# Astrophysical shock-wave solutions of the Einstein equations

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We construct exact, spherically symmetric, shock-wave solutions of the Einstein equations for a perfect fluid. The solutions are obtained by matching a Friedmann-Robertson-Walker metric (a cosmological model for the Universe) to a static Oppenheimer-Tolman metric (a model for the interior of a star) across a shock-wave interface. This is in the spirit of Oppenheimer and Snyder, except, in contrast with the Oppenheimer-Snyder model, the pressure $p$ is nonzero. Our shock-wave solutions model the general relativistic version of an explosion into a *static, singular, isothermal sphere.* Shock waves introduce time irreversibility, loss of information, decay, dissipation, and increase of entropy into the dynamics of a perfect fluid in general relativity. As a corollary, we also obtain a different Oppenheimer-Snyder model for the case $p \equiv 0$.

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## I. INTRODUCTION

In a recent paper [1], the authors constructed a class of exact, shock-wave solutions of the Einstein equations of general relativity. The paper [1] concluded with the derivation of a set of ordinary differential equations (ODE's) that describe the matching of a Friedmann-Robertson-Walker- (FRW cf. [2], p. 412) type metric to an Oppenheimer-Tolman- (OT-)type[1] metric, such that the interface between the two metrics defines a spherically symmetric, fluid dynamical shock wave. In this paper we give an explicit solution of these equations that models a shock wave exploding into the general relativistic version of a *static, singular, isothermal sphere.*

The FRW metric is a uniformly expanding (or contracting) solution of the Einstein gravitational field equations, which is generally accepted as a cosmological model for the universe. The OT solution is a time-independent solution, which models the interior of a star. Both metrics are spherically symmetric, and both are determined by a system of ODE's that close when an equation of state $p = p(\rho)$ for the fluid is specified. In our dynamically matched solution, we imagine the FRW metric as an exploding *inner* core (of a star or the Universe as a whole), and the boundary of this inner core is a shock surface that is driven by the expansion behind the shock into the *outer*, static, OT solution, which we imagine as the outer layers of a star, or the outer regions of the Universe. In the exact solution constructed here, the shock wave emerges from $\bar{r} = 0$ at the initial *(big bang)* singularity in the FRW metric, and thus, our model provides a scenario by which the *big bang begins with a shock-wave explosion.*

The outer static solution is called a *static isothermal sphere* because the metric entries are time independent, and the constant sound speed models a gas at constant temperature. It is *singular* because it has an inverse-square density profile, and thus the density and pressure tend to $\infty$ at the center of the sphere. The Newtonian version of a static singular isothermal sphere is well known and is important in theories of how stars form from gaseous clouds [4]. The idea in the Newtonian case goes as follows: a star begins as a diffuse cloud of gas, which slowly contracts under its own gravitational force by radiating energy out through the gas cloud as gravitational potential energy is converted into kinetic energy. This contraction continues until the gas cloud reaches the point where the mean free path for transmission of light is small enough that light is scattered, instead of transmitted, through the cloud. The scattering of light within the gas cloud has the effect of equalizing the temperature within the cloud. At this point the gas begins to drift toward the most compact configuration of the density that balances the pressure when the equation of state is isothermal; namely, it drifts toward the configuration of a static, singular, isothermal sphere. Since this solution in the Newtonian case is also an inverse square in density and pressure, the density tends to infinity at the center of the sphere, and this ignites thermonuclear reactions. The result is a shock-wave explosion emanating from the center of the sphere, and this signifies the birth of the star. The explicit solution, which we present here, is an exact, *general relativistic version* of such a shock-wave explosion.

[1]We choose this as a name for static, spherically symmetric metrics that solve the Einstein equations for a perfect fluid. It appears that OT solutions have not been given a name in the literature, and we consider this name to be appropriate, cf. [3]. In the special case when the density is constant, this metric is commonly referred to as the *interior Schwarzschild* metric.

 

Our explicit solution applies to a perfect fluid with an isothermal equation of state $p = \sigma\rho$ in the FRW solution, and $\bar{p} = \bar{\sigma}\bar{\rho}$ in the OT solution, where both the inner FRW sound speed $\sqrt{\sigma}$ and the outer OT sound speed $\sqrt{\bar{\sigma}}$ are assumed to be constant. Here $p$ denotes the fluid pressure and $\rho$ the mass-energy density, and we let the unbarred and barred variables refer to the standard coordinate systems for the FRW and OT metrics, respectively. We assume throughout that the speed of light $c = 1$. Our shock-wave solution is constructed from exact solutions of the FRW and OT metrics that exist for these special equations of state. In [1] we showed that, in general, the shock position $\bar{r} = \bar{r}(t)$ is given implicitly by the equation $M(\bar{r}) = (4\pi/3)\rho(t)\bar{r}^3$, where $M(\bar{r})$ denotes the total OT mass inside radius $\bar{r}$, and $\rho(t)$ is the FRW density at the shock. For the exact solution with constant sound speeds constructed here, the shock surface condition implies that $\rho = 3\bar{\rho}$ across the shock. Moreover, in order that conservation of energy and momentum hold across the shock, we show that the sound speeds must be related by an algebraic equation of the form $\bar{\sigma} = H(\sigma)$, where $H'(\sigma) > 0$, $H(0) = 0$, and $H(\sigma) < \sigma$, cf. Fig. 1. Since, at the shock, the inner FRW sound speed and density exceed the outer OT sound speed and density, respectively, we conclude that the outgoing shock wave is the stable one, and the larger sound speed in the FRW metric is interpreted as modeling an isothermal equation of state at a higher temperature (consistent with the heating of the fluid by the shock wave). In the limit $\sigma \to 0$, our model recovers the Newtonian limit of low velocities and weak gravitational fields.

We verify that there exist two distinguished values of $\sigma$, $\sigma_1 \approx 0.458 < \sigma_2 = \sqrt{5}/3 \approx 0.745$, such that, if $0 < \sigma < 1$, then the Lax characteristic condition (that characteristics impinge on the shock [5]), is satisfied if and only if $0 < \sigma < \sigma_1$, and the shock speed is less than the speed of light if and only if $0 < \sigma < \sigma_2$. A calculation gives $\bar{\sigma}_1 \equiv H(\sigma_1) \approx 0.161$, and $\bar{\sigma}_2 \equiv H(\sigma_2) \approx 0.236$. We conclude that, for $\sigma$ between $\sigma_1$ and $\sigma_2$, a different type of shock wave appears in which the shock is supersonic relative to the fluid on both sides of the shock. Thus, in this theory, a fluid with a sound speed no larger than $\sqrt{\sigma_2} \approx \sqrt{0.745}$ can drive shock waves with speeds all the way up to the speed of light. The time reversal and stability properties of these shocks when $\sigma_1 < \sigma < \sigma_2$ remains to be investigated.

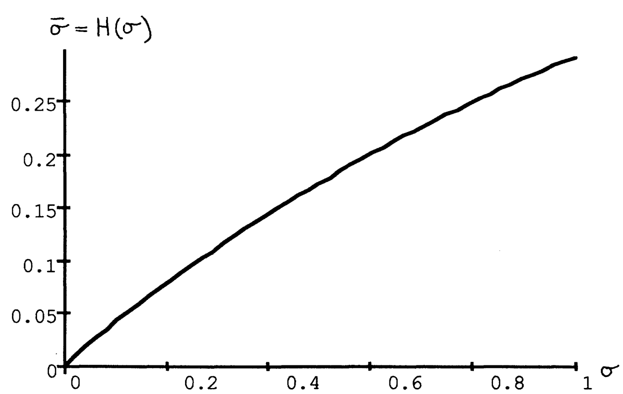


FIG. 1. A plot of $\bar{\sigma}$ vs $\sigma$.

