-\frac{1}{p_1\cdot k_2}\{(\tilde{\epsilon_2^*}\cdot\tilde{\varepsilon_1})(p_2\cdot\tilde{\epsilon_1})(k_2\cdot\tilde{\varepsilon_2^*}) + (\tilde{\epsilon_1}\cdot\tilde{\varepsilon_2^*})(p_2\cdot\tilde{\epsilon_2^*})(k_2\cdot\tilde{\varepsilon_1}) - (\tilde{\epsilon_2}\cdot\tilde{\varepsilon_2^*})(p_2\cdot\tilde{\epsilon_1})(k_2\cdot\tilde{\varepsilon_1})\} + \frac{1}{p_1\cdot k_1}(\tilde{\epsilon_1}\cdot\tilde{\varepsilon_1})(p_2\cdot\tilde{\epsilon_2^*})(k_2\cdot\tilde{\varepsilon_2^*})\Bigg]. \quad (3.42)$$

Consequently, the amplitude $\mathcal{M}_{gW\to\gamma W}$ is expressed in a factorized form as

$$\mathcal{M}_{gW\to\gamma W} = \frac{\kappa}{2e}F\left(\frac{p_2\cdot\tilde{\epsilon_1}}{p_2\cdot k_1}\right)[\mathcal{M}_{\gamma W}]. \quad (3.43)$$

B. Gravitational Compton scattering

In this subsection we investigate the factorization property of the gravitational elastic processes $gX \to gX$ for $X = s$, f, or W. The Feynman diagrams for these processes are shown in Fig. 3, where the solid line is for X. Through this subsection the $k_1(k_2)$ denote four-momenta of the incident(final) graviton and $p_1(p_2)$ denote four-momenta of the incident(final) X particle.

1. $gs \to gs$

The relevant Lagrangian of the process $gs \to gs$ consists of five parts:

$$\mathcal{L}_I^{gs} = \mathcal{L}_g^0 + \kappa\mathcal{L}_g^1 + \mathcal{L}_{gs}^0 + \kappa\mathcal{L}_{gs}^1 + \kappa^2\mathcal{L}_{gs}^2. \quad (3.44)$$

The transition amplitude after absorbing the contact term denoted by the last diagram in Fig. 3 becomes

$$\mathcal{M}_{gs} = \mathcal{M}_a^{gs} + \mathcal{M}_b^{gs} + \mathcal{M}_c^{gs}, \quad (3.45)$$

$$\mathcal{M}_a^{gs} = -\frac{\kappa^2}{4}\frac{1}{(q_1^2 - m^2)}[p_1\cdot k_1(\epsilon_1\cdot\epsilon_2^*) - 2(p_2\cdot\epsilon_2^*)(p_1\cdot\epsilon_1)]^2, \quad (3.46)$$

$$\mathcal{M}_b^{gs} = -\frac{\kappa^2}{4}\frac{1}{(q_2^2 - m^2)}[p_2\cdot k_1(\epsilon_1\cdot\epsilon_2^*) + 2(p_2\cdot\epsilon_1)(p_1\cdot\epsilon_2^*)]^2, \quad (3.47)$$

$$\mathcal{M}_c^{gs} = -\frac{\kappa^2}{4}\frac{1}{q_3^2}\{[(p_2 + p_1)\cdot k_1](\epsilon_1\cdot\epsilon_2^*) + 2(p_2\cdot\epsilon_1)(p_1\cdot\epsilon_2^*) - 2(p_2\cdot\epsilon_2^*)(p_1\cdot\epsilon_1)\}^2. \quad (3.48)$$

It is now straightforward to determine the corresponding factors A_i^{gs} and B_i^{gs} with the same C_i^{gs} $(i = 1, 2, 3)$ as in Eq. (3.8):

$$A_i^{gs} = \kappa^2 B_i^{\gamma s}, \quad B_i^{gs} = -\frac{B_i^{\gamma s}}{4} \quad (i = 1, 2, 3). \tag{3.49}$$

Consequently we obtain the factorized transition amplitude as

$$\mathcal{M}_{gs} = \frac{\kappa^2}{8e^4} F[\mathcal{M}_{\gamma s}]^2. \tag{3.50}$$

Note that the resulting amplitude is exactly the square of the standard scalar-Compoton scattering amplitude.

2. $gf \to gf$

The relevant Lagrangian for the process $gf \to gf$ is made up of five parts:

$$\mathcal{L}_I^{gf} = \mathcal{L}_g^0 + \kappa \mathcal{L}_g^1 + \mathcal{L}_{gf}^0 + \kappa \mathcal{L}_{gf}^1 + \kappa^2 \mathcal{L}_{gf}^2. \tag{3.51}$$

The transition amplitude $\mathcal{M}_{gf}$ for the process in Fig. 3 is reorganized after absorbing the contact term into other three parts as

$$\mathcal{M}_{gf} = \mathcal{M}_a^{gf} + \mathcal{M}_b^{gf} + \mathcal{M}_c^{gf}, \tag{3.52}$$

$$\mathcal{M}_a^{gf} = \frac{\kappa^2}{8(q_1^2 - m^2)} [2(\epsilon_1 \cdot p_1)(\epsilon_2^* \cdot p_2) - (\epsilon_1 \cdot \epsilon_2^*)(p_1 \cdot k_1)] \, [\bar{u}(p_2, s_2) \, [\not\epsilon_2^* (\not q_1 + m) \not\epsilon_1] \, u(p_1, s_1)], \tag{3.53}$$

$$\mathcal{M}_b^{gf} = \frac{\kappa^2}{8(q_2^2 - m^2)} [2(\epsilon_1 \cdot p_2)(\epsilon_2^* \cdot p_1) + (\epsilon_1 \cdot \epsilon_2^*)(p_1 \cdot k_2)] \, [\bar{u}(p_2, s_2) \, [\not\epsilon_1 (\not q_2 + m) \not\epsilon_2] \, u(p_1, s_1)],$$

$$\begin{aligned} \mathcal{M}_c^{gf} = & -\frac{\kappa^2}{4q_3^2} [2(\epsilon_1 \cdot p_2)(\epsilon_2^* \cdot p_1) - 2(\epsilon_1 \cdot p_1)(\epsilon_2^* \cdot p_2) + (\epsilon_1 \cdot \epsilon_2^*)(p_1 + p_2) \cdot k_1)] \\ & \times [\bar{u}(p_2, s_2) \, [\not\epsilon_1 (\epsilon_2^* \cdot k_1) + \not\epsilon_2^* (\epsilon_1 \cdot k_2) - (\epsilon_1 \cdot \epsilon_2^*) \not k_2] \, u(p_1, s_1)]. \end{aligned} \tag{3.54}$$

We obtain the factors A_i^{gf} and B_i^{gf} $(i = 1, 2, 3)$ as

$$A_i^{gf} = \kappa^2 B_i^{\gamma s}, \quad B_i^{gf} = \frac{B^{\gamma f}}{4} \quad (i = 1, 2, 3). \tag{3.55}$$

As a result the transition amplitude is reduced to the factorized form

$$\mathcal{M}_{gf} = \frac{\kappa^2}{8e^4} F \, [\mathcal{M}_{\gamma s}] \, [\mathcal{M}_{\gamma f}], \tag{3.56}$$

where the expressions of $\mathcal{M}_{\gamma s}$ and $\mathcal{M}_{\gamma f}$ are given in Eqs. (3.26) and (3.36), respectively. Note that the transition amplitude of graviton-fermion scattering is factorized into the transition amplitude of scalar-Compton scattering amplitude and the fermion-Compton scattering amplitude.

