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Quantum optics in static spacetimes: How to sense a cosmic string

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We study the influence of static gravitational fields on the spontaneous emission and the Lamb shift of atoms. To illustrate the procedure we consider a two-level atom coupled by a dipole interaction to a massless scalar quantum field in a general static Riemann space and work out the Einstein coefficient and the radiative energy shift. To treat an example, the general scheme is applied to a cosmic string spacetime. The possibility is discussed to detect, at least in principle, the cosmic string via the modified spontaneous emission.

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I. INTRODUCTION

It is well known that the decay rate for spontaneous emission, and accordingly the respective Einstein coefficient, is not an inherent property of the atom but depends on the particular vacuum in which the atom is located. Similarly the shift of the atom's energy level, the Lamb shift, depends on the surroundings. In a confined space, for example, realized by a cavity, for mirrors causing boundary conditions and in dielectric media the mode structure of the electromagnetic field is modified as compared to the empty Minkowski space. Consequently, the decay rate and Lamb shift differ from their free-space values. These values refer to atoms at rest or in uniform motion. For accelerated atoms, spontaneous excitation and radiative energy shifts have been worked out in [1] and [2] with the intention to analyze quantitatively the distinct contribution of vacuum fluctuations and radiation reaction.

The basis of all the considerations mentioned above was a flat spacetime. The inclusion of the influence of inhomogeneous gravitational fields amounts to the transition to a curved Riemann space. In the following we restrict ourselves to static gravitational fields or spacetimes, respectively. We will give a general discussion of the spontaneous decay and Lamb shift in this physical situation. An important application could be found in the astrophysical context where the Einstein coefficient is modified for example in the strong gravitational field of collapsed stars. This modification enters the interpretation of the spectroscopical data.

Below we will study a simpler application of our general scheme in choosing as static Riemann space the spacetime of a straight cosmic string which is flat everywhere outside the string but shows a conical topology. This demonstrates how a nontrivial topology influences spontaneous emission and Lamb shift. A cosmic string could, only in principle, of course, be sensed this way.

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We shall consider a model consisting of a two-level atom and a scalar quantum field interacting via a coupling of dipole type. The more realistic calculation based on an electromagnetic vector potential would follow the same lines. We are interested in the total amount of considered effects only. Therefore, we do not discuss the contributions of vacuum fluctuations and radiation reaction separately.

The structure of the paper is as follows. In Sec. II we quantize in the canonical way a Klein-Gordon field which is minimally coupled to the gravitational field. We introduce the two-level atom and solve the equations of motion for the coupled atom-field system in the Heisenberg picture. In a perturbation approach we find the spontaneous decay rate and the energy shift for the atom and discuss the Minkowski limit. In Sec. III we apply this general scheme to the case of a cosmic string spacetime. We study the quantized Klein-Gordon field and find the Lamb shift as well as the decay rate for a two-level atom. The results are compared with those in the literature. Finally we point out the possibility of sensing a cosmic string.

We use units such that $\hbar = c = 1$. Greek indices run from 0 to 3 and latin indices from 1 to 3.

II. SPONTANEOUS EMISSION AND LAMB SHIFT IN STATIC SPACETIMES

Spontaneous emission and the Lamb shift are quantum optical effects which are caused by the coupling of the atom to the quantum vacuum. We give a general treatment of both effects for cases in which the vacuum mode structure is not the standard one attributed to an inertial reference system in infinite Minkowski space. We include gravity in changing to a static Riemannian spacetime. The scheme is flexible enough to allow for boundaries such as fixed or moving mirrors and cavities. To make the calculation transparent, we restrict our attention to a real scalar massless quantum field coupled to a two-level atom.

A. Scalar quantum field

A static spacetime is characterized by the existence of a timelike Killing vector field which is hypersurface orthogonal. For our purpose it is more convenient to use the following characterization: There exists a coordinate system such that the components of the metric tensor $g_{\mu\nu}$ [signature $(+,-,-,-)$] do not depend on the timelike coordinate $t = x^0$. Furthermore, the components g_{0j} for $j \neq 0$ vanish. With regard to field quantization we follow closely the scheme of Fulling [3,4], which is analogous to the Minkowski case.

The minimally coupled massless Klein-Gordon equation in curved spacetime reads

$$\Box_g \phi = 0 \ , \tag{1}$$

where $\Box_g = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu) = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d'Alembert operator. Equation (1) can be derived from the Lagrangian $\mathcal{L} = \frac{1}{2} \sqrt{|g|} g^{\mu\nu} \partial_\nu \phi \partial_\nu \phi$. The conjugated momentum π and the Hamilton function H are defined as

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}, \quad H = \int d^3x (\pi \partial_0 \phi - \mathcal{L}) \ . \tag{2}$$

To solve Eq. (1), one separates variables with regard to the time coordinate t in making the ansatz

$$\phi(t, \vec{x}) = \psi_j(\vec{x}) e^{\pm E'_j t} \tag{3}$$

[$\vec{x}$ denotes the spatial coordinates (x^1, x^2, x^3)], which leads to the eigenvalue equation

$$K \psi_j = \frac{1}{\sqrt{|g|}} g_{00} \partial_l (\sqrt{|g|} g^{lk} \partial_k) \psi_j = E'^2_j \psi_j \ . \tag{4}$$