Since Lax-type shock waves are time-irreversible solutions of the equations because of the increase of entropy (in a generalized sense, cf. [6]) and consequent loss of information (effected by the impinging of characteristics on the shock), we infer from the mathematical theory of shock waves that when $0 < \sigma < \sigma_1$, many solutions must decay time asymptotically to the same shock wave. Thus, in contrast to the pure FRW solution, in our model we should not expect a unique time reversal of the solution all the way back to the initial big bang singularity when the sound speeds lie in the range $0 < \sigma < \sigma_1$.

We note that the OT solution when $\bar{p} = \bar{\sigma}\bar{\rho}$ is, by itself, of limited physical value because $\bar{p} = \infty$ at $\bar{r} = 0$. We interpret this as saying that this exact solution is unstable because it requires an infinite pressure at $\bar{r} = 0$ to "hold it up." In contrast, our shock-wave solution here removes the singularity at $\bar{r} = 0$ (for times after some initial time) and thus demonstrates that a shock wave in the core can supply the pressure required to *stabilize* an OT solution by *holding it up.*

As a final comment, we note that in Sec. VII we also construct a different Oppenheimer-Snyder-type solution [7] having $p \equiv 0$ that models gravitational collapse to a black hole. This model is based on Friedmann-Robertson-Walker metrics that are flat at each time $t$.

## II. PRELIMINARIES

We consider the Einstein gravitational field equations

$$G = \kappa T \ , \tag{2.1}$$

where $G$ denotes the Einstein curvature tensor for the spacetime metric $g$, $T$ denotes the stress-energy tensor for a perfect fluid,

$$T = (\rho + p)u \otimes u + pg \ , \tag{2.2}$$

and $\kappa = 8\pi\mathcal{G}$. (We assume the speed of light $c = 1$.) Here $u$ is the four-velocity of the fluid, $\mathcal{G}$ is Newton's gravitational constant, and we assume a baryotropic equation of state of the form $p = p(\rho)$. In a given coordinate system, $T$ takes the form

$$T_{ij} = pg_{ij} + (p + \rho)u_i u_j \ , \tag{2.3}$$

where $i, j$ are assumed to run from 0 to 3, and we use the Einstein summation convention throughout. The Einstein tensor $G$ is constructed from the Riemann curvature tensor so as to satisfy $\text{div}G = 0$. Thus, on solutions of (2.1), $\text{div}T = 0$, and this is the relativistic version of the classical Euler equations for compressible fluid flow. The compressible Euler equations provide the setting for the mathematical theory of shock waves. We now briefly recall the FRW and OT metrics, and the results of [1].

The FRW metric describes a spherically symmetric spacetime that is homogeneous and maximally symmetric at each fixed time. In coordinates, the FRW metric is given by [2]

$$ds^2 = -dt^2 + R^2(t)\left\{\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right\} , \quad (2.4)$$

where $t \equiv x^0$, $r \equiv x^1$, $\theta \equiv x^2$, $\varphi \equiv x^3$, $R \equiv R(t)$ is the "cosmological scale factor," and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ denotes the standard metric on the unit 2 sphere. The constant $k$ can be chosen to be $+1$, $-1$, or 0 by appropriately rescaling the radial variable, and each of the three cases is qualitatively different. We assume that the fluid is perfect, and that the fluid is comoving with the metric. The fluid is said to be comoving relative to a background diagonal metric $g_{ij}$ if $u^i = 0$, $i = 1, 2, 3$, so that $g$ diagonal and $u$ having length one imply [8]

$$u^0 = \sqrt{-g_{00}} . \quad (2.5)$$

Substituting (2.4) into the field equations, and making the assumption that the fluid is perfect and comoving with the metric, yields the following constraints on the unknown functions $R(t)$, $\rho(t)$, and $p(t)$ [2,1]:

$$3\ddot{R} = -4\pi\mathcal{G}(\rho + 3p)R , \quad (2.6)$$

$$R\ddot{R} + 2\dot{R}^2 + 2k = 4\pi\mathcal{G}(\rho - p)R^2 , \quad (2.7)$$

together with

$$\dot{p}R^3 = \frac{d}{dt}\{R^3(p+\rho)\} . \quad (2.8)$$

Equation (2.8) is equivalent to

$$p = -\rho - \frac{R\dot{\rho}}{3\dot{R}} . \quad (2.9)$$

Substituting (2.6) into (2.7), we get

$$\dot{R}^2 + k = \frac{8\pi\mathcal{G}}{3}\rho R^2 . \quad (2.10)$$

Since $\rho$ and $p$ are assumed to be functions of $t$ alone in (2.4), Eqs. (2.9) and (2.10) give two equations for the two unknowns $R$ and $\rho$ under the assumption that the equation of state is of the form $p = p(\rho)$. It follows from (2.9) and (2.10) that $(R(t), \rho(t))$ is a solution if and only if $(R(-t), \rho(-t))$ is a solution. Moreover, it follows that [1]

$$\dot{\rho}\dot{R} < 0 . \quad (2.11)$$

Thus, to every expanding solution there exists a corresponding contracting solution, and vice versa.

The OT metric describes a time-independent, spherically symmetric solution that models the interior of a star. In coordinates the components of the metric are given by

$$d\bar{s}^2 = -B(\bar{r})d\bar{t}^2 + A(\bar{r})^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2 . \quad (2.12)$$

We write this metric in bar coordinates so that it can be distinguished from the unbarred coordinates when we do the matching of metrics below. Assuming the stress tensor is that of a perfect fluid, which is comoving with the metric, and substituting (2.12) into the field equations (2.1), yields (cf. [2])

$$A(\bar{r}) = \left(1 - \frac{2\mathcal{G}M}{\bar{r}}\right) , \quad (2.13)$$

where $M \equiv M(\bar{r})$, $\bar{\rho} \equiv \bar{\rho}(\bar{r})$, and $\bar{p} \equiv \bar{p}(\bar{r})$ satisfy the following system of ordinary differential equations in the unknown functions $(\bar{\rho}(\bar{r}), p(\bar{r}), M(\bar{r}))$:

$$\frac{dM}{d\bar{r}} = 4\pi\bar{r}^2\bar{\rho} , \quad (2.14)$$

$$-\bar{r}^2\frac{d}{d\bar{r}}\bar{p} = \mathcal{G}M\bar{\rho}\left\{1 + \frac{\bar{p}}{\bar{\rho}}\right\}\left\{1 + \frac{4\pi\bar{r}^3\bar{p}}{M}\right\}\left\{1 - \frac{2\mathcal{G}M}{\bar{r}}\right\}^{-1} . \quad (2.15)$$

Equation (2.15) is called the Oppenheimer-Volkov equation, and is referred to by Weinberg as "the fundamental equation of Newtonian astrophysics, with general-relativistic corrections supplied by the last three factors" ([2], p. 301). Assuming the equation of state $\bar{p} = \bar{p}(\bar{\rho})$, Eqs. (2.14) and (2.15) yield a system of two ODE's in the two unknowns $(\bar{\rho}, M)$. The total mass $M$ inside radius $\bar{r}$ is then defined by

$$M(\bar{r}) = \int_0^{\bar{r}} 4\pi\xi^2\bar{\rho}(\xi)d\xi . \quad (2.16)$$

The metric component $B \equiv B(\bar{r})$ is determined from $\bar{\rho}$ and $M$ through the equation

$$\frac{B'(\bar{r})}{B} = -2\frac{\bar{p}'(\bar{r})}{\bar{p} + \bar{\rho}} . \quad (2.17)$$

In [1], we described a procedure for constructing a coordinate transformation $(\bar{t}, \bar{r}) \to (t, r)$ such that the FRW metric (2.4) matches the OT metric (2.12) Lipschitz continuously across a shock surface $\Sigma$. This shock surface is given implicitly by the equation