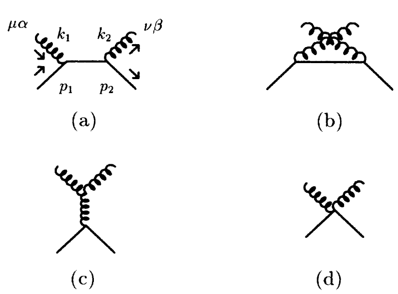

FIG. 3. Feynman diagrams for the process $gX \to gX$. The curly line is for a graviton. X, denoted by a solid line, can be a scalar s, a fermion f, or a vector boson W.

3. $gW \to gW$

The relevant Lagrangian for the process $gW \to gW$ is composed of five parts up to $O(\kappa^2)$:

$$\mathcal{L}_I^{gW} = \mathcal{L}_g^0 + \kappa \mathcal{L}_g^1 + \mathcal{L}_{gW}^0 + \kappa \mathcal{L}_{gW}^1 + \kappa^2 \mathcal{L}_{gW}^2. \tag{3.57}$$

The amplitude expression of the process is so complicated that the explicit presentation will be omitted. Instead, we write down just the factors A_i^{gW} and B_i^{gW} $(i = 1, 2, 3)$ given by the relations

$$A_i^{gW} = -\kappa^2 B_i^{\gamma s}, \quad B_i^{gW} = \frac{B_i^{\gamma W}}{4}. \tag{3.58}$$

It is now clear that the amplitude of graviton scattering with a vector boson can be written in a factorized form as

$$\mathcal{M}_{gW} = \frac{\kappa^2}{8e^4} F \, [\mathcal{M}_{\gamma s}] \, [\mathcal{M}_{\gamma W}], \tag{3.59}$$

where $\mathcal{M}_{\gamma s}$ and $\mathcal{M}_{\gamma W}$ are the same as Eqs. (3.26) and (3.42), respectively.

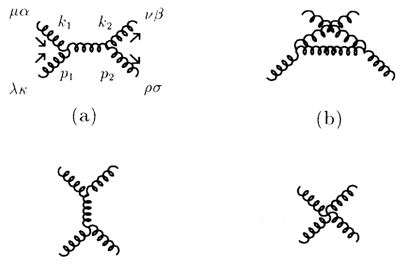

FIG. 4. Feynman diagrams for the process $gg \to gg$. The curly line is for a graviton.

C. Graviton-graviton elastic scattering

The process $gg \to gg$ is a pure gravitational process of order of κ^2. The relevant Lagrangian for the process $gg \to gg$ is made up of three terms as

$$\mathcal{L}_I^{gg} = \mathcal{L}_g^0 + \kappa\mathcal{L}_g^1 + \kappa^2\mathcal{L}_g^2. \tag{3.60}$$

As in other cases, after absorbing the contact term denoted by the last diagram in Fig. 4, we obtain the factors A_i^{gg}, B_i^{gg}, and C_i^{gg} for the process $gg \to gg$ as

$$A_i^{gW} = -\kappa^2 B_i, \quad B_i^{gW} = \frac{B_i}{16} \quad (i = 1, 2, 3). \tag{3.61}$$

Likewise, the transition amplitude of graviton-graviton scattering [22] is factorized as

$$\mathcal{M}_{gg} = \frac{\kappa^2}{8e^4} F \left[\mathcal{M}_{\gamma v}\right]^2. \tag{3.62}$$

Here, $p_1(p_2)$ and $\varepsilon_1^\mu(\varepsilon_2^\mu)$ are the four-momentum and wave vector of another initial(final) graviton, respectively.

D. Summary

In the present subsection, we summarize our results for the transition amplitudes obtained from the factorization procedure.

(1) The transition amplitudes $\mathcal{M}_{gs\to\gamma s}, \mathcal{M}_{gf\to\gamma f}$ and $\mathcal{M}_{gW\to\gamma W}$ have a common factor

$$\left(\frac{p_2 \cdot \tilde{\epsilon}_1}{p_2 \cdot k_1}\right). \tag{3.63}$$

(2) The transition amplitudes $\mathcal{M}_{gs\to gs}$, $\mathcal{M}_{gf\to gf}$, and $\mathcal{M}_{gW\to gW}$ have as a common factor

$$\mathcal{M}_{\gamma s} = 2e^2 \left(\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2^*\right). \tag{3.64}$$

(3) On the other hand, the graviton-graviton scattering amplitude $\mathcal{M}_{gg\to gg}$ is proportional to the square of the amplitude $\mathcal{M}_{\gamma v}$.

(4) All the transition amplitudes have as a common kinematical factor

$$F = \frac{(p_1 \cdot k_1)(p_1 \cdot k_2)}{(k_1 \cdot k_2)}. \tag{3.65}$$

(5) The other factors are exactly of the same form as the amplitudes $\mathcal{M}_{\gamma s}$, $\mathcal{M}_{\gamma f}$, and $\mathcal{M}_{\gamma W}$ except for their overall coupling constants according to the number of involved gravitons.

(6) While the form of Eq. (3.26) is independent of the choice of $\tilde{\epsilon}_i$, the form of the photon-vector boson scattering amplitude $\mathcal{M}_{\gamma v}$ can be modified if the $\tilde{\epsilon}_i$ and $\tilde{\varepsilon}_i$ are defined in a different way. Nevertheless, we find that the transiton amplitude $\mathcal{M}_{gg}$ satisfies Bose and crossing symmetries as expected from the cyclic property of the factorization. The completely symmetric expression of the amplitude $\mathcal{M}_{gg}$ can be found in Ref. [22].

IV. POLARIZATION

Clearly factorization will allow us to describe every four-body graviton interaction with a scalar, a fermion, a photon, a vector boson, or a graviton itself through well-known four-body photon interaction processes in the ordinary QED. One noteworthy advantage from factorization is a simple explanation for polarization phenomena in the graviton processes.

A natural Cartesian basis for a polarization vector $\epsilon^\mu(\lambda)$ with a momentum k_1 can be given in terms of two arbitrary four momenta p_1 and p_2. For simplicity, we use k_2, p_1, and p_2 satisfying the constraints $k_2^2 = 0$ and $p_1^2 = p_2^2 = m^2$. Then we can choose the basis consisting of two orthonomal four-vectors n_1 and n_2 such that [23,24]

$$n_1^\mu = \frac{N}{2}\left[(p_1 + p_2)^\mu - \frac{p_1 \cdot (k_1 + k_2)}{k_1 \cdot k_2}(k_1 + k_2)^\mu\right],$$
$$n_2^\mu = N\frac{\epsilon^{\mu\nu\alpha\beta} p_{1\nu} k_{2\alpha} k_{1\beta}}{k_1 \cdot k_2}, \tag{4.1}$$

with the conditions

$$k_i \cdot n_j = 0, \quad p_i \cdot n_2 = 0, \quad n_i \cdot n_j = -\delta_{ij} \quad (i, j = 1, 2), \tag{4.2}$$

and the normalization factor $N = 1/\sqrt{2F - m^2}$. In the basis we can introduce, as a polarization vector,

$$\epsilon^\mu(\lambda) = \frac{1}{\sqrt{2}}\left[n_1^\mu + i\lambda n_2^\mu\right], \tag{4.3}$$

where $\lambda = \pm 1$ is for the right- and left-handed polarization, respectively. Note that the scalar product $(n_1 \cdot \epsilon)$ of the polarization vector $\epsilon^\mu(\lambda)$ and the four-vector n_1 is independent of the helicity value λ. Certainly one can take another set of (n_1', n_2') as a basis, but it is different from the set (n_1, n_2) simply by a complex phase, which can be neglected without any change in physical observables.