Therein K is a positive, time-independent differential operator that is Hermitian with respect to the scalar product:

$$(F_1, F_2) = \int d^3x \sqrt{|g|} g^{00} F_1^*(\vec{x}) F_2(\vec{x}) \ . \tag{5}$$

We assume it to be self-adjoint and possessing a complete set of eigenfunctions $\psi_j(\vec{x})$ which may obey certain boundary conditions. We take the eigenvalues E'_j to be non-negative. Then any function F_i in the Hilbert space can be written as $F_i = \int d\mu(j) \tilde{f}_i(j) \psi_j(\vec{x})$, where the measure $\mu(j)$ is defined such that the scalar product (5) becomes

$$(F_1, F_2) = \int d\mu(j) \tilde{f}_1^*(j) \tilde{f}_2(j) \ . \tag{6}$$

The index j may belong to a point spectrum or a continuous spectrum or both. The symbol $\int d\mu(j)$ may therefore contain sums and integrals. A nontrivial example is given below in Eq. (54). Equations (5) and (6) lead to the orthonormality and completeness relations

$$\int d^3x \sqrt{|g|} g^{00} \psi_j^*(\vec{x}) \psi_k(\vec{x}) = \delta(j,k) \ , \tag{7}$$

$$\int d\mu(j) \psi_j^*(\vec{x}) \psi_j(\vec{y}) = \frac{\delta(\vec{x} - \vec{y})}{\sqrt{|g|} g^{00}} \ , \tag{8}$$

where $\int d\mu(k) \delta(j,k) \tilde{f}(k) = \tilde{f}(j)$. The general solution of the Klein-Gordon equation (1) may then be written as

$$\phi(t, \vec{x}) = \int d\mu(j) \frac{1}{\sqrt{2E'_j}} \times \left[a_j \psi_j(\vec{x}) e^{-iE'_j t} + a_j^\dagger \psi_j^*(\vec{x}) e^{iE'_j t} \right] \ . \tag{9}$$

Imposing the canonical commutation relations for the field

$$[\phi(t, \vec{x}), \phi(t, \vec{y})] = [\pi(t, \vec{x}), \pi(t, \vec{y})] = 0 \ , \tag{10}$$

$$[\phi(t, \vec{x}), \pi(t, \vec{y})] = \delta(\vec{x} - \vec{y}) \ , \tag{11}$$

leads to the commutation relations for the creation and annihilation operators of Klein-Gordon particles:

$$[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, \quad [a_j, a_k^\dagger] = \delta(j,k) \ . \tag{12}$$

$N_j = a_j^\dagger a_j$ is the respective number operator for particles in the mode j. Its expectation value is a constant of motion. Particles are thereby defined as having the wave functions $\phi(t, \vec{x})$ in first quantization. The renormalized operator

$$H = \int d\mu(j) E'_j a_j^\dagger a_j \tag{13}$$

gives the evolution of the free field with respect to the coordinate time t. The vacuum is defined as the state $|0\rangle$ which satisfies $a_j|0\rangle = 0$ for all j.

B. Two-level atoms

To obtain a static situation as far as physical processes are concerned, we consider a single pointlike two-level atom at a fixed space point specified by coordinates $\vec{x}_a =$ const. The physical time attributed to the atom is its proper time τ which is related to the time coordinate t according to $d\tau/dt = \sqrt{g_{00}(\vec{x}_a)} =$ const. The atom possesses stationary energy eigenstates $|+\rangle$ and $|-\rangle$ with energies $\pm\frac{1}{2}E_0$. In realistic situations the non-Minkowskian spacetime will act as a gravitational field causing shifts of the atomic energy levels [5–7]. They are assumed to be included in E_0 and must not be confused with the radiative energy shift discussed below.

The Hamilton operator of the free atom which describes its dynamics with respect to τ is given by

$$H^A(\tau) = E_0 R_3(\tau) \ , \tag{14}$$

where $R_3 = \frac{1}{2}|+\rangle\langle+| - \frac{1}{2}|-\rangle\langle-|$. We introduce the operators $R_\pm \equiv |\pm\rangle\langle\mp|, R_1 \equiv \frac{1}{2}(R_- + R_+)$, and $R_2 \equiv \frac{i}{2}(R_- - R_+)$. The latter models the dipole momentum operator. They obey the commutation relations

$$[R_3(\tau), R_\pm(\tau)] = \pm R_\pm(\tau), \quad [R_+(\tau), R_-(\pi)] = 2R_3(\tau),$$
$$[R_2(\tau), R_3(\tau)] = iR_1(\tau) \ . \quad (15)$$

C. Heisenberg equations of motion for the coupled atom-field system

We work in the Heisenberg picture. The atom is locally coupled to the quantum field via the scalar analogon to the electric dipole interaction

$$H^I(\tau) = dR_2(\tau)\phi(t(\tau), \vec{x}_a) \ , \quad (16)$$

where d is a small positive coupling constant. The time evolution of the free quantum field with respect to the atom's proper time τ is determined by

$$H^F(\tau) = \int d\mu(j) E'_j a^\dagger_j a_j \frac{dt}{d\tau} \ . \quad (17)$$