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\bar{r}^3 . \quad (2.18)$$

Equation (2.18) defines the radial coordinate $\bar{r}$ of the OT metric as a function of the time coordinate $t$ of the FRW metric along the shock surface $\Sigma$, and this applies when any equation of state $p = p(\rho)$ is assigned to the FRW metric and any equation of state $\bar{p} = \bar{p}(\bar{\rho})$ is assigned to the OT metric. The transformation $\bar{r} = \bar{r}(t, r)$ is given by

$$\bar{r} = R(t)r , \quad (2.19)$$

in the mapping $(\bar{t}, \bar{r}) \to (t, r)$, but the transformation $\bar{t} = \bar{t}(t, r)$ is more complicated, and its existence is demonstrated in [1]; we will not need any explicit information about the $\bar{t}$ transformation here. The identification (2.19) (together with the implicit identification of the angular coordinates $\theta$ and $\phi$), guarantees that the areas of the spheres of symmetry agree under this coordinate identification. Since, in our construction in [1], the FRW metric matches the OT metric Lipschitz continuously across the shock (2.18), the following general theorem, which is proved in [1], theorem 4, applies (see also [9]):

*Theorem 1.* Let $\Sigma$ denote a smooth, three-dimensional shock surface in spacetime with spacelike normal vector $\mathbf{n}$. Assume that the components $g_{ij}$ of the gravitational metric $g$ are smooth on either side of $\Sigma$ (continuous up to the boundary on either side separately), and Lipschitz continuous across $\Sigma$ in some fixed coordinate system. Then the following statements are equivalent.

(i) $[K] = 0$ at each point of $\Sigma$. (Here $[f]$ denotes the jump in the quantity $f$ across the surface $\Sigma$, and $K$ denotes the extrinsic curvature, or second fundamental form, which is determined separately on each side of the shock surface $\Sigma$ by the metric $g$.)

(ii) The curvature tensors $R^i_{jkl}$ and $G_{ij}$, viewed as second-order operators on the metric components $g_{ij}$, produce no $\delta$-function sources on $\Sigma$.

(iii) For each point $P \in \Sigma$ there exists a $C^{1,1}$ coordinate transformation defined in a neighbor of $P$, such that, in the new coordinates (which can be taken to be the Gaussian normal coordinates for the surface), the metric components are $C^{1,1}$ functions of these coordinates. (By $C^{1,1}$ we mean that the first derivatives are Lipschitz continuous.)

(iv) For each $P \in \Sigma$, there exists a coordinate frame that is locally Lorentzian at $P$, and can be reached from the original coordinates by a $C^{1,1}$ coordinate transformation. [A coordinate frame is locally Lorentzian at a point $P$ if $g_{ij}(P) = \mathrm{diag}(-1,1,1,1)$ and $g_{ij,k}(P) = 0$ for all $i,j,k = 0,\ldots,3$.]

Moreover, if any one of these equivalencies hold, then the Rankine-Hugoniot jump conditions, $[G^{ij}]n_i = 0$, hold at each point at $\Sigma$. (This expresses the weak form of conservation of energy and momentum across $\Sigma$ when $G = \kappa T$.)

In the case of spherical symmetry, the conservation condition $[G^{ij}]n_i = 0$ reduces to the single condition $[G^{ij}]n_i n_j = 0$, and this implies the equivalencies in theorem 1. In fact, we have the following ( [1], proposition 9).

*Theorem 2.* Assume that $g$ and $\bar{g}$ are two spherically symmetric metrics that match Lipschitz continuously across a three-dimensional shock interface $\Sigma$ to form the matched metric $g \cup \bar{g}$. That is, assume that $g$ and $\bar{g}$ are Lorentzian metrics given by

$$ds^2 = -a(t,r)dt^2 + b(t,r)dr^2 + c(t,r)d\Omega^2$$

and

$$d\bar{s}^2 = -\bar{a}(\bar{t},\bar{r})d\bar{t}^2 + \bar{b}(\bar{t},\bar{r})d\bar{r}^2 + \bar{c}(\bar{t},\bar{r})d\Omega^2 \ ,$$

and that there exists a smooth coordinate transformation $\Psi$: $(t,r) \to (\bar{t},\bar{r})$, defined in a neighborhood of the shock surface $\Sigma$ given by $r = r(t)$, such that the metrics agree on $\Sigma$. (We implicitly assume that $\theta$ and $\varphi$ are continuous across the surface.) Assume that

$$c(t,r) = \bar{c}(\Psi(t,r)) \ ,$$

in an open neighborhood of the shock surface $\Sigma$, so that, in particular, the areas of the two-spheres of symmetry in the barred and unbarred metrics agree on the shock surface. Assume also that the shock surface $r = r(t)$ in unbarred coordinates is mapped to the surface $\bar{r} = \bar{r}(\bar{t})$ by $(\bar{t},\bar{r}(\bar{t})) = \Psi(t,r(t))$. Assume, finally, that the normal $\mathbf{n}$ to $\Sigma$ is non-null, and that $\mathbf{n}(c) \neq 0$, where $\mathbf{n}(c)$ denotes the derivative of the function $c$ in the direction of the vector $\mathbf{n}$. Then the following are equivalent to the statement that the components of the metric $g \cup \bar{g}$ in any Gaussian-normal coordinate system are $C^{1,1}$ functions of these coordinates across the surface $\Sigma$:

$$[G^i_j]n_i = 0 \ , \tag{2.20}$$

$$[G^{ij}]n_i n_j = 0 \ , \tag{2.21}$$

$$[K] = 0 \ . \tag{2.22}$$

Here, $[f] = \bar{f} - f$ denotes the jump in the quantity $f$ across $\Sigma$, and $K$ denotes the second fundamental form on the shock interface.

It is straightforward to check that the conditions in the above theorem on the functions $c$ and $\bar{c}$ are met when $\bar{c} = \bar{r}$, $c = Rr$, and $\bar{r}(t,r) = R(t)r$. In light of (2.20) and (2.21), we conclude that conservation across the shock surface (2.18) is equivalent to the condition that the equation $[T^{ij}]n_i n_j = 0$ holds across $\Sigma$. In [1] we derived the identity

$$[T^{ij}]n_i n_j = (\rho + p)n_0^2 - (\bar{\rho} + \bar{p})\frac{\bar{n}_0^2}{B} + (p - \bar{p})|\mathbf{n}|^2 \ . \tag{2.23}$$

Here $n^i$ and $\bar{n}^i$ denote the components of the normal vector $\mathbf{n}$ to $\Sigma$ in the $(t,r)$ and $(\bar{t},\bar{r})$ coordinate systems, respectively. Equation (2.23) represents the additional constraint imposed by conservation across the shock surface (2.18). Using the expressions for the components $n_i$ and $\bar{n}_i$ of $\mathbf{n}$, we readily obtain the equivalent expression (see Eq. (5.34) of [1])

$$[T^{ij}]n_i n_j = (\bar{p} + \rho)\dot{r}^2 - (\bar{\rho} + \bar{p})\frac{1 - kr^2}{AR^2}\dot{\bar{r}}^2 + (p - \bar{p})\frac{1 - kr^2}{R^2} = 0 \ . \tag{2.24}$$

Here, $\dot{r}$ and $\dot{\bar{r}}$ denote the shock speeds $dr/dt$ and $d\bar{r}/dt$, respectively. In [1], we used Eq. (2.9) to eliminate $p$ from (2.24), and thereby derived an autonomous system of ODE's in $(R,r)$ as a function of $t$ that determines the inner FRW metric and the shock position $r(t)$ in terms of any given OT metric. (cf. (5.46)–(5.49) of [1].) Thus, for any assignment of equation of state $\bar{p} = \bar{p}(\bar{\rho})$ and initial conditions for an OT metric, our system of ODE's determines the FRW metrics, $R(t)$, $\rho(t)$, and $p(t)$, that match the given OT metric Lipschitz continuously across the shock surface (2.18), such that conservation holds across the surface.