Factorization allows us to use the well-known polarization effects in the ordinary QED for the investigation of polarization effects in the graviton processes. As a preliminary part, we consider the process $\gamma s \to Ys$ where Y is a massless scalar. The transition amplitude of the process $\gamma s \to Ys$ is

$$\mathcal{M}(\gamma_\lambda s \to Ys) = e^2\epsilon^\mu(\lambda)\left[\frac{p_1}{k_1\cdot p_1} - \frac{p_2}{k_1\cdot p_2}\right]_\mu = e^2\frac{1}{\sqrt{2}FN}, \quad (4.4)$$

where the coupling constant of the Yss vertex is assumed to be e. Note that the amplitude is completely independent of photon helicity λ.

One can now show that the process $gX \to \gamma X$ with $\mathcal{M}(\gamma s \to Ys)$ as a factor exhibits the same polarization property as $\gamma X \to \gamma X$ with X a scalar, a fermion, or a vector boson. However, the kinematical factor in the center-of-mass frame takes the form

$$F\frac{1}{\sqrt{2}FN} = \sqrt{F - \frac{m^2}{2}} = \sqrt{\frac{s}{2}}\cot\frac{\theta}{2}, \quad \theta = \angle(g,\gamma), \quad (4.5)$$

so that it makes the angular distribution of the graviton process different from the corresponding QED process. The former is more forwardly peaked than the latter. In addition, the graviton cross section increases and might violate unitarity at very high energy due to the $\sqrt{s}$ factor in Eq. (4.5).

Let us now consider the elastic photon-scalar scattering process $\gamma s \to \gamma s$. The transition amplitude is given by

$$\mathcal{M}(\gamma s \to \gamma s) = 2e^2\left[(\epsilon_1\cdot\epsilon_2^*) - \frac{(p_1\cdot\epsilon_1)(p_2\cdot\epsilon_2^*)}{p_1\cdot k_1} + \frac{(p_2\cdot\epsilon_1)(p_1\cdot\epsilon_2^*)}{p_2\cdot k_1}\right]. \quad (4.6)$$

While in general two different helicity bases are needed for two photons, only one helicity basis can be employed in the present case:

$$\epsilon_1^\mu(\lambda) = \epsilon_2^\mu(\lambda) = \frac{1}{\sqrt{2}}(n_1^\mu + i\lambda n_2^\mu). \quad (4.7)$$

These enable us to derive the result [25]

$$\mathcal{M}\left(\gamma_\lambda s \to \gamma_{\lambda'} s\right) = 2e^2\left(\delta_{\lambda\lambda'} - \frac{m^2}{2F}\right). \quad (4.8)$$

Despite scattering the photon helicity is preserved in the massless case.

Combined with the factorization property in the previous section, Eq. (4.8) leads to

$$\mathcal{M}(g_\lambda X \to g_{\lambda'} X) \propto F\left(\delta_{\lambda\lambda'} - \frac{m^2}{2F}\right) \times[\mathcal{M}(\gamma_\lambda X \to \gamma_{\lambda'} X)], \quad (4.9)$$

with X a scalar, a fermion, or a vector boson. The result yields an interesting fact that in the massless case the final graviton helicity should be the same as the initial graviton helicity irrespective of the spin configuration of matter fields.

On the other hand, the transition amplitude for the process $gg \to gg$ has neither $\mathcal{M}(\gamma s \to Ys)$ nor $\mathcal{M}(\gamma s \to \gamma s)$. As a result graviton helicity in the process $gg \to gg$ might be not preserved unlike in the processes $gX \to \gamma X$ and $gX \to gX$.

From now on we investigate in more detail polarization effects in the graviton scattering processes. As shown above, the helicity formalism permits a simple and general understanding of the polarization effects in the graviton scattering processes. However, it is often convenient to employ the so-called covariant polarization density matrix formalism, especially for a mixed state. In the massive case, the helicity formalism requires fixing the reference frame and has more complicated crossing symmetries. These problems can be avoided by the covariant density matrix formalism. In the light of these advantageous aspects the covariant density matrix formalism is employed in the present work to get general information on polarization effects in arbitrary reference frame.

The polarization of a photon (or a massless spin-1 particle) beam is completely described [24,26] in terms of the so-called Stokes parameters (SP's) ξ_i^γ $(i = 1,2,3)$. In the helicity basis, ξ_2^γ is the degree of circular polarization and the others are degrees of linear polarization. Because a graviton has only two helicity values, one can introduce the so-called graviton SP's ξ_i^g $(i = 1,2,3)$ [27]. Similarly, ξ_2^g is for the degree of graviton circular polarization and the others are for degrees of graviton linear polarization. On the whole, the photon or graviton polarization density matrix ρ_V $(V = \gamma, g)$ is given in the helicity basis by

$$\rho_V = \frac{1}{2}\begin{bmatrix} 1+\xi_2^V & -\xi_3^V + i\xi_1^V \\ -\xi_3^V - i\xi_1^V & 1-\xi_2^V \end{bmatrix}, \quad (4.10)$$

and, for a given polarization density matrix, the SP's are determined by the relations

$$\xi_1^V = -\mathrm{Tr}(\sigma_2\rho_V), \quad \xi_2^V = \mathrm{Tr}(\sigma_3\rho_V), \quad \xi_3^V = -\mathrm{Tr}(\sigma_1\rho_V), \quad (4.11)$$

where σ_i $(i = 1,2,3)$ are the Pauli matrices.

In the covariant density matrix formalism the photon projection operator $\epsilon^\mu(\lambda)\epsilon^{*\nu}(\lambda')$ for an incident photon beam is replaced by its photon covariant density matrix

$$\rho_\gamma^{\mu\nu} = \frac{1}{2}[(n_1^\mu n_1^\nu + n_2^\mu n_2^\nu) - (n_1^\mu n_2^\nu + n_2^\mu n_1^\nu)\xi_1^\gamma + i(n_2^\mu n_1^\nu - n_1^\mu n_2^\nu)\xi_2^\gamma + (n_2^\mu n_2^\nu - n_1^\nu n_1^\mu)\xi_3^\gamma]. \quad (4.12)$$

In the graviton case the covariant density matrix $\rho_g^{\mu\alpha:\nu\beta}$, which should substitute for the graviton projection operator $\epsilon^{\mu\alpha}(\lambda)\epsilon^{*\nu\beta}(\lambda')$, is written in terms of the graviton SP's ξ_i^g $(i = 1,2,3)$ [27] as

$$\begin{aligned}\rho^{\mu\alpha:\nu\beta} &= \sum_{\lambda\lambda'}\epsilon^{\mu\alpha}(p,\lambda)\rho_{\lambda\lambda'}\epsilon^{*\nu\beta}(p,\lambda') \\ &= \frac{1}{4}\{(n_1^\mu n_1^\nu + n_2^\mu n_2^\nu)(n_1^\alpha n_1^\beta + n_2^\alpha n_2^\beta) - (n_2^\mu n_1^\nu - n_1^\mu n_2^\nu)(n_2^\alpha n_1^\beta - n_1^\alpha n_2^\beta) \\ &\quad -[(n_2^\mu n_2^\nu - n_1^\mu n_1^\nu)(n_2^\alpha n_2^\beta - n_1^\alpha n_1^\beta) - (n_1^\mu n_2^\nu + n_2^\mu n_1^\nu)(n_1^\alpha n_2^\beta + n_2^\alpha n_1^\beta)]\xi_3^g \\ &\quad +i[(n_1^\mu n_1^\nu + n_2^\mu n_2^\nu)(n_2^\alpha n_1^\beta - n_1^\alpha n_2^\beta) + (n_2^\mu n_1^\nu - n_1^\mu n_2^\nu)(n_1^\alpha n_1^\beta + n_2^\alpha n_2^\beta)]\xi_2^g \\ &\quad -[(n_2^\mu n_2^\nu - n_1^\mu n_1^\nu)(n_1^\alpha n_2^\beta + n_2^\alpha n_1^\beta) - (n_1^\mu n_2^\nu + n_2^\mu n_1^\nu)(n_2^\alpha n_2^\beta - n_1^\alpha n_1^\beta)]\xi_1^g\}. \end{aligned} \quad (4.13)$$

In a process with an incident graviton the transition amplitude can be written as

$$\mathcal{M}_{\mathcal{I}} = \epsilon_{\mu\nu}\mathcal{A}_{\mathcal{I}}^{\mu\nu}, \quad (4.14)$$

and then the absolute square of the amplitude is given by

$$|\mathcal{M}_{\mathcal{I}}|^2 = \epsilon^{\mu\alpha}\epsilon^{*\nu\beta}\mathcal{A}_{\mathcal{I}\mu\alpha}\mathcal{A}^*_{\mathcal{I}\nu\beta}. \quad (4.15)$$

Polarization effects of an incident graviton beam are determined by replacing $\epsilon^{\mu\alpha}\epsilon^{*\nu\beta}$ in Eq. (4.15) by the covariant density matrix $\rho^{\mu\alpha:\nu\beta}$ in Eq. (4.13).