The total Hamilton operator of the complete atom-field system then reads

$$H = H^A + H^I + H^F \ . \quad (18)$$

Herewith we can write down the Heisenberg equations of motion:

$$\begin{aligned}\frac{d}{d\tau}a_j(\tau) &= -iE_j a_j(\tau) - id[a_j(\tau), R_2(\tau)\phi(\tau, \vec{x}_a)] \\ &= -iE_j a_j(\tau) - id\frac{\psi^*_j(\vec{x}_a)}{\sqrt{2E'_j}}R_2(\tau) \ , \quad (19)\\ \frac{d}{d\tau}R_3(\tau) &= -id[R_3(\tau), R_2(\tau)\phi(\tau, \vec{x}_a)] \\ &= -dR_1(\tau)\phi(\tau, \vec{x}_a) \ , \quad (20)\\ \frac{d}{d\tau}R_\pm(\tau) &= \pm iE_0 R_\pm(\tau) - id[R_\pm(\tau), R_2(\tau)\phi(\tau, \vec{x}_a)] \\ &= \pm iE_0 R_\pm(\tau) + dR_3(\tau)\phi(\tau, \vec{x}_a) \ , \quad (21)\end{aligned}$$

where we have introduced

$$E_j \equiv E'_j \frac{dt}{d\tau} = E'_j \frac{1}{\sqrt{g_{00}}} \ , \quad (22)$$

which is the energy of the photon as measured in the reference frame of the atom.

This set of coupled differential equations cannot be solved analytically. Therefore we follow an approximation scheme with regard to the coupling constant d (compare [8–10]). We first solve each equation formally and then expand it to first order in d. These solutions are then used to solve the other equations up to second order in d. The solution to each of these equations will thereby be split into a free part (index f) which is present also without a coupling ($d = 0$) and a source part (index s), which results from the interaction between atom and field.

Let us first treat the field operators a_j. Formal integration of Eq. (19) gives

$$a^f_j(\tau) = e^{-iE_j(\tau - \tau_0)} a^f_j(\tau_0) \ , \quad (23)$$
$$a^s_j(\tau) = -id\frac{\psi^*_j(\vec{x}_a)}{\sqrt{2E'_j}} \int_{\tau_0}^{\tau} d\tau' R_2(\tau') e^{-iE_j(\tau - \tau')} \ , \quad (24)$$

where the corresponding Green function has been used. In order to calculate $a^s_j(t)$ to first order in d we may replace R_2 in Eq. (24) by its free part (determined below). Then one gets, in the limit $\tau \to \infty$,

$$\begin{aligned}a^s_j(\tau) = i\frac{d}{2}\frac{\psi^*_j(\vec{x}_a)}{\sqrt{2E'_j}}[&\zeta^*(E_j + E_0)R^f_+(\tau) \\ &-\zeta^*(E_j - E_0)R^f_-(\tau)] \ . \quad (25)\end{aligned}$$

Here we made use of the ζ function

$$\zeta(x) = \lim_{T\to\infty}(-i)\int_0^T d\tau \ e^{ix\tau} = \lim_{\epsilon\to 0}\frac{1}{x - i\epsilon} = \frac{\mathcal{P}}{x} - i\pi\delta(x), \quad (26)$$

where $\mathcal{P}$ denotes the principal part of the corresponding integrals.

In the same way we get, for the atomic operators,

$$R^f_3(\tau) = R^f_3(\tau_0) \ , \quad (27)$$

$$\begin{aligned}R^s_3(\tau) = i\frac{d}{2}\int d\mu(j)\frac{1}{\sqrt{2E'_j}}\Big\{&\psi_j a^f_j(\tau)\left[\zeta^*(E_j - E_0)R^f_+(\tau) + \zeta^*(E_j + E_0)R^f_-(\tau)\right] \\ &-\psi^*_j a^{f\dagger}_j(\tau)\left[\zeta(E_j + E_0)R^f_+(\tau) + \zeta(E_j - E_0)R^f_-(\tau)\right]\Big\} \ , \quad (28)\end{aligned}$$

$$R^f_\pm(\tau) = R^f_\pm(\tau_0)e^{\pm iE_0(\tau - \tau_0)} \ , \quad (29)$$

$$R^s_\pm(\tau) = -id\int d\mu(j)\frac{1}{\sqrt{2E'_j}}\left[\psi_j a^f_j(\tau)\zeta^*(E_j \mp E_0) - \psi^*_j a^{f\dagger}_j(\tau)\zeta(E_j \pm E_0)\right] R^f_3(\tau) \ . \quad (30)$$

Again, the source part solutions $R^s_3, R^s_\pm$ are correct only to first order in d.