In this paper, we proceed somewhat differently. Here we will solve our differential equations by working with an equivalent form of (2.24) which we now derive. Thus, differentiating (2.18) with respect to $t$ and applying (2.14) yields

$$\dot{\rho} = \frac{3}{\bar{r}}(\bar{\rho} - \rho)\dot{\bar{r}} \ . \tag{2.25}$$

Solving for $\dot{\rho}$ in (2.9) yields

$$\dot{\rho} = -\frac{3\dot{R}}{R}(\rho + p) \ . \tag{2.26}$$

Combining (2.25) and (2.26) thus gives

$$\dot{\bar{r}} = \dot{R}r\frac{\rho + p}{\rho - \bar{\rho}} \ . \tag{2.27}$$

Differentiating $\bar{r} = Rr$ with respect to $t$ in (2.27) and solving for $\dot{r}$ gives

$$\dot{r} = \frac{\dot{R}r}{R}\frac{\bar{\rho} + p}{\rho - \bar{\rho}} \ . \tag{2.28}$$

Substituting (2.27) and (2.28) into (2.24), we obtain the following equation, which is equivalent to the conservation condition $[T^{ij}]n_i n_j = 0$:

$$0 = \left(\frac{1}{1 - kr^2}\right)(\rho + \bar{p})(p + \bar{\rho})^2 - \frac{1}{A}(\bar{\rho} + \bar{p})(\rho + p)^2 + \frac{1}{r^2\dot{R}^2}(p - \bar{p})(\rho - \bar{\rho})^2 \ . \tag{2.29}$$

Equation (2.29) expresses conservation at the shock surface (2.18). But from Eq. (4.54) of [1], we know that the identity

$$\dot{R}^2 r^2 = -A + (1 - kr^2) \tag{2.30}$$

holds on the shock surface, and thus we can transform (2.29) into the final form,

$$0 = (1 - \Theta)(\rho + \bar{p})(p + \bar{\rho})^2 + \left(1 - \frac{1}{\Theta}\right)(\bar{\rho} + \bar{p})(\rho + p)^2 + (p - \bar{p})(\rho - \bar{\rho})^2 \ , \tag{2.31}$$

where

$$\Theta = \frac{A}{1 - kr^2} \ . \tag{2.32}$$

It is this functional form of (2.24) that we shall analyze.

## III. AN EXACT SOLUTION OF THE OT TYPE

We now construct exact solutions of the OT type, which represent the general relativistic version of static, singular isothermal spheres. First assume the equation of state,

$$\bar{p} = \bar{\sigma}\bar{\rho} \ , \tag{3.1}$$

for the OT metric, and assume that the density is of form

$$\bar{\rho}(\bar{r}) = \frac{\gamma}{\bar{r}^2} \ , \tag{3.2}$$

for some constant $\gamma$. In this case, an exact solution of the OT type was first found by Tolman,[2] [3]; namely, by (2.16),

$$M(\bar{r}) = 4\pi\gamma\bar{r} \ . \tag{3.3}$$

Putting (3.1)–(3.3) into (2.15) and simplifying, yields the identity

$$\gamma = \frac{1}{2\pi\mathcal{G}}\left(\frac{\bar{\sigma}}{1 + 6\bar{\sigma} + \bar{\sigma}^2}\right) \ . \tag{3.4}$$

From (2.13), we obtain

$$A = 1 - 8\pi\mathcal{G}\gamma \ . \tag{3.5}$$

To solve for $B$, start with (2.17) and write

$$\frac{1}{B}\frac{dB}{d\rho}\frac{d\rho}{d\bar{r}} = -\frac{2\bar{\sigma}}{(1 + \bar{\sigma})\bar{\rho}}\frac{d\bar{\rho}}{d\bar{r}} \ , \tag{3.6}$$

which simplifies to

$$\frac{dB}{B} = -\frac{2\bar{\sigma}}{(1 + \bar{\sigma})}\frac{d\bar{\rho}}{\bar{\rho}} \ . \tag{3.7}$$

This equation has the explicit solution

$$B = B_0\left(\frac{\bar{\rho}}{\bar{\rho}_0}\right)^{-2\bar{\sigma}/(1+\bar{\sigma})} = B_0\left(\frac{\bar{r}}{\bar{r}_0}\right)^{4\bar{\sigma}/(1+\bar{\sigma})} \ . \tag{3.8}$$

By rescaling the time coordinate, we can take $B_0 = 1$ at $\bar{r}_0 = 1$, in which case (3.8) reduces to

$$B = \bar{r}^{4\bar{\sigma}/(1+\bar{\sigma})} \ . \tag{3.9}$$

We conclude that when (3.4) holds, (3.1)–(3.5) and (3.8) provide an exact solution of the Einstein field equations (2.1) of OT type. Note that, since $\sqrt{\bar{\sigma}}$ is the sound speed of the fluid, (3.1)–(3.3) provide exact solutions for any sound speed $0 \leq \bar{\sigma} \leq 1$. Note also that when $\bar{\sigma} = \frac{1}{3}$, the extreme relativistic limit for free particles [2], (3.4) yields $\gamma = 3/56\pi\mathcal{G}$ [cf., [2], Eq. (11.4.13)]. These exact solutions by themselves are not so interesting physically because the density and pressure are infinite at $\bar{r} = 0$ at every value of time. Our shock-wave construction, given below, removes the singularity at $\bar{r} = 0$ in these solutions, after some initial time.

## IV. AN EXACT SOLUTION OF THE FRW TYPE

We now construct exact solutions of the FRW type. We restrict ourselves to the case $k = 0$ in (2.4), so that the metric takes the simple (conformally flat) form

[2] In the case $\bar{\sigma} = \frac{1}{3}$, this solution was rediscovered by Misner and Zapolsky, cf. [2], p. 320.

$$ds^2 = -dt^2 + R^2(t)\{dr^2 + r^2 d\Omega^2\} . \tag{4.1}$$

Now assume an arbitrary equation of state of the form $p = p(\rho)$. We will obtain a closed form solution of the Einstein equations (2.1) in this case. By (2.9) and (2.10), it suffices to solve the system of two ODE's,

$$\dot{R}^2 = \frac{8\pi\mathcal{G}}{3}\rho R^2 \tag{4.2}$$

and

$$p(\rho) = -\rho - \frac{R\dot{\rho}}{3\dot{R}} . \tag{4.3}$$

Rewrite (4.2) as

$$\dot{R} = \pm\left(\frac{8\pi\mathcal{G}\rho}{3}\right)^{1/2} R , \tag{4.4}$$

and substitute into (4.3) to obtain

$$p = -\rho \mp \frac{\dot{\rho}}{\sqrt{24\pi\mathcal{G}\rho}} . \tag{4.5}$$

[The upper and lower plus or minus signs will always correspond to the two cases represented by the upper and lower plus or minus sign in (4.4), respectively.] The point to be noted here is that when $p = p(\rho)$ is assigned, (4.5) is independent of $R$, and thus we can integrate it explicitly; namely, since

$$dt = \mp\frac{d\rho}{(\rho + p)\sqrt{24\pi\mathcal{G}\rho}} , \tag{4.6}$$

we obtain

$$t - t_0 = \mp\int_{\rho_0}^{\rho} \frac{d\xi}{[\xi + p(\xi)]\sqrt{24\pi\mathcal{G}\xi}} . \tag{4.7}$$

Formula (4.7) gives $t$ as a function of $\rho$, and we can use this, together with (4.2), to obtain a closed form expression for $R$ as a function of $\rho$. Thus, since

$$\dot{R} = \frac{d\rho}{dt}\frac{dR}{d\rho} = \mp(\rho + p)\sqrt{24\pi\mathcal{G}\rho}\frac{dR}{d\rho} , \tag{4.8}$$

if we combine this with (4.2), we get

$$\frac{dR}{R} = \frac{-d\rho}{3(\rho + p)} , \tag{4.9}$$

which has the explicit solution

$$R = R_0 \exp\left(\int_{\rho_0}^{\rho} \frac{-1}{3[\xi + p(\xi)]}d\xi\right) . \tag{4.10}$$

## V. AN EXPLICIT SHOCK-WAVE SOLUTION OF THE EINSTEIN EQUATIONS

We now use the theory developed in [1] to match the above OT- and FRW-type metrics at a spherical interface across which the metrics join Lipschitz continuously, and such that the conservation constraint (2.31) holds at the interface. The resulting solution is interpreted as a fluid dynamical shock wave in which the increase of entropy in the fluids drives a time-irreversible gravitational wave.