On the other hand, the scattering amplitude for a graviton production is in general written as

$$\mathcal{M}_{\mathcal{F}}(\lambda) = \epsilon^*_{\mu\nu}(p,\lambda)\mathcal{A}_{\mathcal{F}}^{\mu\nu}. \quad (4.16)$$

Then the final spin-2 polarization density matrix $\rho_{\lambda\lambda'}$ is determined through the relation

$$\rho_{\lambda\lambda'} = \frac{\mathcal{M}_{\mathcal{F}}(\lambda)\mathcal{M}^*_{\mathcal{F}}(\lambda')}{\sum_{\lambda\lambda'}\mathcal{M}_{\mathcal{F}}(\lambda)\mathcal{M}^*_{\mathcal{F}}(\lambda')}. \quad (4.17)$$

After such manipulation, Eq. (4.11) is used to obtain the final graviton SP.

In the following we present the differential cross sections in a 2×2 matrix form in order to consider the beam interference effects and to relate directly those expressions with the final polarization density matrices through the relation (4.17).

A. Graviton conversion into a photon

In this subsection, we use the factorized amplitudes for the processes $gX \to \gamma X$ obtained in Sec. III A in order to consider the polarization effects. Since those processes have the amplitude $\mathcal{M}_{\gamma s\to Ys}$ as a common factor, it is obvious that the polarization effects of the initial graviton beam should be identical to those of the initial photon beam in the process $\gamma X \to \gamma X$.

1. $gs \to \gamma s$

First of all, we consider the simplest process $gs \to \gamma s$. Following the procedure described before, we obtain the differential cross section of the process $gs \to \gamma s$ in a 2×2 matrix form as

$$\frac{d\sigma^{\gamma s}}{dt}(\lambda\lambda') = \frac{\pi\alpha\alpha_g}{4}\frac{(su-m^4)}{(s-m^2)^2t}\left(F_0^{\gamma s} + \sum_{i=1}^{3}F_i^{\gamma s}\chi_i\right)_{\lambda\lambda'}, \quad (4.18)$$

$$F_0^{\gamma s} = 1+2f+2f^2+2f(1+f)\xi_3^g,\ \ F_1^{\gamma s} = (1+2f)\xi_1^g,$$
$$F_2^{\gamma s} = (1+2f)\xi_2^g,\ \ F_3^{\gamma s} = (1+2f+2f^2)\xi_3^g + 2f(1+f), \quad (4.19)$$

where we have introduced the notations

$$\alpha = \frac{e^2}{4\pi},\ \ \alpha_g = \frac{\kappa^2}{4\pi},\ \ f = -\frac{m^2}{2F},$$
$$s = (p_1+k_1)^2,\ \ u = (p_1-k_2)^2,\ \ t = (p_1-p_2)^2. \quad (4.20)$$

Here the χ_i $(i=1,2,3)$ are three 2×2 matrices related with the Pauli matrices σ_i as

$$\chi_1 = -\sigma_2,\ \ \chi_2 = \sigma_3,\ \ \chi_3 = -\sigma_1. \quad (4.21)$$

Then the final photon SP can be obtained from Eq. (4.19) as

$$\xi_i'^{\gamma} = \frac{F_i^{\gamma s}}{F_0^{\gamma s}}. \quad (4.22)$$

The polarization of the final photon depends on that of the incident graviton in general. One can now check that the final photon SP's are identical to those of the initial graviton beam in the massless case, i.e., $\xi_i'^{\gamma} = \xi_i^g$ $(i=1,2,3)$. This result is due to the fact that the amplitude has not only the factor $\mathcal{M}_{\gamma s\to Ys}$ but also the factor $\mathcal{M}_{\gamma s\to\gamma s}$.

2. $gf \to \gamma f$

In the process $gf \to \gamma f$ we can in principle consider the case where all the particles are polarized. But in order to look into the implications from factorization to the polarizations, it will be sufficient to consider the case where all the other particles except the final fermion are polarized.

For notational convenience we first introduce the invariants

$$a_1 = \frac{(s_1\cdot k_1)}{m},\ \ a_2 = \frac{(s_1\cdot k_2)}{m},\ \ \epsilon = \frac{\varepsilon_{\mu\nu\rho\sigma}s_1^\mu k_1^\nu p_1^\rho k_2^\sigma}{m(s-m^2)}, \quad (4.23)$$

where s_1 is the incident fermion spin four-vector and m is the fermion mass. k_i and p_i $(i=1,2)$ are defined in the same way as in Sec. III.

The differential cross section of the process $gf \to \gamma f$ is obtained in a 2×2 matrix form as

$$\frac{d\sigma^{\gamma f}}{dt}(\lambda\lambda') = \frac{\pi\alpha\alpha_g}{4}\frac{(su-m^4)}{(s-m^2)^2t}\left(F_0^{\gamma f}+\sum_{i=1}^{3}F_i^{\gamma f}\chi^i\right)_{\lambda\lambda'}, \tag{4.24}$$

$$\begin{aligned} F_0^{\gamma f} &= -h_1+4f(1+f)(1-\xi_3^g) \\ &\quad -2f\left[(1+2f)a_1+a_2\right]\xi_2^g, \\ F_1^{\gamma f} &= 2(1+2f)\xi_1^g-4fh_2\epsilon\xi_2^g, \\ F_2^{\gamma f} &= -(1+2f)\left[h_1\xi_2^g+2(1-\xi_3^g)fa_1\right]-2\frac{fa_2}{h_2}\xi_3^g \\ &\quad -4f\epsilon\xi_1^g, \\ F_3^{\gamma f} &= -4f(1+f)+2\left[1+2f(1+f)\right]\xi_3^g \\ &\quad +2f[(1+2f)a_1+h_2a_2]\xi_2^g, \end{aligned} \tag{4.25}$$

where in addition to f we introduce two Lorentz invariant functions

$$h_1 = \frac{s-m^2}{u-m^2}+\frac{u-m^2}{s-m^2}, \quad h_2=\frac{s-m^2}{u-m^2}. \tag{4.26}$$

When averaged over the initial spin states, Eq. (4.24) gives the same results as in Ref. [7]. From the ratio of $F_i^{\gamma f}$ to $F_0^{\gamma f}$ in Eq. (4.25) similar to Eq. (4.22), we can obtain the SP of the final photon beam. We note that when the initial graviton SP's are used in place of the photon SP's, the final photon polarization is identical to that [24] of the QED Compton scattering process. This is a result expected from the factorization property. For the case of massless and unpolarized fermion, we obtain the photon SP's as

$$\begin{aligned} \xi_1'^{\gamma} &= -\left(\frac{2su}{s^2+u^2}\right)\xi_1^g, \quad \xi_2'^{\gamma}=\xi_2^g, \\ \xi_3'^{\gamma} &= -\left(\frac{2su}{s^2+u^2}\right)\xi_3^g. \end{aligned} \tag{4.27}$$

As a result, the degree of circular polarization is preserved despite scattering even in the graviton-fermion scattering process. However, the degrees of linear polarization are reduced according to the scattering angle.