Now, in order to solve Eq. (20) to second order in d it is convenient to make use of the fact that equal-time atomic and field operators commute and rewrite (20) in the form

$$\frac{d}{d\tau}R_3(\tau) = -d\int d\mu(j)\frac{1}{\sqrt{2E_j'}}\left[R_1(\tau)\psi_j a_j(\tau) + \psi_j^\dagger a_j^\dagger(\tau)R_1(\tau)\right] . \tag{31}$$

We insert the first-order expressions for a_j and $R_\pm$ we have just found and obtain, for the differential equation,

$$\begin{aligned}\frac{d}{d\tau}R_3(\tau) = &-\int d\mu(j)\frac{d}{\sqrt{2E_j'}}\left[\psi_j R_1^f(\tau)a_j^f(\tau) + \psi_j^* a_j^{f\dagger}(\tau)R_1^f(\tau)\right]\\ &-\frac{\pi}{4}\int d\mu(j)\frac{d^2|\psi_j|^2}{2E_j'}\left[\delta(E_j-E_0)-\delta(E_j+E_0)\right] - \frac{\pi}{2}\int d\mu(j)\frac{d^2|\psi_j|^2}{2E_j'}\left[\delta(E_j-E_0)+\delta(E_j+E_0)\right]R_3^f(\tau)\\ &-\frac{i}{2}\int d\mu(j)\int d\mu(k)\frac{d^2}{\sqrt{2E_j'}\sqrt{2E_k'}}\Big\{\psi_j^*\psi_k\left[\zeta^*(E_k-E_0)+\zeta^*(E_k+E_0)\right]a_j^{f\dagger}(\tau)a_k^f(\tau)\\ &-\psi_j\psi_k^*\left[\zeta(E_k-E_0)+\zeta(E_k+E_0)\right]a_k^{f\dagger}(\tau)a_j^f(\tau)\Big\}R_3^f(\tau) ,\end{aligned} \tag{32}$$

where

$$[\zeta(x)-\zeta^*(x)] \pm [\zeta(y)-\zeta^*(y)] = -2\pi i\,[\delta(x)\pm\delta(y)] \tag{33}$$

has been used. We also dropped terms $a_j a_k$ and $a_j^\dagger a_k^\dagger$ because they will not make any contributions in what follows.

For the expectation value in an arbitrary atom state (which may be a combination of $|+\rangle$ and $|-\rangle$) and a field state which is an eigenstate of the number operator representing n_j particles in the mode j we obtain, to second order in d,

$$\frac{d}{d\tau}\langle R_3(\tau)\rangle = -\left[\tfrac{1}{2}\Gamma_0 + \Gamma_\beta\langle R_3(\tau)\rangle\right] , \tag{34}$$

with

$$\Gamma_0 \equiv \frac{\pi}{2}\int d\mu(j)\left|\Omega_j^s(x_a)\right|^2\left[\delta(E_j-E_0)-\delta(E_j+E_0)\right] , \tag{35}$$

$$\Gamma_\beta \equiv \frac{\pi}{2}\int d\mu(j)\left|\Omega_j^s(x_a)\right|^2\left[\delta(E_j-E_0)+\delta(E_j+E_0)\right](1+2n_j) , \tag{36}$$

$$n_j = \langle a_j^\dagger a_j\rangle , \tag{37}$$

which contain the characteristic quantity

$$\Omega_j^s(\vec{x}_a) \equiv \frac{d}{\sqrt{2E_j'}}\psi_j(\vec{x}_a) . \tag{38}$$

n_j is the expectation value of the particle number in the field mode j. Note that the second δ function in Eqs. (35) and (36) makes no contribution because both energies E_j and E_0 are positive. Γ_0 and Γ_β then agree for $n_j = 0$. $\Omega_j^s(\vec{x}_a)$ characterizes the coupling of the atom with the field mode j. It depends on the atom (via the coupling constant d) as well as on the quantum field at the position $\vec{x}_a$ of the atom [via the mode functions $\psi_j(\vec{x}_a)$].

In the same way we treat the differential equation (21) and obtain, to second order in d,

$$\begin{aligned}\frac{d}{d\tau}R_\pm(\tau) &= \pm iE_0R_\pm(\tau) + d\phi(\tau,\vec{x}_a)R_3(\tau)\\ &= \pm iE_0R_\pm(\tau) + \int d\mu(j)\frac{d}{\sqrt{2E_j'}}\left[\psi_j R_3^f(\tau)a_j^f(\tau)+\psi_j^* a_j^{f\dagger}(\tau)R_3^f(\tau)\right]\\ &+\frac{i}{4}\int d\mu(j)\frac{d^2|\psi_j|^2}{2E_j'}\Big\{\left[\zeta^*(E_j+E_0)-\zeta(E_j-E_0)\right]R_+^f(\tau)+\left[\zeta^*(E_j-E_0)-\zeta(E_j+E_0)\right]R_-^f(\tau)\Big\}\\ &+\frac{i}{2}\int d\mu(j)\int d\mu(k)\frac{d^2}{\sqrt{2E_j'}\sqrt{2E_k'}}\Big\{\psi_j\psi_k^* a_k^{f\dagger}(\tau)a_j^f(\tau)\left[\zeta(E_k+E_0)R_+^f(\tau)+\zeta(E_k-E_0)R_-^f(\tau)\right]\\ &+\psi_j^*\psi_k a_j^{f\dagger}(\tau)a_k^f(\tau)\left[\zeta^*(E_k-E_0)R_+^f(\tau)+\zeta^*(E_k+E_0)R_-^f(\tau)\right]\Big\} .\end{aligned} \tag{39}$$

Thus, we get, for the expectation value for an arbitrary atom-field state,

$$\frac{d}{d\tau}\langle R_\pm(\tau)\rangle = \pm i\left(E_0 + E_\beta \pm \frac{i}{2}\Gamma_\beta\right)\langle R_\pm(\tau)\rangle \mp i\left(E_\beta \mp \frac{i}{2}\Gamma_\beta\right)\langle R_\mp(\tau)\rangle \ , \qquad (40)$$

where we have introduced

$$E_\beta \equiv \tfrac{1}{4}\int d\mu(j)\,|\Omega_j^s(x_a)|^2\left(\frac{\mathcal{P}}{E_j + E_0} - \frac{\mathcal{P}}{E_j - E_0}\right) \times(1-2n_j) \ , \qquad (41)$$

as another characteristic quantity depending on $\Omega_j^s(\vec{x}_a)$ of Eq. (38).