Assume now that the equation of state for the OT metric is taken to be

$$\bar{p} = \bar{\sigma}\bar{\rho}$$

for some constant $\bar{\sigma}$, and that the fixed OT solution is given by (3.2)–(3.5) and (3.8). Then, given an arbitrary FRW metric, our results in [1] imply that we can construct a coordinate mapping $(\bar{t}, \bar{r}) \to (t, r)$ such that the FRW metric matches the OT metric Lipschitz continuously across the shock surface (2.18). This applies, in principle, to *any* equation of state $p = p(\rho)$ chosen for the FRW metric. Using (3.3) and solving for $\rho$ gives $\rho$ on the shock surface $\bar{r}(t) = r(t)R(t)$:

$$\rho = \frac{3}{4\pi}\frac{M}{\bar{r}(t)^3} = \frac{3\gamma}{\bar{r}(t)^2} = 3\bar{\rho} . \tag{5.1}$$

To meet the additional conservation condition, we restrict to FRW metrics with $k = 0$, and we use (2.31) to determine the pressure. Substituting $\Theta = A = 1 - 8\pi\mathcal{G}\gamma \equiv$ const into (2.31), we see that the resulting equation is homogeneous of degree three in the $\rho, \bar{\rho}$ and $p, \bar{p}$ variables. Since $\bar{p} = \bar{\sigma}\bar{\rho}$, and

$$\rho = 3\bar{\rho}$$

on the shock surface, it is clear from homogeneity that (2.31) can be met if and only if $p = \sigma\rho$ for some constant $\sigma$. Substituting this into (2.31) gives the following relation between $\sigma$ and $\bar{\sigma}$ (cf. Fig. 1):

$$\bar{\sigma} = \tfrac{1}{2}\sqrt{9\sigma^2 + 54\sigma + 49} - \tfrac{3}{2}\sigma - \tfrac{7}{2} \equiv H(\sigma) . \tag{5.2}$$

Alternatively, we can solve for $\sigma$ in (5.2) and write this relation as

$$\sigma = \frac{\bar{\sigma}(\bar{\sigma} + 7)}{3(1 - \bar{\sigma})} .$$

This guarantees that conservation holds across the shock surface, and thus by theorem 2, the results of theorem 1 apply. Note that $H(0) = 0$, and, to leading order,

$$\bar{\sigma} = H(\sigma) = \tfrac{3}{7}\sigma + O(\sigma^2) , \tag{5.3}$$

as $\sigma \to 0$. It is easy to verify that within the physical region $0 \le \sigma, \bar{\sigma} \le 1$, $H'(\sigma) > 0$, and $\bar{\sigma} < \sigma$, as would be expected physically because $\rho = 3\bar{\rho} > \bar{\rho}$ at the shock surface. One can verify that when $\sigma = \frac{1}{3}$, we have

$$\bar{\sigma} = \sqrt{17} - 4 = 0.1231\ldots ,$$

and when $\sigma = 1$, we have

$$\bar{\sigma} = \frac{\sqrt{112}}{2} - 5 = 0.2915\ldots .$$

We now obtain formulas for $\rho(t)$, $R(t)$, and the shock positions $r(t)$ and $\bar{r}(t) = r(t)R(t)$. Substituting $p = \sigma\rho$ into (4.6) and (4.9) yields

$$dt = \mp\frac{1}{\sqrt{24\pi\mathcal{G}(1+\sigma)}}\rho^{-3/2}d\rho \tag{5.4}$$

and

$$\frac{dR}{R} = -\frac{1}{3(1+\sigma)}\frac{d\rho}{\rho} . \tag{5.5}$$

Using (5.1) we obtain

$$\rho^{-3/2}d\rho = -\frac{2}{\sqrt{3\gamma}}d\bar{r} . \tag{5.6}$$

Putting this into (5.4) gives

$$dt = \mp\frac{1}{(1+\sigma)}\frac{1}{\sqrt{18\pi\mathcal{G}\gamma}}d\bar{r} . \tag{5.7}$$

Integrating Eq. (5.7) gives the formula for the shock position:

$$\bar{r}(t) = \pm\sqrt{18\pi\mathcal{G}\gamma}(1+\sigma)(t-t_0) + \bar{r}_0 . \tag{5.8}$$

Thus, (5.1) gives $\rho$ in terms of $t$:

$$\rho(t) = \frac{3\gamma}{\bar{r}(t)^2} = \frac{3\gamma}{[\pm\sqrt{18\pi\mathcal{G}\gamma}(1+\sigma)(t-t_0)+\bar{r}_0]^2} . \tag{5.9}$$

Finally, we can use (5.5) to obtain $R(t)$, and the shock position $r(t) = \bar{r}(t)R(t)^{-1}$:

$$R(t) = R_0\left(\frac{\rho}{\rho_0}\right)^{-1/3(1+\sigma)} = R_0\left(\frac{\bar{r}(t)}{\bar{r}_0}\right)^{2/3(1+\sigma)} , \tag{5.10}$$

$$r(t) = \bar{r}(t)R(t)^{-1} = \bar{r}(t)R_0^{-1}\left(\frac{\bar{r}(t)}{\bar{r}_0}\right)^{-2/3(1+\sigma)}$$
$$= \bar{r}_0R_0^{-1}\left(\frac{\bar{r}(t)}{\bar{r}_0}\right)^{(1+3\sigma)/(3+3\sigma)} . \tag{5.11}$$

Differentiating (5.8) and (5.11) gives the speeds of the shock $\dot{\bar{r}}$ and $\dot{r}$ in the $(t,\bar{r})$- and $(t,r)$-coordinate systems, respectively:

$$\dot{\bar{r}} = 3(1+\sigma)\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right)^{1/2} , \tag{5.12}$$

$$\dot{r} = \frac{1+3\sigma}{R(t)}\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right)^{1/2} , \tag{5.13}$$

where again, $\bar{\sigma} = H(\sigma)$ is given in (5.2).

Note that the solution (5.8)–(5.11) contains two arbitrary constants $\bar{r}_0, R_0$ or $r_0, R_0$, as it should from the initial value problem (4.4) and (4.5). Note also that for an outgoing shock wave we choose the plus sign in (4.4) and (5.8), and in this case there is a singularity in backward time:

$$t_* = t_0 - \frac{\bar{r}_0}{\sqrt{18\pi\mathcal{G}\gamma}(1+\sigma)} . \tag{5.14}$$

As $t \to t_*$, it is clear that $\bar{r} \to 0, \rho, \bar{\rho}, p, \bar{p}$ all tend to infinity, and $R, r$ tend to zero. If we take this as a cosmological model, then $t = t_*$ represents the initial *big bang* singularity in which a shock wave emerges from $\bar{r} = 0$.