3. $gW\to\gamma W$

In order to describe the process with all polarized particles, we have to introduce four different sets of Stokes parameters. For simplicity, we consider the case where the initial and final massive vector bosons are unpolarized. Then in a 2×2 matrix form, the differential cross section of the process $gW\to\gamma W$ is given by

$$\frac{d\sigma^{\gamma W}}{dt}(\lambda\lambda') = \frac{\pi\alpha\alpha_g}{12}\frac{(su-m^4)}{(s-m^2)^2t} \times\left(F_0^{\gamma W}+\sum_{i=1}^{3}F_i^{\gamma W}\chi^i\right)_{\lambda\lambda'}, \tag{4.28}$$

$$\begin{aligned} F_0^{\gamma W} &= 6(1+\xi_3^g)f(1+f)+3+4h_1+2h_1^2, \\ F_1^{\gamma W} &= -3(1+2f)\xi_1^g, \quad F_2^{\gamma W}=(1+2f)(5+4h_1)\xi_2^g, \\ F_3^{\gamma W} &= 3(1+2f+2f^2)\xi_3^g+6f(1+f). \end{aligned} \tag{4.29}$$

In the massless case where $f=0$, the final photon SP's are shown to be

$$\begin{aligned} \xi_1'^{\gamma} &= -\left(\frac{3}{2h_1^2+4h_1+3}\right)\xi_1^g, \\ \xi_2'^{\gamma} &= \left(\frac{4h_1+5}{2h_1^2+4h_1+3}\right)\xi_2^g, \\ \xi_3'^{\gamma} &= \left(\frac{3}{2h_1^2+4h_1+3}\right)\xi_3^g. \end{aligned} \tag{4.30}$$

Note that even in the massless limit the final photon helicity is not equal to the initial graviton helicity.

B. Gravitational Compton scattering

In this subsection we consider polarization effects in elastic graviton scattering processes $gX\to gX$, by using the results obtained in Sec. III B. Since the factorization forces every amplitude to have the amplitude $\mathcal{M}_{\gamma s\to\gamma s}$ as a factor, the graviton helicity should be preserved when the particle X is massless. At first, we present the results in the massive case and then discuss the massless limit to check graviton helicity preservation.

1. $gs\to gs$

The simplest one of elastic graviton-matter scattering processes is the graviton-scalar scattering process $gs\to gs$. The differential cross section is written in a 2×2 matrix form as

$$\frac{d\sigma^{gs}}{dt}(\lambda\lambda') = \frac{\pi\alpha_g^2}{16}\left(\frac{u-m^2}{t}\right)^2\left(F_0^{gs}+\sum_{i=1}^{3}F_i^{gs}\chi_i\right)_{\lambda\lambda'}, \tag{4.31}$$

$$\begin{aligned} F_0^{gs} &= \frac{1}{2}(1+2f)^2+f^2(1+f)^2(1+\xi_3^g), \\ F_1^{gs} &= \frac{1}{2}(1+2f)(1+f)^2\xi_1^g, \\ F_2^{gs} &= \frac{1}{2}(1+2f)(1+f)^2\xi_2^g, \\ F_3^{gs} &= \frac{1}{2}(1+2f)^2\xi_3^g+f^2(1+f)^2(1+\xi_3^g). \end{aligned} \tag{4.32}$$

We can see the difference in the polarization of the outgoing graviton from the polarization of the outgoing photon in the process $gs\to\gamma s$ by comparison of Eq. (4.19) with Eq. (4.32). When interacting gravitons are unpolarized, the differential cross section of Eq. (4.31) gives the same results as in Ref. [15]. In the massless case, where f vanishes, we find that $\xi_i'^g=\xi_i^g$ $(i=1,2,3)$ as in the process $gs\to\gamma s$ so that graviton polarization is preserved in spite of scattering.

2. $gf \to gf$

The next simplest process is $gf \to gf$. The differential cross section of a 2×2 matrix form is given by

$$\frac{d\sigma^{gf}}{dt}(\lambda\lambda') = \frac{\pi\alpha_g^2}{16}\left(\frac{u-m^2}{t}\right)^2 \left(F_0^{gf} + \sum_{i=1}^{3} F_i^{gf}\chi_i\right)_{\lambda\lambda'}, \tag{4.33}$$

$$\begin{aligned}
F_0^{gf} &= -f(1+f)(3h_1-2) - h_1 + 4f^2(1+f)^2(1+\xi_3^g) - \left[(1+2f+2f^2)\left\{(1+2f)a_1 + a_2\right\} + 2(1+2f)\epsilon\right] f\xi_2^g \\
F_1^{gf} &= 2\left[(1+f)^4 - f^4\right]\xi_1^g + 4(1+f)f^2 h_2\epsilon\xi_2^g, \\
F_2^{gf} &= -\left[(1+2f+2f^2)\left\{(1+2f)a_1 + a_2\right\} + 2(1+2f)h_2\epsilon\right] f - (1+2f)\left[f^2 h_1 + (h_1-2f)(1+f)\right]\xi_2^g \\
&\quad -2(1+f)f^2\left[2\epsilon\xi_1^g - \left\{(1+2f)a_1 + \frac{a_2}{h_2}\right\}\xi_3^g\right], \\
F_3^{gf} &= 4f^2(1+f)^2(\xi_3^g - 1) + 2(1+2f^2+2f^2)\xi_3^g + 2f^2(1+f)\left\{(1+2f)a_1 + h_2 a_2\right\}\xi_2^g,
\end{aligned} \tag{4.34}$$

where the spin states of the final fermion are summed. When averaged over the initial spin states, Eq. (4.33) gives the same results as in Ref. [7]. Then the SP of the outgoing graviton can be obtained from Eq. (4.34):

$$\xi_i'^g = \frac{F_i^{gf}}{F_0^{gf}}. \tag{4.35}$$

One can see the polarization of the outgoing graviton is influenced by the polarization of the incident graviton as well as incident fermion. In the massless case, the final graviton SP reduces to

$$\xi_1'^g = -\left(\frac{2su}{s^2+u^2}\right)\xi_1^g, \quad \xi_2'^g = \xi_2^g$$
$$\xi_3'^g = -\left(\frac{2su}{s^2+u^2}\right)\xi_3^g. \tag{4.36}$$

It is straightforward to show that graviton helicity is preserved in the massless case as in the process $gf \to \gamma f$.

3. $gW \to gW$

We consider only the case where the massive vector boson W is unpolarized. The differential cross section of the process $gW \to gW$ is written in a 2×2 matrix form as

$$\frac{d\sigma^{gW}}{dt}(\lambda\lambda') = \frac{\pi\alpha_g^2}{6}\left(\frac{u-m^2}{t}\right)^2 \left(F_0^{gW} + \sum_{i=1}^{3} F_i^{gW}\chi_i\right)_{\lambda\lambda'}, \tag{4.37}$$

$$\begin{aligned}
F_0^{gW} &= 6f^4 + 12f^3 + 2f^2(h_1-3)^2 + 2f(h_1^2 - 2h_1 - 2) \\
&\quad + (h_1^2-1) + 6f^2(1+f)^2\xi_3^g, \\
F_1^{gW} &= -3(1+4f+6f^2+4f^3)\xi_1^g, \\
F_2^{gW} &= [2f^2(2h_1+1)(2f+3) - 2f(h_1^2-2h_1-2) \\
&\quad + (h_1^2-1)]\xi_2^g, \\
F_3^{gW} &= 6f^2(1+f)^2 + 3[f^4 + (1+f)^4]\xi_3^g.
\end{aligned} \tag{4.38}$$

In the massless limit the final graviton SP's become

$$\xi_1'^g = -\left(\frac{3}{h_1^2-1}\right)\xi_1^g, \quad \xi_2'^g = \xi_2^g, \quad \xi_3'^g = \left(\frac{3}{h_1^2-1}\right)\xi_3^g. \tag{4.39}$$

Clearly graviton helicity is preserved in the massless case as shown in Eq. (4.39).