It is now straightforward to solve the equations of motions (34) and (40) of the expectation values. Integrating (34) yields

$$\langle R_3(\tau)\rangle = -\frac{1}{2}\frac{\Gamma_0}{\Gamma_\beta} + \left(\langle R_3(0)\rangle + \frac{1}{2}\frac{\Gamma_0}{\Gamma_\beta}\right)e^{-\Gamma_\beta\tau} \ . \qquad (42)$$

Equation (42) describes the time dependence of the mean value of the energy of the atom. Let us assume that the particle state is the vacuum ($n_j = 0, \Gamma_\beta = \Gamma_0$). If the atom is initially ($\tau = 0$) in the upper state $|+\rangle$, Eq. (42) describes the exponential decay to the lower state $|-\rangle$ at a rate Γ_0. Accordingly, Γ_0 of (35) with (38) gives the vacuum Einstein $A_\downarrow$ coefficient

$$A_\downarrow = \Gamma_0 \qquad (43)$$

describing the rate of spontaneous emission for the $|+\rangle \to |-\rangle$ transition. The lineshape is Lorentzian because it is an exponential decay. Similarly, (42) shows that for $n_j = 0$ the ground state is stable. For $n_j \neq 0$ ground state stability and the decay rate are correspondingly modified.

We turn to Eq. (40). In a first approximation we may ignore the second term on the right-hand side. Then we find

$$\langle R_\pm(\tau)\rangle = \langle R_\pm(0)\rangle e^{\pm i(E_0+E_\beta)\tau}e^{-\Gamma_\beta\tau/2} \ . \qquad (44)$$

Let us discuss the case $n_j = 0, \Gamma_\beta = \Gamma_0$ first. Regard some state which is a superposition of $|+\rangle$ and $|-\rangle$ and note that the imaginary part of $\langle R_+(\tau)\rangle$ gives $\langle R_2(\tau)\rangle$ of (16). Equation (44) then shows that the dipole oscillates with $E_0 + E_\beta$ and decays in amplitude. The latter is a consequence of the decay to the ground state $|-\rangle$ with rate Γ_0 as discussed above. E_β represents a shift in the transition frequency. It is present although the particle field is in its vacuum state. E_β of (41) with $n_j = 0$ may accordingly be identified with the vacuum *Lamb shift* of the two-level atom in the general physical situation discussed here. If real particles are present ($n_j \neq 0$) the modified shift is again given by (41).

D. The Minkowski case

For later use and to test our results we compute the decay rate and Lamb shift for the Minkowski case. In Cartesian coordinates the mode functions read $\psi_j = (2\pi)^{-3/2}e^{i\vec{k}\vec{x}}$ with $E_j = |\vec{k}|$. We assume that the atom is at rest and immersed in a thermal bath with temperature $T = (k_B\beta)^{-1}$. Because the particles are represented by a scalar bosonic field their average number in the mode j is given by $n_j = (e^{\beta E_j} - 1)^{-1}$. Then we get for the decay rate and frequency shift of a two-level atom

$$\Gamma_0^{\text{min.}} = \frac{d^2}{8\pi}E_0 \qquad (45)$$

$$\Gamma_\beta^{\text{min.}} = \frac{d^2}{8\pi}E_0\left(1 + \frac{2}{e^{\beta E_0} - 1}\right) \ , \qquad (46)$$

$$E_\beta^{\text{min.}} = \frac{d^2}{16\pi^2}\int_0^\infty dx\, x\left(\frac{\mathcal{P}}{x + E_0} - \frac{\mathcal{P}}{x - E_0}\right) \times\left(1 - \frac{2}{e^{\beta x} - 1}\right) \ . \qquad (47)$$

The equilibrium value for the atom's energy is

$$\langle H^A\rangle = \langle R_3\rangle_{\text{eq}}E_0 = \left(-\frac{1}{2} + \frac{1}{e^{\beta E_0} + 1}\right)E_0 \ , \qquad (48)$$

which exhibits the fermionic nature of the two-level atom. These results agree with results obtained in [1] and [2]. They correspond to those of the electromagnetic case [8–10].