We summarize these results in the following theorem.

*Theorem 3.* Assume an equation of state of the form $\bar{p} = \bar{\sigma}\bar{\rho}$ for the OT metric, and $p = \sigma\rho$ for the FRW metric, assume (5.2) holds, and take $k = 0$. Then the OT solution given by (3.2), (3.3), (3.5), and (3.8), will match the FRW solution given by (5.9) and (5.10), across the shock surface (5.8), such that conservation of energy and momentum hold across the surface. The coordinate identification $(t,r) \to (\bar{t},\bar{r})$ is given by $\bar{r} = Rr$, together with a smooth function $\bar{t} = \bar{t}(t,r)$ whose existence (in a neighborhood of the shock surface) is demonstrated in [1].

By theorem 2, all of the equivalencies in theorem 1 hold across the shock surface. In the next section we show that the shock speeds are less than the speed of light, and we determine when the Lax characteristic conditions hold.

## VI. THE LAX SHOCK CONDITIONS

To complete the analysis of our shock-wave solution discussed in the last section, it remains to analyze the shock speed and characteristic speeds on both sides of the shock. In classical gas dynamics, characteristics (in the same family of a shock) impinge on the shock from both sides, leading to an increase of entropy and consequent loss of information. This is also the source of the well-known time irreversibility, as well as the stability, of gas dynamical shock waves. This interpretation carries over to a general system of hyperbolic conservation laws. Indeed, this characteristic condition has been proposed by Lax [5,6], as a stability criterion for shock waves in settings other than gas dynamics. This "Lax characteristic condition" can be easily applied in general systems, where either a physical entropy is difficult to work with or has not been identified [6]. Since in gas dynamics the density and pressure are always larger behind (stable) shock waves, and in our example $\rho = 3\bar{\rho}$ [cf. (5.1)], we restrict our attention to the case of an outgoing shock wave in which the FRW metric is on the inside and the OT metric is on the outside of the shock. This is equivalent to taking the plus sign in (4.4) [and the corresponding upper sign in Eqs. (4.5)–(4.7)], and we therefore restrict our attention to this case.

The goal of this section is to show that, in this case, there exist values $0 < \sigma_1 < \sigma_2 < 1$ ($\sigma_1 \approx 0.458$, $\sigma_2 = \sqrt{5}/3 \approx 0.745$), such that, for $0 < \sigma < 1$, the Lax characteristic condition holds at the shock if and only if $0 < \sigma < \sigma_1$; and the shock speed is less than the speed of light if and only if $0 < \sigma < \sigma_2$. We conclude that our gravitational shock wave represents a new type of fluid dynamical shock wave when $\sigma_1 < \sigma < \sigma_2$. For the outgoing shock waves with $\sigma$ in this interval, the shock speed exceeds all of the characteristic speeds on either side of the shock because both the fast and slow characteristics cross the shock wave from the OT side to the FRW side of the shock. Our first result is the following lemma.

*Lemma 1.* For $0 < \sigma < 1$, the shock speed, relative to the FRW fluid particles, is given by

$$s = (1+3\sigma)\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right)^{1/2} \equiv s(\sigma) \ , \tag{6.1}$$

where $s(\sigma)$ is the function of $\sigma$ obtained by substituting (5.2) for $\bar{\sigma}$ in (6.1).

The function $s(\sigma)$ is plotted in Fig. 2. By numerical calculation we obtain that $1 - s(\sigma)$ is monotone for $0 < \sigma < 1$ and becomes negative above $\sigma = \sigma_2$, where, using computer algebra, we find

$$\sigma_2 = \sqrt{5}/3 \approx 0.745 \ . \tag{6.2}$$

Therefore, by general covariance, the following theorem is a consequence of lemma 1.

*Theorem 4.* For $0 < \sigma < 1$, the shock speed is less than the speed of light if and only if $\sigma < \sigma_2$.

To prove lemma 1, we recall that the "speed" of a shock is a coordinate-dependent quantity that can be interpreted in a special relativistic sense at a point $P$ in coordinate systems for which $g_{ij}(P) = \text{diag}(-1,1,1,1)$. [We call such coordinate frames "locally Minkowskian" to distinguish these from "locally Lorentzian" frames in which $g_{ij,k}(P) = 0$ as well. Since we are dealing only with velocities and not accelerations, we do not need to invoke the additional condition $g_{ij,k}(P) = 0$ for a local Lorentzian coordinate frame in order to recover a special relativistic interpretation for velocities.] Moreover, since we are dealing only with radial motion, it suffices to work with coordinate systems that are locally Minkowskian in the $(t,r)$ variables alone. In such coordinate frames, a "speed" at $P$ transforms according to the special relativistic velocity transformation law when a Lorentz transformation is performed. We now determine the shock speed at a point $P$ on the shock in a locally Minkowskian frame that is comoving with the FRW metric. To this end, let $(t,r)$ coordinates correspond to the FRW metric with $k=0$ in (4.1). Let $(t,\tilde{r})$ coordinates correspond to a locally Minkowskian system obtained from $(t,r)$ by a transformation of the form $r = \varphi(\tilde{r})$, so that, in $(t,r)$ coordinates,

$$ds^2 = -dt^2 + R(t^2)^2(\varphi')^2 d\tilde{r}^2.$$

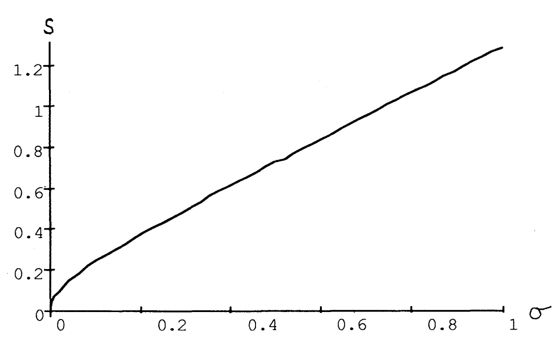


FIG. 2. A plot of the shock speed $s$ vs $\sigma$.

Choose $\varphi$ so that $R^2(\varphi')^2 = 1$ at the point $P$; i.e., at $P = P(t,r)$, set $\varphi'(r) = 1/R(t)$. Thus, in the $(t,\tilde{r})$ coordinates

$$ds^2 = -dt^2 + d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$$

at the point $P$, and so the $(t,\tilde{r})$ coordinates represent the class of locally Minkowskian coordinate frames that are fixed relative to the fluid particles of the FRW metric at the point $P$. (That is, any two members of this class of coordinate frames will differ by higher-order terms that do not affect the calculation of radial velocities at $P$.) Therefore, the speed $d\tilde{r}/dt$ of a particle in $(t,\tilde{r})$ coordinates gives the value of the speed of the particle relative to the FRW fluid in the special relativistic sense. Since

$$\frac{dr}{dt} = \frac{dr}{d\tilde{r}}\frac{d\tilde{r}}{dt} = \varphi'\frac{d\tilde{r}}{dt} = \frac{1}{R}\frac{d\tilde{r}}{dt} \ , \tag{6.3}$$

we conclude that if the speed of a particle in $(t,r)$ coordinates is $dr/dt$, then its geometric speed relative to observers fixed with the FRW fluid (and hence also fixed relative to the radial coordinate $r$ of the FRW metric because the fluid is comoving) is equal to $R(dr/dt)$.

Now consider the shock wave (5.11) which moves with speed [cf. (5.13)]:

$$\frac{dr}{dt} \equiv \dot{r} = \frac{1+3\sigma}{R(t)}\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right)^{1/2} \ . \tag{6.4}$$

Then by (6.3), the speed of the shock $s$ relative to the FRW fluid particles must be given by (6.1). A graph of $s(\sigma)$ is given in Fig. 2, from which we conclude that the shock speed moves with a speed less than one relative to the FRW fluid if and only if $\sigma < \sigma_2$ holds; and for $0 < \sigma < 1$, $s(\sigma) = 1$ if and only if $\sigma = \sigma_2$, where numerical symbolic algebra gives $\sigma_2 = \sqrt{5}/3 \approx 0.745$. This completes the proof of lemma 1.