4. $g\gamma \to g\gamma$

The differential cross section of the process $g\gamma \to g\gamma$ is written in a 2×2 matrix form as

$$\frac{d\sigma^{g\gamma}}{dt}(\lambda\lambda') = \frac{\pi\alpha_g^2}{32s^2}\left(F_0^{g\gamma} + \sum_{i=1}^{3} F_i^{g\gamma}\chi_i\right)_{\lambda\lambda'}, \tag{4.40}$$

$$\begin{aligned}
F_0^{g\gamma} &= \frac{1}{2t^2}\left[(1+\xi_2^g\xi_2^\gamma)s^4 + (1-\xi_2^g\xi_2^\gamma)u^4\right], \\
F_1^{g\gamma} &= \left(\frac{su}{t}\right)^2\xi_1^g, \\
F_2^{g\gamma} &= \frac{1}{2t^2}\left[(\xi_2^g + \xi_2^\gamma)s^4 + (\xi_2^g - \xi_2^\gamma)u^4\right], \\
F_3^{g\gamma} &= \left(\frac{su}{t}\right)^2\xi_3^g,
\end{aligned} \tag{4.41}$$

where ξ_i^γ and ξ_i^g $(i = 1,2,3)$ are the SP's of the incident photon beam and the incident graviton beam, respectively, and the final photon polarization is summed. When interacting gravitons and photons are unpolarized, the differential cross section of the Eq. (4.40) gives the same results as in Ref. [15]. Then the explicit form of the outgoing graviton SP can be obtained as

$$\xi_i'^g = \frac{F_i^{g\gamma}}{F_0^{g\gamma}}. \tag{4.42}$$

Similarly, we can obtain the change of polarization of the photon colliding with a graviton. After taking an average over the initial photon polarization, we obtain the final graviton SP's as

$$\xi_1'^g = \left(\frac{2s^2u^2}{s^4+u^4}\right)\xi_1^g, \quad \xi_2'^g = \xi_2^g, \quad \xi_3'^g = \left(\frac{2s^2u^2}{s^4+u^4}\right)\xi_3^g. \tag{4.43}$$

We also find that the graviton helicity is preserved for the unpolarized incident and final photon beams, but in general the final graviton SP depends on both initial photon and graviton SP's.

C. Graviton-graviton scattering

In the elastic graviton-graviton scattering process, three-graviton vertices and a four-graviton vertex are involved. Even though the vertices are so complicated as shown explicitly in the Appendix, we obtain the very simple differential cross section of the process $gg \to gg$ as

$$\frac{d\sigma^{gg}}{dt}(\lambda\lambda') = \frac{\pi\alpha_g^2}{32s^4u^2t^2}\left(F_0^{gg} + \sum_{i=1}^{3} F_i^{gg}\chi_i\right)_{\lambda\lambda'}, \tag{4.44}$$

$$\begin{aligned} F_0^{gg} &= \frac{1}{2}(1+\xi_2^{g_1}\xi_2^{g_2})s^8 + \frac{1}{2}(1-\xi_2^{g_1}\xi_2^{g_2})(u^8+t^8) \\ &\quad +(\xi_3^{g_1}\xi_3^{g_2} + \xi_1^{g_1}\xi_1^{g_2})u^4t^4, \\ F_1^{gg} &= (\xi_1^{g_2}u^4 + \xi_1^{g_1}t^4)s^4, \\ F_2^{gg} &= \frac{1}{2}(\xi_2^{g_2} - \xi_2^{g_1})(u^8 - t^8) + \frac{1}{2}(\xi_2^{g_2} + \xi_2^{g_1})s^8, \\ F_3^{gg} &= (\xi_3^{g_2}u^4 + \xi_3^{g_1}t^4)s^4. \end{aligned} \tag{4.45}$$

Here $\xi_i^{g1}(\xi_i^{g2})$ are the incident graviton SP's with momentum $k_1(p_1)$ and the polarization of one of two final graviton beams is summed. When interacting gravitons and photons are unpolarized, the differential cross section of the Eq. (4.44) gives the same results as in Ref. [15]. In contrast to all the other processes under consideration in the present section, the graviton helicity in the process $gg \to gg$ is not preserved. This reflects that the transition amplitude contains neither $\mathcal{M}_{\gamma s\to Ys}$ nor $\mathcal{M}_{\gamma s\to\gamma s}$ as a common factor, which can lead to the graviton helicity preservation.

V. SUMMARY AND DISCUSSION

Gravitational gauge invariance and graviton transversality force the transition amplitudes of four-body graviton interactions to be factorized:

$$\left\{\begin{array}{c}\mathcal{M}_{gs\to\gamma s}\\ \mathcal{M}_{gf\to\gamma f}\\ \mathcal{M}_{gW\to\gamma W}\end{array}\right\} = -\frac{\kappa}{2e^3}F\left[\mathcal{M}_{\gamma s\to Ys}\right]\times\left\{\begin{array}{c}\mathcal{M}_{\gamma s}\\ \mathcal{M}_{\gamma f}\\ \mathcal{M}_{\gamma W}\end{array}\right\}, \tag{5.1}$$

$$\left\{\begin{array}{c}\mathcal{M}_{gs}\\ \mathcal{M}_{gf}\\ \mathcal{M}_{gW}\end{array}\right\} = \frac{\kappa^2}{8e^4}F\left[\mathcal{M}_{\gamma s}\right]\times\left\{\begin{array}{c}\mathcal{M}_{\gamma s}\\ \mathcal{M}_{\gamma f}\\ \mathcal{M}_{\gamma W}\end{array}\right\}, \tag{5.2}$$

$$\mathcal{M}_{gg} = \frac{\kappa^2}{8e^4}F\left[\mathcal{M}_{\gamma v}\right]\times\left[\mathcal{M}_{\gamma v}\right]. \tag{5.3}$$

The introduction of manifestly gauge invariant four-vectors $\tilde{\epsilon}_i$ and $\tilde{\varepsilon}_i$ $(i=1,2)$ renders each amplitude expression simplified. This simplification with the factorization property justifies why, with all the very complicated three-graviton and four-graviton vertices [15,16], the final form of transition amplitudes is so simple.

The factorized transition amplitudes facilitate the investigation of polarization effects in the four-body graviton interactions. The transition amplitudes for the graviton interactions with a photon or a matter field, $gX \to \gamma X$, where X is a scalar, a fermion, or a vector boson, have essentially the same transition amplitude structure as those involving a photon instead of the graviton, apart from a simple overall kinematical factor. As a result, the polarization effects involving the graviton are identical to those for the corresponding photon if the graviton Stokes parameters are used in place of the photon Stokes parameters. But the kinematical factor makes the angular distribution of the graviton process different from that of the corresponding photon process. On the other hand, the processes $gX \to gX$ have as a common factor the elastic photon-scalar scattering amplitude $\mathcal{M}_{\gamma s}$ with the scalar mass equal to the X mass in their amplitude expressions. This leads to the conclusion that, when the particle X is massless, the graviton helicity is preserved due to the photon helicity conservation of the process $\gamma s \to \gamma s$ in the massless limit. Only the mass terms cause the graviton helicity to be flipped.

The process $gg \to gg$ does not contain $\mathcal{M}_{\gamma s}$ as a common factor. This point is reflected in the fact that the graviton helicity is not preserved in the process $gg \to gg$ in spite of the masslessness of graviton.