E. Discussion

We have discussed spontaneous emission and Lamb shift for a two-level atom at rest in a static spacetime. Boundaries such as mirrors or cavity walls may be present. For vacuum ($n_j = 0$) the results are given by the Einstein coefficient (43) and the Lamb shift (41). The generalization to the nonvacuum case is based on (35), (36), and (41) (cf. Sec. VII of [1]). These formulas representing the interaction of the atom with the particle field are generally valid. They reflect the fact that the vacuum and the particle modes in general, represent a global structure whereas the atom and its interaction with the field are localized pointlike. Accordingly we have to solve the respective eigenvalue problem in (4) and (7) of the classical particle field. (For photons one would refer to the respective equations for the vector potential.) This specifies the eigenvalues E'_j, the eigenfunctions $\psi_j(\vec{x})$ and the integration measure $d\mu(j)$. These are the quantities which are influenced by the gravitational field (represented as static spacetime), by boundaries and by topological peculiarities of the spacetime. Their modification represents the deviation from the simplest situation in empty flat Minkowski spacetime. The localized field-atom interaction enters the final formulas via the modulus squared of the composed quantity $\Omega_j^s(\vec{x}_a) = d\psi_j(\vec{x}_a)(2E'_j)^{-1}$ taken at the position $\vec{x}_a$ of the atom. It is the fact that the classical modes $\psi_j(\vec{x})$ enter the final formulas in such a simple way, which facilitates the evaluation of the physical effects considerably. We demonstrate this in the following example.

III. HOW TO SENSE A COSMIC STRING

In this section we will apply the above scheme to the case of the spacetime of a cosmic string. We consider a static straight string that lies along the z axis. The metric then reads [13,14]

$$ds^2 = dt^2 - dz^2 - dr^2 - b^2r^2d\varphi^2 \ , \tag{49}$$

where the range of variables is $0 \leq \varphi < 2\pi$, $0 < r < \infty$, $z, t \in (-\infty, +\infty)$, and the parameter b is given by $b = 1 - 4G\mu$, where μ is the mass per unit length of the string and G is Newton's gravitational constant. For $b < 1$ the (ρ, φ) surface gets a conical topology. The spacetime is locally flat (except for the apex of the cone) and can be interpreted to have a deficit angle $\delta\varphi = 8\pi G\mu$. For $b = 1$, equation (49) describes Minkowski spacetime in cylindrical coordinates. It is therefore the topology and accordingly the symmetry of the Minkowski space which has been changed. For grand unified theory (GUT) strings which are to be expected cosmologically the parameter b is very close to 1: $(1 - b) \approx 10^{-6} \ll 1$.

Many authors have studied quantum field theory in cosmic string spacetimes. For a treatment of the scalar quantum field we refer to [15]. There and in the references quoted therein one also finds a discussion of Killing vectors and conserved quantities. See [16] for a treatment of the self-adjointness problem for scalar fields. Of special interest for us are Refs. [17–19], where the Unruh-DeWitt detector in the spacetime of a cosmic string was investigated. To evaluate complicated mathematical expressions, in all three cases a quantization condition has been introduced in restricting b^{-1} to be integer. By this physically realistic strings are excluded. We will show that because of the simplicity of our approach such a condition need not be imposed to make the calculation tractable.

A. Quantum fields in cosmic string spacetime

The Klein-Gordon equation in the cosmic string spacetime (49) takes the form

$$\left(\partial_t^2 - \frac{1}{r}\partial_r(r\partial_r) - \partial_z^2 - \frac{1}{b^2r^2}\partial_\varphi^2\right)\phi(t,\vec{x}) = 0 \ . \tag{50}$$

Following [15] we expand ϕ in terms of cylindrical modes. Using the notation of the previous sections, one finds

$$\phi(t,\vec{x}) = \int d\mu_j \frac{1}{\sqrt{2E_j}}\left(a_j\psi_j + a_j^\dagger\psi_j^*\right) \ , \tag{51}$$

where

$$\psi_j = \frac{1}{2\pi\sqrt{b}}e^{i\kappa z}e^{il\varphi}J_{|l|/b}(\zeta r) \ . \tag{52}$$

Therein $J_\nu(z)$ denotes Bessel functions and the entities $E_j = (\zeta^2 + \kappa^2)^{1/2}$, κ, and lb^{-1} correspond to energy, z-linear momentum, and z component of angular momentum. ζ gives the momentum perpendicular to the string. For the quantum numbers $j = \{\kappa, l, \zeta\}$ one has $l \in Z$, $\kappa \in (-\infty, +\infty)$, and $\zeta \in (0, +\infty)$. The commutation relations for the operators $a_j^\dagger$ and a_j read

$$\left[a_j, a_{j'}^\dagger\right] = \delta(j,j') = \delta_{ll'}\delta(\kappa - \kappa')\frac{\delta(\zeta - \zeta')}{\sqrt{\zeta\zeta'}} \ . \tag{53}$$

In Eq. (51) the measure is defined such that $\int d\mu_j \delta(j,j') = 1$, i.e.,

$$\int d\mu_j = \sum_{l=-\infty}^{\infty}\int_0^\infty d\zeta\,\zeta\int_{-\infty}^{\infty}d\kappa \ . \tag{54}$$