We next determine when the Lax characteristic condition holds at the shock. To this end, we first determine the speed of the characteristics relative to the fixed FRW fluid particles. By (6.3), the characteristic speeds on the FRW side of the shock must equal the sound speeds $\pm\sqrt{\sigma}$ in the $(t,\tilde{r})$ coordinate frame because the FRW fluid is comoving with respect to the $(t,\tilde{r})$ coordinates. (The characteristic speed is obtained from the fluid speed and sound speed by the special relativistic summation formula for velocities [10].) We conclude that the FRW characteristic speeds $\tilde{\lambda}^-_{\text{FRW}}$, $\tilde{\lambda}^+_{\text{FRW}}$ (the speeds of the characteristics relative to the FRW fluid) are given, respectively, by the formula

$$\tilde{\lambda}^{\pm}_{\text{FRW}} \equiv \pm\frac{d\tilde{r}}{dt} = \pm\sqrt{\sigma} \ . \tag{6.5}$$

By (6.3),

$$\lambda^{\pm}_{\text{FRW}} = \tilde{\lambda}^{\pm}_{\text{FRW}}\frac{1}{R} = \pm\frac{\sqrt{\sigma}}{R} \ .$$

Thus, since the $(t,r)$ coordinates are also comoving with the fluid, the sound waves in the $(t,r)$ coordinates of the FRW metric must move at coordinate speed

$$\frac{dr}{dt} = \pm\frac{\sqrt{\sigma}}{R} \ .$$

We refer to the $-,+$ characteristics as being in the 1,2-characteristic families, respectively. Now in the one-space–one-time–dimensional theory of conservation laws the Lax characteristic condition states that the characteristic curves in the family of the shock impinge upon the shock from both sides, while all other characteristic curves cross the shock, cf. [6]. Since in our example the shock is outward moving with respect to $r$ and $\bar{r}$, it follows that on the FRW side, only the 2 characteristic can impinge on the shock, and thus we must identify the shock wave as a 2 shock. Thus the Lax characteristic condition must be interpreted as meaning that the following inequalities hold:

$$s < \tilde{\lambda}^{+}_{\mathrm{FRW}} \tag{6.6}$$

and

$$\tilde{\lambda}^{+}_{\mathrm{OT}} < s \ . \tag{6.7}$$

Here $\tilde{\lambda}^{+}_{\mathrm{OT}}$ refers to the speed of the faster characteristic on the OT side of the shock as measured in the $(t,\bar{r})$-coordinate system, which is related to the $(\bar{t},\bar{r})$-coordinate system through the $(t,r)\rightarrow(\bar{t},\bar{r})$ coordinate identification. By (6.1) and (6.5), (6.6) is equivalent to the condition

$$\begin{aligned} \tilde{\lambda}^{+}_{\mathrm{FRW}} - s(\sigma) &\equiv \Delta(\sigma) \\ &= \sqrt{\sigma} - (1+3\sigma)\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right)^{1/2} \\ &> 0 \ . \end{aligned} \tag{6.8}$$

A numerical plot of the function $\Delta(\sigma)$, given in Fig. 3, shows that $\Delta(\sigma)$ changes from positive to negative at the unique point $\sigma=\sigma_1$, where

$$\sigma_1 \approx 0.458 \ . \tag{6.9}$$

We are now ready to prove the following theorem.

*Theorem 5.* For $0<\sigma<1$, the Lax characteristic conditions (6.6) and (6.7) hold across the shock if and only if $0<\sigma<\sigma_1$.

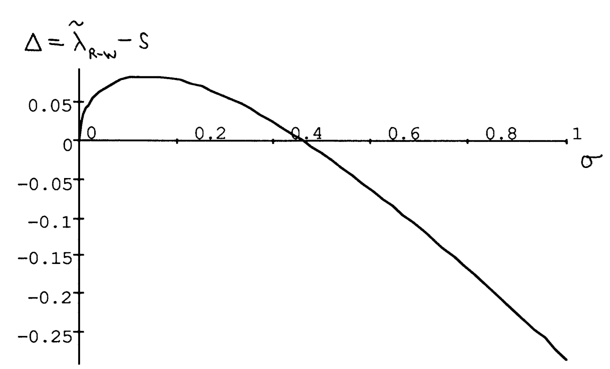


FIG. 3. A plot of the difference between the inner characteristic speed $\tilde{\lambda}_{\mathrm{FRW}}$ and the shock speed $s$ as a function of $\sigma$.

Since (6.6) follows from (6.8) and (6.9), the proof of theorem 5 will be complete once we prove the following lemma, which immediately implies (6.7).

*Lemma 2.* The inequality

$$\tilde{\lambda}^{-}_{\mathrm{OT}} < \tilde{\lambda}^{+}_{\mathrm{OT}} < 0 \tag{6.10}$$

holds for all $0<\sigma<1$.

The next theorem is another immediate consequence of lemma 2.

*Theorem 6.* If $\sigma_1<\sigma<\sigma_2$, then the following inequalities hold:

$$\tilde{\lambda}^{-}_{\mathrm{FRW}} < \tilde{\lambda}^{+}_{\mathrm{FRW}} < s(\sigma) \tag{6.11}$$

and

$$\tilde{\lambda}^{-}_{\mathrm{OT}} < \tilde{\lambda}^{+}_{\mathrm{OT}} < s(\sigma) \ . \tag{6.12}$$

Note that when $\sigma_1<\sigma<\sigma_2$, (6.11) and (6.12) describe a different kind of shock wave in which the 1 and 2 characteristics both cross the shock because the shock speed exceeds the characteristic speeds on both sides of the shock. This occurs even though the sound speeds and shock speed all remain less than the speed of light. In words, theorem 6 states that in general relativity, a sound speed $\sqrt{\sigma}\approx\sqrt{0.744}$ can drive the shock speed all the way up to the speed of light.

It remains only to give the proof of lemma 2. Let $\bar{\mathbf{u}}$ denote the velocity vector for the fluid on the OT side of the shock, and let $\alpha=0,1$ refer to components in the $(\bar{t},\bar{r})$-coordinate frame and $i=0,1$ to components in the $(t,r)$-coordinate frame. Then a velocity vector tangent to the particle paths of the fluid on the OT side of the shock is given by $(\bar{u}^0,\bar{u}^1)=(1,0)$ in barred coordinates because the fluid is comoving relative to the barred coordinate system on the OT side of the shock; for brevity we write $\bar{u}^{\alpha}=(1,0)^{\alpha}$. (Since our aim is to compute the characteristic speed, which is a ratio of two vector components, a tangent vector of any length will suffice.) Let $x^i\equiv(t,r)^i$ and $\bar{x}^{\alpha}\equiv(\bar{t},\bar{r})^{\alpha}$. Then

$$u^i = \frac{\partial x^i}{\partial\bar{x}^{\alpha}}\bar{u}^{\alpha} = \frac{\partial x^i}{\partial\bar{x}^0}\bar{u}^0 = \frac{\partial x^i}{\partial\bar{x}^0} \ . \tag{6.13}$$

Thus the speed of the OT fluid as measured in the FRW coordinates $(t,r)$ is given by

$$u \equiv \frac{u^1}{u^0} = \frac{\partial x^1/\partial\bar{x}^0}{\partial x^0/\partial\bar{x}^0} = \frac{(\partial r/\partial t)(\bar{t},\bar{r})}{(\partial t/\partial\bar{t})(\bar{t},\bar{r})} \ . \tag{6.14}$$