The validity of factorization can become more concrete through further extensive investigation. We point out a few aspects worth further investigating. (i) A formal proof of factorization might be presented. There is a factorization property of the same type in closed string theories [28]. From the fact that a closed string theory reduces to a supergravity theory in the infinite string tension limit [29], one can conclude that this is a real proof of the factorization in the linearized gravity. However, factorization in the string theory is due to the independence of left-moving modes and right-moving modes, while only gravitational gauge invariance and Lorentz invariance are imposed in the linearized gravity. Still, the relationship between two concepts are to be established. (ii) Factorization is expected to hold even if matter particles have different masses. As an example, the process $ge \to W\nu_e$ can be considered to check this point. (iii) It will be an interesting question whether factorization survives against any loop effects [30].

To conclude, factorization has such a generic property in any Lorentz-invariant gauge theory that its more intensive and extensive investigation is expected to provide us with some clues for the unification of gravity with other interactions.

ACKNOWLEDGMENTS

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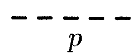
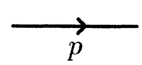
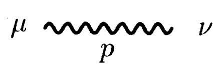
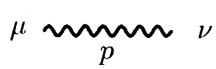
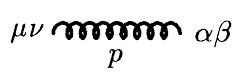
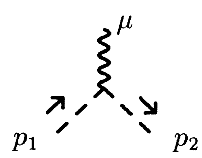
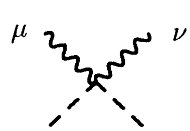
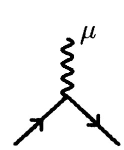
APPENDIX

In this appendix we present all the Feynman rules for propagators and vertices needed in the present work. The Landau gauge is chosen for the photon propagator and the de Donder gauge for the graviton propagator. A dashed line is for a scalar and a directed solid line for a fermion. A vector boson is denoted by a wiggly line and a graviton by a curly line.

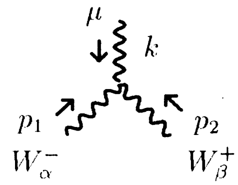
In the $ffgg$ vertex [Sym] represents symmetrization between μ and ν and between λ and κ while the symbol [Per] indicates permutation among both $(k_1\mu\nu)$ and $(k_2\lambda\kappa)$.

In the last two vertices, the symbol [Sym] means symmetrization between μ and α, between ν and β, and between σ and γ, respectively, for the three-graviton vertex, or between μ and α, between ν and β, between σ and γ, and between ρ and λ, respectively, for the four-graviton vertex. The P indicates permutation among $(k_1\mu\alpha), (k_2\nu\beta), (k_3\sigma\gamma)$ for the three-graviton vertex, or among $(k_1\mu\alpha), (k_2\nu\beta), (k_3\sigma\gamma)$, and $(k_4\rho\lambda)$ for the four-graviton vertex, and each subscript in P is for the number of independent permutations. As an example, $P_3(k_1 \cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) = (k_1 \cdot k_2)\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma} + (k_2 \cdot k_3)\eta_{\nu\sigma}\eta_{\beta\gamma}\eta_{\mu\alpha} + (k_3 \cdot k_1)\eta_{\sigma\mu}\eta_{\gamma\alpha}\eta_{\nu\beta}$.

Feynman Rules

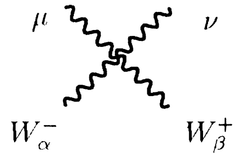
Scalar propagator:		$\frac{i}{p^2-m^2}$
Fermion propagator:		$\frac{i}{\not{p}-m}$
W boson propagator:		$\frac{-i(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{m^2})}{p^2-m^2}$
Photon propagator:		$\frac{-i\eta^{\mu\nu}}{p^2}$
Graviton propagator:		$\frac{i}{2}\frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}}{p^2}$
$ss\gamma$ vertex:		$-ie(p_1+p_2)_\mu$
$ss\gamma\gamma$ vertex:		$2ie^2\eta_{\mu\nu}$
$ff\gamma$ vertex:		$-ie\gamma_\mu$

$WW\gamma$ vertex:



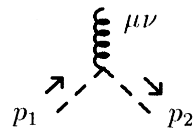
$$-ie[(p_1-p_2)_\mu\eta_{\alpha\beta} +(p_2-k)_\alpha\eta_{\mu\beta} +(k-p_1)_\beta\eta_{\alpha\mu}]$$

$WW\gamma\gamma$ vertex:



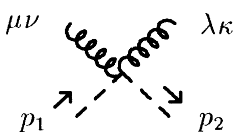
$$-ie^2(2\eta_{\mu\nu}\eta_{\alpha\beta}-\eta_{\mu\alpha}\eta_{\nu\beta}-\eta_{\mu\beta}\eta_{\nu\alpha})$$

ssg vertex:



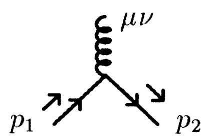
$$\frac{i}{2}\kappa[\eta_{\mu\nu}(p_1\cdot p_2-m^2) -p_{1\mu}p_{2\nu}-p_{1\nu}p_{2\mu}]$$

$ssgg$ vertex:



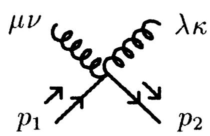
$$\begin{aligned}\tfrac{i}{4}\kappa^2[&(\eta_{\mu\nu}\eta_{\lambda\kappa}-\eta_{\mu\lambda}\eta_{\nu\kappa}-\eta_{\mu\kappa}\eta_{\nu\lambda})(p_1\cdot p_2-m^2)\\ &+\eta_{\nu\lambda}(p_{1\mu}p_{2\kappa}+p_{1\kappa}p_{2\mu})+\eta_{\mu\kappa}(p_{1\lambda}p_{2\nu}+p_{1\nu}p_{2\lambda})\\ &+\eta_{\mu\lambda}(p_{1\nu}p_{2\kappa}+p_{1\kappa}p_{2\nu})+\eta_{\nu\kappa}(p_{1\lambda}p_{2\mu}+p_{1\mu}p_{2\lambda})\\ &-\eta_{\mu\nu}(p_{1\lambda}p_{2\kappa}+p_{1\kappa}p_{2\lambda})-\eta_{\lambda\kappa}(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu})]\end{aligned}$$

ffg vertex:



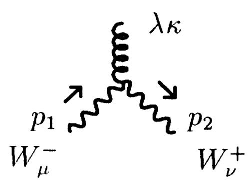
$$\tfrac{i}{8}\kappa[2\eta_{\mu\nu}(\not{p}_1+\not{p}_2-2m) -(p_1+p_2)_\mu\gamma_\nu-\gamma_\mu(p_1+p_2)_\nu]$$

$ffgg$ vertex:



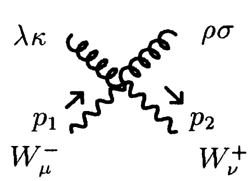
$$i\kappa^2[\mathrm{Sym}][\mathrm{Per}][\tfrac{3}{16}\eta_{\mu\lambda}\gamma_\kappa(\mathrm{p}_1+\mathrm{p}_2)_\nu +\tfrac{1}{16}\eta_{\mu\lambda}k_{2\nu}\gamma_\kappa+\tfrac{1}{32}\eta_{\mu\lambda}\not{k}_2(\gamma_\kappa\gamma_\nu-\gamma_\nu\gamma_\kappa)]$$

WWg vertex:



$$\begin{aligned}-\tfrac{i}{2}\kappa[&\eta_{\lambda\kappa}\eta_{\mu\nu}(p_1\cdot p_2-m^2)-\eta_{\lambda\kappa}p_{1\nu}p_{2\mu}\\ &+\eta_{\kappa\mu}p_{1\nu}p_{2\lambda}-\eta_{\mu\nu}p_{1\kappa}p_{2\lambda}+\eta_{\lambda\nu}p_{1\kappa}p_{2\mu}\\ &-\eta_{\kappa\mu}\eta_{\lambda\nu}(p_1\cdot p_2-m^2)+\eta_{\kappa\nu}p_{1\lambda}p_{2\mu}\\ &-\eta_{\mu\nu}p_{1\lambda}p_{2\kappa}+\eta_{\lambda\mu}p_{1\nu}p_{2\kappa}\\ &-\eta_{\kappa\nu}\eta_{\lambda\mu}(p_1\cdot p_2-m^2)]\end{aligned}$$