B. Two-level atom in the spacetime of a cosmic string

We will consider an atom that is at rest relative to the string ($dt = d\tau$). We get, from (38) and (52),

$$\left|\Omega_j^{s,cs}(\vec{x}_a)\right|^2 = \left|\frac{d\psi_j(\vec{x}_a)}{2E_j}\right|^2 = \frac{d^2}{8\pi^2bE_j}J_{|l|/b}^2(\zeta r_a) \ , \tag{55}$$

where r_a denotes the distance between the atom and the string. Putting this expression into Eqs. (36) and (41) and assuming that the atom is immersed in a thermal bath with temperature $T = (k_B\beta)^{-1}$ yields

$$\begin{aligned}\Gamma_\beta^{cs}(r_a) &= \frac{\pi}{2}\int_{-\infty}^{\infty}d\kappa\int_0^\infty d\zeta\,\zeta\sum_{l=-\infty}^{\infty}\frac{d^2}{8\pi^2bE_j}J_{|l|/b}^2(\zeta r_a)(1+2n_j)\delta(E_j - E_0)\\ &= \frac{d^2}{8\pi b}E_0\left(1 + \frac{2}{e^{\beta E_0} - 1}\right)\int_0^1 dx\frac{x}{\sqrt{1-x^2}}\sum_{l=-\infty}^{+\infty}J_{|l|/b}^2(xE_0r_a) \ ,\end{aligned} \tag{56}$$

$$\begin{aligned}E_\beta^{cs}(r_a) &= \frac{d^2}{32\pi^2b}\int_0^\infty d\zeta\int_{-\infty}^{\infty}d\kappa\frac{\zeta}{E_j}\\ &\quad\times\sum_{l=-\infty}^{\infty}J_{|l|/b}^2(\zeta r_a)\left(1 - \frac{2}{e^{\beta E_j} - 1}\right)\left(\frac{\mathcal{P}}{E_j + E_0} - \frac{\mathcal{P}}{E_j - E_0}\right)\\ &= \frac{d^2}{32\pi^2b}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\cos\theta\int_0^\infty dx x\\ &\quad\times\sum_{l=-\infty}^{+\infty}J_{|l|/b}^2(xr_a\cos\theta)\left(1 - \frac{2}{e^{\beta x} - 1}\right)\left(\frac{\mathcal{P}}{x + E_0} - \frac{\mathcal{P}}{x - E_0}\right) \ .\end{aligned} \tag{57}$$

Equations (56) and (57) give the exact expressions for the decay rate and the frequency shift of a two-level atom at rest in a thermal bath in the spacetime of a cosmic string. In general (i.e., for $b \neq 1$), they depend on the radial distance r_a between atom and string but never on the coordinates φ and z what was to be expected because of the symmetry properties of the spacetime.

The Minkowskian case is easily recovered. It follows from $\sum_{l=-\infty}^{+\infty} J_{|l|}^2(\zeta r_a) = 1$ [20] that the decay rate (56) and frequency shift (57) take the Minkowskian form (46) and (47) for $b = 1$.

Let us discuss the dependence of the decay rate on the distance r_a to the string. In the limiting case $r_a = 0$, i.e., when the atom is located on the string, one can make use of $\sum_{l=-\infty}^{+\infty} J_{|l|/b}^2(0) = J_0^2(0) = 1$, to obtain

$$\Gamma_\beta^{cs}(0) = \frac{1}{b}\Gamma_\beta^{\text{min.}} , \tag{58}$$

$$E_\beta^{cs}(0) = \frac{1}{b}E_\beta^{\text{min.}} . \tag{59}$$

For small distances $E_0 r_a \ll 1$ we may expand the Bessel functions [20]

$$J_{|l|/b}(E_0 r_a) \approx \left(\frac{E_0 r_a}{2}\right)^{|l|/b} \frac{1}{\Gamma(1+|l|/b)} \quad \text{for } l \neq 0 , \tag{60}$$

$$J_0(E_0 R_a) \approx 1 - (E_0 r_a)^2 + \mathcal{O}((E_0 r_a)^4) . \tag{61}$$

and get, for realistic values of b close to 1,

$$\Gamma_\beta^{cs}(r_a) \approx \frac{1}{b}\Gamma_\beta^{\text{min.}} \left[1 - E_0^2 r_a^2 \left(\frac{1}{3} - \frac{2(E_0^2 r_a^2)^{1/b-1}}{\Gamma(2+2/b)}\right)\right] . \tag{62}$$

For large distances $E_0 r_a \gg 1$, on the other hand, one gets, for the asymptotic behavior of the Bessel functions [20],

$$J_{|l|/b}(E_0 r_a) \approx \frac{1}{\sqrt{2\pi \frac{|l|}{b}}} \exp\left[-\frac{|l|}{b}\ln\left(\frac{2\frac{|l|}{b}}{eE_0 r_a}\right)\right] \quad \text{for } \frac{|l|}{b} > E_0 r_a, \ \frac{|l|}{b} \gg 1 , \tag{63}$$

$$\approx \frac{2^{1/3}}{3^{2/3}\Gamma(\frac{2}{3})} \frac{1}{(E_0 r_a)^{1/3}} \quad \text{for } \frac{|l|}{b} \approx E_0 r_a, \ \frac{|l|}{b} \gg 1 , \tag{64}$$

$$\approx \sqrt{\frac{2}{\pi E_0 r_a}} \cos\left(E_0 r_a - \frac{\pi}{2}\frac{|l|}{b} - \frac{\pi}{4}\right) \quad \text{for } \frac{|l|}{b} \ll E_0 r_a . \tag{65}$$