But

$$\frac{\partial t}{\partial\bar{t}}(\bar{t},\bar{r}) = \frac{1}{(\partial\bar{t}/\partial t)(t,\bar{r})} \ , \tag{6.15}$$

so

$$u \equiv \frac{\partial r}{\partial\bar{t}}(\bar{t},\bar{r})\frac{\partial\bar{t}}{\partial t}(t,\bar{r}) = \frac{\partial r}{\partial t}(t,\bar{r}) \ . \tag{6.16}$$

Since

$$r(t,\bar{r}) = \frac{\bar{r}}{R(t)} ,$$

and this holds in a neighborhood of the shock surface, we have

$$u = \frac{\partial r}{\partial t}(t,\bar{r}) = \frac{\partial}{\partial t}\frac{\bar{r}}{R(t)} = -\frac{\bar{r}\dot{R}}{R^2} . \qquad (6.17)$$

But by (5.10),

$$\dot{R} = \frac{2}{r(t)}\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right)^{1/2} , \qquad (6.18)$$

so

$$u \equiv -\frac{\bar{r}}{R(t)^2}\dot{R}(t) = -\frac{2}{R}\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right)^{1/2} . \qquad (6.19)$$

Thus, by (6.3),

$$\tilde{u} = -2\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right)^{1/2} , \qquad (6.20)$$

and this gives the OT fluid speed in the locally Minkowskian frame, which is fixed with the FRW fluid particles. But $\sqrt{\sigma}$ is the sound speed for the OT metric; thus, $\sqrt{\bar{\sigma}}$ is the sound speed as measured in the frame obtained from the $(t,\tilde{r})$ coordinates by the Lorentz transformation for $\tilde{u}$. Therefore, to obtain the OT characteristic speed $\tilde{\lambda}^+_{\rm OT}$ in the frame $(t,\tilde{r})$, we use the relativistic addition of velocities formula:

$$\tilde{\lambda}^+_{\rm OT} = \frac{d\tilde{r}}{dt} = \frac{\tilde{u}+\sqrt{\bar{\sigma}}}{1+\tilde{u}\sqrt{\bar{\sigma}}} , \qquad (6.21)$$

and this implies that

$$\lambda^+_{\rm OT} = \frac{dr}{dt} = \frac{1}{R}\frac{\tilde{u}+\sqrt{\bar{\sigma}}}{1+\tilde{u}\sqrt{\bar{\sigma}}} . \qquad (6.22)$$

We now calculate $\tilde{\lambda}^+_{\rm OT}$. By (6.20), we have

$$\tilde{\lambda}^+_{\rm OT} = -\frac{2-\sqrt{1+6\bar{\sigma}+\bar{\sigma}^2}}{\sqrt{1+6\bar{\sigma}+\bar{\sigma}^2}-2\bar{\sigma}}\sqrt{\bar{\sigma}} \equiv \tilde{\lambda}^+_{\rm OT}(\sigma) , \qquad (6.23)$$

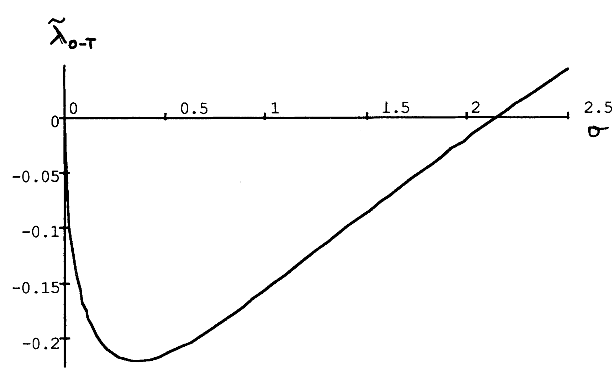


FIG. 4. A plot of the outer characteristic speed as a function of $\sigma$.

where again we use (5.2) to eliminate $\bar{\sigma}$ in favor of $\sigma$. A numerical plot of $\tilde{\lambda}^+_{\rm OT}(\sigma)$ vs $\sigma$ is given in Fig. 4. This verifies that $\tilde{\lambda}^+_{\rm OT}(\sigma) < 0$ for $0 < \sigma < 1$, and thus completes the proof of lemma 2 in light of the inequality $\tilde{\lambda}^-_{\rm OT} < \tilde{\lambda}^+_{\rm OT}$.

## VII. CONCLUDING REMARKS

We first remark that our example here is inherently a theory of *strong* shock waves because the condition $\rho = 3\bar{p}$ implies that $[\rho] \to 0$ if and only if $\rho \to 0$, the latter being a singular limit; cf. [6]. We also note that when $k > 0$, our general shock-wave solutions described in [1] reduce to the well-known model of Oppenheimer and Snyder (OS) when $\bar{p} \equiv 0$, in which case the general ODE's derived in [1] reproduce the OS equation of state $p \equiv 0$. Thus, our shock waves provide a natural generalization of the OS model to the case of nonzero pressure. An important difference between the OS solution and our shock-wave solutions is that the OS interface is a time-reversible *contact discontinuity*, while the interfaces in our models describe true, *time-irreversible, fluid dynamical, shock waves.* Indeed, for a contact discontinuity, a smooth regularization of the solution at a fixed time will propagate as a nearby smooth solution for all times thereafter. In contrast, it is well known from the theory of hyperbolic conservation laws that shock-wave solutions cannot be approximated globally by smooth, shock-free solutions of the hyperbolic equations [6]. It is interesting to note, however, that the OS model *reduces* to flat Minkowski space when we take $k \to 0$ in the OS solution [see Weinberg [2] p. 344, Eqs. (11.9.23) and (11.9.21)]. Moreover, when we take $\bar{\sigma} \to 0$ in our solution (5.8)–(5.11), we also get flat Minkowski space. However, the first limit is singular [because $\dot{R} = 0$ implies $R \equiv$ const when $k = 0$; cf. [2], p. 344, Eq. (11.9.22)]; the second limit is only one way to impose $\bar{\rho}$=0. Indeed, we can obtain a *different*, time-reversible, OS-type contact discontinuity for the case $k = 0$ by noting first from (2.31) that $\bar{p} \equiv 0 \equiv \bar{p}$ implies $p = 0$, and thus we can integrate (4.7) and (4.10) in the case $p = 0$ to obtain the formulas

$$\rho(t) = \frac{1}{(\pm\sqrt{6\pi\mathcal{G}}(t-t_0)+1/\sqrt{\rho_0})^2} , \qquad (7.1)$$

$$R(t) = R_0\left(\frac{\rho(t)}{\rho_0}\right)^{-1/3} . \qquad (7.2)$$

The shock surface is then given by

$$\bar{r}(t) = \left(\frac{3}{4\pi}\frac{M}{\rho(t)}\right)^{1/3} , \qquad (7.3)$$

where $M \equiv$ const when we assume empty space $\bar{\rho} \equiv$

$0 \equiv \bar{p}$. We conclude that (7.1)–(7.3) defines a nontrivial, time-reversible general relativistic model that corresponds to the exact shock-wave solution given in (5.8)–(5.11), and thus defines a different OS-type model of gravitational collapse; cf. [2], p. 345, Eq. (11.9.25).

As a final comment we note also that once values for $\sigma$ and $\bar{\sigma} = H(\sigma)$ are specified, the formulas (5.8)–(5.11) determine a *unique* shock-wave solution despite the appearance of two free parameters, say $R_0$ and $\bar{r}_0$. To see this, note that after fixing the shock position $\bar{r}_0$, the freedom in $R_0$ is only a coordinate freedom due to the fact that $R(t) \to \alpha^{-1}R(t)$ under the coordinate rescaling $r \to \alpha r$ in the FRW metric (4.1) when $k = 0$.

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