$WWgg$ vertex:

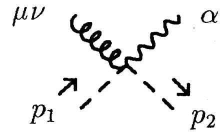


$$\begin{aligned}-\tfrac{i}{4}\kappa^2[&(\eta_{\lambda\kappa}\eta_{\rho\sigma}-2\eta_{\lambda\rho}\eta_{\kappa\sigma})(\eta_{\mu\nu}(p_1\cdot p_2-m^2)-p_{1\nu}p_{2\mu})\\ &-\eta_{\lambda\kappa}(T_{\mu\nu\rho\sigma}+T_{\mu\nu\sigma\rho})-\eta_{\rho\sigma}(T_{\mu\nu\lambda\kappa}+T_{\mu\nu\kappa\lambda})\\ &+2\eta_{\kappa\rho}(T_{\mu\nu\lambda\sigma}+T_{\mu\nu\sigma\lambda})+2\eta_{\lambda\sigma}(T_{\mu\nu\rho\kappa}+T_{\mu\nu\kappa\rho})\\ &+2(\eta_{\rho\mu}\eta_{\nu\sigma}p_{1\lambda}p_{2\kappa}-\eta_{\mu\lambda}\eta_{\nu\sigma}p_{1\rho}p_{2\kappa}-\eta_{\mu\rho}\eta_{\nu\kappa}p_{1\lambda}p_{2\sigma}\\ &\quad+\eta_{\mu\sigma}\eta_{\nu\rho}p_{1\kappa}p_{2\lambda}-\eta_{\mu\kappa}\eta_{\nu\rho}p_{1\sigma}p_{2\lambda}-\eta_{\mu\sigma}\eta_{\nu\lambda}p_{1\kappa}p_{2\rho}\\ &\quad+\eta_{\mu\lambda}\eta_{\nu\kappa}p_{1\rho}p_{2\sigma}+\eta_{\mu\kappa}\eta_{\nu\lambda}p_{1\sigma}p_{2\rho})]\end{aligned}$$

where

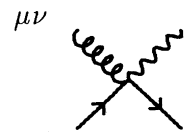
$$T_{\mu\nu\rho\sigma}=\eta_{\mu\nu}p_{1\rho}p_{2\sigma}-\eta_{\mu\rho}p_{1\nu}p_{2\sigma} -\eta_{\nu\sigma}p_{1\rho}p_{2\mu}+\eta_{\mu\rho}\eta_{\nu\sigma}(p_1\cdot p_2-m^2)$$

$ss\gamma g$ vertex:



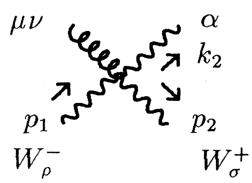
$$\tfrac{i}{2}e\kappa[\eta_{\mu\nu}(p_1+p_2)_\alpha - \eta_{\alpha\mu}(p_1+p_2)_\nu - \eta_{\alpha\nu}(p_1+p_2)_\mu]$$

$ff\gamma g$ vertex:



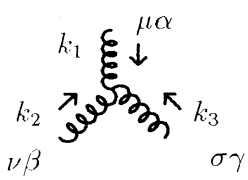
$$-\tfrac{i}{4}e\kappa[2\eta_{\mu\nu}\gamma_\alpha - \eta_{\alpha\mu}\gamma_\nu - \eta_{\alpha\nu}\gamma_\mu]$$

$WW\gamma g$ vertex:



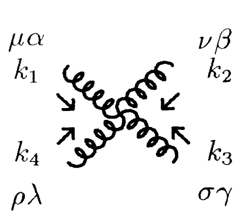
$$\begin{aligned}\tfrac{i}{2}e\kappa[&\eta_{\mu\nu}(\eta_{\rho\sigma}(p_1+p_2)_\alpha - \eta_{\alpha\sigma}p_{2\rho} - \eta_{\alpha\rho}p_{1\sigma})\\ &-\eta_{\rho\sigma}((p_1+p_2)_\mu\eta_{\nu\alpha} + (p_1+p_2)_\nu\eta_{\mu\alpha})\\ &-(p_1+p_2)_\alpha(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\nu\sigma}\eta_{\mu\rho})\\ &+\eta_{\alpha\rho}[(p_1+k_2)_\mu\eta_{\nu\sigma} + (p_1+k_2)_\nu\eta_{\mu\sigma}]\\ &+\eta_{\alpha\sigma}[(p_2-k_2)_\mu\eta_{\nu\rho} + (p_2-k_2)_\nu\eta_{\mu\rho}]\\ &+(p_1+k_2)_\sigma(\eta_{\mu\alpha}\eta_{\nu\rho} + \eta_{\nu\alpha}\eta_{\mu\rho})\\ &+(p_2-k_2)_\rho(\eta_{\mu\sigma}\eta_{\nu\alpha} + \eta_{\nu\sigma}\eta_{\mu\alpha})]\end{aligned}$$

ggg vertex:



$$\begin{aligned}&i\kappa\ [\mathrm{Sym}][\tfrac{1}{2}\mathrm{P}_6(\mathrm{k}_1\cdot\mathrm{k}_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma})\\ &-P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\\ &+\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_{1\sigma}k_{2\mu}\eta_{\alpha\nu}\eta_{\beta\gamma})\\ &-\tfrac{1}{4}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma})]\end{aligned}$$

$gggg$ vertex:



$$\begin{aligned}&i\kappa^2[\mathrm{Sym}][\tfrac{1}{4}\mathrm{P}_6(\mathrm{k}_1\cdot\mathrm{k}_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})\\ &+2P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\rho}\eta_{\beta\lambda}) - \tfrac{1}{2}P_6(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})\\ &-4P_{12}(k_{1\sigma}k_{2\mu}\eta_{\alpha\nu}\eta_{\beta\rho}\eta_{\gamma\lambda}) + \tfrac{1}{2}P_{24}(k_{1\sigma}k_{2\mu}\eta_{\alpha\nu}\eta_{\beta\gamma}\eta_{\rho\lambda})\\ &-P_{12}(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\sigma}\eta_{\beta\gamma}\eta_{\rho\lambda}) + 4P_6(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\sigma}\eta_{\beta\rho}\eta_{\gamma\lambda})\\ &-P_{12}(k_1\cdot k_2\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\nu\rho}\eta_{\beta\lambda}) + 2P_6(k_1\cdot k_2\eta_{\mu\sigma}\eta_{\alpha\rho}\eta_{\nu\gamma}\eta_{\beta\lambda})\\ &-\tfrac{1}{2}P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\rho\lambda}) + P_{12}(k_{1\sigma}k_{2\rho}\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\gamma\lambda})\\ &-2P_{12}(k_{1\sigma}k_{2\rho}\eta_{\mu\nu}\eta_{\alpha\gamma}\eta_{\beta\lambda}) + \tfrac{1}{4}P_{24}(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}\eta_{\rho\lambda})\\ &-\tfrac{1}{2}P_{24}(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\rho}\eta_{\gamma\lambda}) + \tfrac{1}{2}P_{24}(k_{1\beta}k_{2\sigma}\eta_{\mu\alpha}\eta_{\nu\rho}\eta_{\gamma\lambda})\\ &-P_6(k_{1\nu}k_{2\mu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda}) - \tfrac{1}{8}P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})\\ &+\tfrac{1}{2}P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})]\end{aligned}$$

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