One sees that in this case the Bessel functions with $|l| > bE_0 r_a$ fall off rapidly. This is a consequence of the *localized absence* of the mode functions discussed in detail in Ref. [15]. Furthermore the number of Bessel functions with $|l| \approx bE_0 r_a$ is very small so that the sum in (56) may be approximated as

$$\sum_{l=-\infty}^{+\infty} J_{|l|/b}^2(xE_0 r_a) \approx \sum_{l=-E_0 b r_a}^{+E_0 b r_a} \frac{2}{\pi x E_0 r_a} \cos^2\left(xE_0 r_a - \frac{\pi}{2}\frac{|l|}{b} - \frac{\pi}{4}\right)$$
$$\approx \sum_{l=-E_0 b r_a}^{+E_0 b r_a} \frac{1}{\pi x E_0 r_a}\left[1 + \sin\left(2xE_0 r_a - \pi\frac{|l|}{b}\right)\right] . \tag{66}$$

When integrating over x we can neglect the rapidly oscillating term in (66) and get

$$\Gamma_\beta^{cs} \approx \frac{1}{b}\Gamma_\beta^{\text{min.}} \int_0^1 dx \frac{x}{\sqrt{1-x^2}} \sum_{l=-E_0 b r_a}^{+E_0 b r_a} J_{|l|/b}^2(xE_0 r_a)$$
$$\approx \frac{1}{b}\Gamma_\beta^{\text{min.}} \sum_{l=-E_0 b r_a}^{+E_0 b r_a} \int_0^1 dx \frac{1}{\sqrt{1-x^2}} \frac{1}{\pi E_0 r_a}$$
$$\approx \Gamma_\beta^{\text{min.}} . \tag{67}$$

The emission rate tends to the Minkowski value as the distance grows. If we do not neglect the second term in (66) we would find that the emission rate is oscillating depending on the distance.

It is also possible to eliminate the integral in Eq. (56). For this we use the following relation which can be proved with the series expansion of the Bessel functions:

$$\int_0^1 dx \frac{x}{\sqrt{1-x^2}} J_\mu^2(ax) = \frac{1}{a}\sum_{k=0}^{\infty} J_{1+2\mu+2k}(2a) . \tag{68}$$

Then we can write down the following exact expression for the emission rate:

$$\Gamma_\beta^{cs}(r_a) = \frac{1}{b}\Gamma_\beta^{\rm min.}\frac{1}{E_0 r_a}\sum_{l=-\infty}^{\infty}\sum_{k=0}^{\infty} J_{1+2|l|/b+2k}(2E_0 r_a)\ . \tag{69}$$

It can be taken as a starting point for numerical evaluations. Figure 1 shows the emission rate of a two-level atom, compared to the Minkowski value, as a function of $E_0 r_a$ for $b = 0.9$.

For the unrealistic case of a "quantized string" with $b^{-1} = n \in \mathbb{N}$, we use the generalized addition theorem for the Bessel functions [17]. It leads to

$$\sum_{l=-\infty}^{\infty} J^2_{|l|/b}(xE_0 r) = b\sum_{k=0}^{1/b-1} J_0(2xE_0 r\sin\pi kb)\ . \tag{70}$$

In this special case we obtain, for the decay rate,

$$\Gamma_\beta^{cs}(r_a) = \frac{1}{b}\Gamma_\beta^{\rm min.}\sum_{k=0}^{1/b-1}\int_0^1 dx\frac{x}{\sqrt{1-x^2}}J_0(2xE_0 r_a\sin\pi kb)$$
$$= \frac{1}{b}\Gamma_\beta^{\rm min.}\left(1+\sum_{k=1}^{1/b-1}\frac{\sin(2E_0 r_a\sin\pi kb)}{2E_0 r_a\sin\pi kb}\right)\ . \tag{71}$$

The result (71) is equivalent to that found in [17–19].

C. A detector for cosmic strings

The decay rate of an atom in a cosmic string spacetime depends on its distance from the string. Therefore it is, in principle, possible to use radiating atoms to detect cosmic strings. By varying the position of the atom one can determine the location of the string and the string parameter b. Note, however, that the change of the decay rate is significant mainly for distances smaller than the particle's wavelength $r_a < 1/E_0$ (see Fig. 1). Heuristically one may say that the atom senses the string only if the emitted (or absorbed) particle characterized by its wavelength "overlaps" with the string. Because we are in the dipole approximation this does not imply that the atom itself has to "overlap" the string. But of course one has to be aware of the fact that the modification of the Minkowskian result essentially amounts to a factor $1/b$ which makes it very small. For us the cosmic string spacetime served only as an example. In generalizing already published results we wanted to demonstrate the easy handling of our expressions. The modification of the Einstein coefficients in strong gravitational fields of stars would be a more relevant astrophysical application of the scheme.

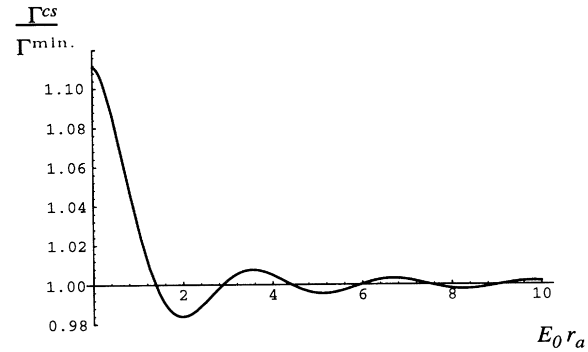

FIG. 1. Decay rate of a two-level atom as a function of the distance of the atom from the cosmic string for $b = 0.9$.

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