Second-post-Newtonian generation of gravitational radiation

Luc Blanchet

Département d'Astrophysique Relativiste et de Cosmologie, CNRS (UPR 176), Observatoire de Paris, 92195 Meudon Cedex, France

(Received 12 August 1994)

This paper derives the expressions of the multipole moments of an isolated gravitating source with an accuracy corresponding to the second-post-Newtonian (2PN) approximation of general relativity. The moments are obtained by a procedure of matching the external gravitational field of the source to the inner field, and are found to be given by integrals extending over the stress-energy distribution of the matter fields and the gravitational field. Although this is not manifest in their expressions, the moments have a compact support limited to the material source only (they are thus perfectly well defined mathematically). From the multipole moments we deduce the expressions of the asymptotic gravitational waveform and associated energy generated by the source at the 2PN approximation. This work, together with a forthcoming work devoted to the application to coalescing compact binaries, will be used in the future observations of gravitational radiation by laser interferometric detectors.

PACS number(s): 04.30.Db, 04.25.Nx

I. INTRODUCTION

A. Motivation and relation to other works

The problem of the generation of gravitational radiation by a slowly moving isolated system is presently solved with an accuracy corresponding to the 1.5-post-Newtonian (in short 1.5PN) approximation of general relativity [1–3]. This approximation takes into account all contributions in the radiation field at large distances from the system up to the relative order ε^3, where $\varepsilon \sim c^{-1}$ is a usual post-Newtonian parameter. The objective of this paper is to go one step further in the resolution of this problem by deriving the expression of the gravitational radiation field with an accuracy corresponding to the second-post-Newtonian (2PN) approximation, taking into account all contributions up to the order ε^4. In a forthcoming paper [4] we shall consider specifically the case of the radiation generated by a coalescing binary system made of compact stars (neutron stars or black holes).

It is now well known that post-Newtonian effects in the radiation generated by a coalescing compact binary system should be detectable by future gravitational-wave interferometers such as the Laser Interferometric Gravitational Wave Observatory (LIGO) and VIRGO [5–10]. It has even been realized in recent years that an extremely accurate theoretical signal is required in order to achieve the full potential precision on the measurement of the binary's parameters [8]. This remarkable state of affairs is made possible by the fact that the signal will spend hundreds to thousands of cycles in the detector bandwidth, and that as a result the instantaneous *phase* of the signal will be amenable to a very precise determination. The main influence of the higher-order post-Newtonian corrections is thus located in the phase of the signal, which is itself mainly determined by the rate of decay of the binary's orbit resulting from the gravitational radiation reaction. Note that this potential high precision on the extraction of the binary's parameters is interesting not only for doing astrophysical measurements of masses, spins, etc., of stars, but also for testing some aspects of the nonlinear structure of general relativity [11].

Analytical and numerical computations of the radiation generated by a small mass in circular orbit around a large one have indicated that post-Newtonian corrections beyond the 3PN approximation may be needed [12–15]. The 2PN approximation worked out in this paper and its sequel [4] thus does not yet reach the ideal precision required by the observations of coalescing binaries, but still represents an appreciable improvement to the existing situation.

The investigation of this paper will be based on a "multipolar-post-Minkowskian" formalism for dealing with the gravitational field in the external vacuum region of the source. Such a formalism combines a multipole moment expansion, valid in the exterior of the source, with a post-Minkowskian expansion, i.e., a nonlinearity expansion or expansion in powers of Newton's constant G, valid wherever the field is weak. This double expansion formalism is originally due to Bonnor and collaborators [16–19], and has been later refined and clarified by Thorne [20]. In recent years, the formalism has been implemented in an explicit and constructive way [21,22] (with the help of some mathematical tools such as analytic continuation), and applications have been made to the problems of gravitational radiation reaction [23,24] and wave generation [1–3]. The field being determined only in the *exterior* of the source, the multipolar-post-Minkowskian formalism must be supplemented by a method of matching to the field *inside* the source. We shall use a variant of the well-known method of matching of asymptotic expansions (see e.g., [25]) which has already, on several occasions, found its way in general relativity [26–29].

Inspection of the solution of the wave generation problem at the 1.5PN approximation [1–3] readily shows what is needed for extension at the 2PN approximation. Indeed, the works [1] and [2] have obtained, respectively, the expressions of the mass-type and current-type multipole moments of the source with relative precision ε^2, the next-order correction being of order $O(\varepsilon^4)$. The inclusion done in [1] of the terms ε^2 in the mass multipole moments permits solving the wave generation problem at the 1PN approximation (in fact, only the terms ε^2 in the quadrupole mass moment are needed). This has set on a solid (and well-defined) footing previous works by Epstein and Wagoner [30], and Thorne [20] who obtained formally correct but divergent expressions of the moments at the 1PN approximation (see [1] for discussion). In the work [3] the nonlinear effects in the radiation field were added and shown to arise at the level ε^3, essentially due to the "tails" of gravitational waves. Higher-order nonlinear effects were shown to be of order $O(\varepsilon^5)$ at least. Now recall that the contribution of a moment with multipolarity l scales like ε^l in the radiation field, and that the current moments always carry an additional factor ε as compared with the corresponding mass moments. Therefore, we conclude that what is needed for solving the 2PN wave generation problem is only to find the extension of the expression of the *mass quadrupole* moment of the source with relative precision ε^4. Note that the computation of the moments of the source can be done in the *near zone* of the source.

This paper will thus mainly focus on the *matching* between the external and internal gravitational fields in the near zone of the source (both fields are expressed in the form of a post-Newtonian expansion when $\varepsilon \to 0$) with a precision consistent with the inclusion of the terms ε^4 in the mass multipole moments. (Although the quadrupole mass moment is sufficient for our purpose, we shall compute the terms ε^4 in all mass moments of arbitrary multipolarity l.) The matching performed in [1] was based on a particularly simple closed form of the internal gravitational field of the source including 1PN corrections. On the other hand, the matching performed in [2] made use of some specific distributional kernels to deal with the quadratic nonlinearities of Einstein's equations. None of these methods can be applied to our problem, which necessitates the inclusion of the full 2PN corrections in the field, depending not only on quadratic but also on *cubic* nonlinearities of Einstein's equations. In this paper we shall employ a matching procedure which is far more general than the ones followed in [1,2]. In particular, we shall show how one can deal with the cubic nonlinearities of Einstein's equations without any use of distributional kernels.

The end result obtained in this paper expresses the multipole moments of the source as integrals extending over the distribution of stress energy of the matter fields in the source *and* of the gravitational field. This result is similar to the one we would obtain by using as the source of the radiation field the total stress-energy (pseudo)tensor of the matter and gravitational fields, and by considering formally that this tensor has a spatially compact support limited to the material source. It is well known that by proceeding in this formal way (i.e., in the manner of Epstein, Wagoner, and Thorne), we obtain integral expressions of the multipole moments which are *divergent*, because of the noncompact-supported distribution of the gravitational field. (Indeed, the integral expression of a moment with multipolarity l involves a large power $\sim |\mathbf{x}|^l$ of the radial distance to the source, which blows up at infinity.) We prove in this paper that the correct expressions of the multipole moments must involve also an analytic continuation factor $|\mathbf{x}|^B$, where B is a complex number, and a "finite part at $B = 0$" prescription to deal with the *a priori* bad behavior of the integrals at spatial infinity. In such a way the expressions of the moments are perfectly well defined mathematically. When a pole $\sim 1/B$ occurs in an integral due to the bad behavior of the integral at spatial infinity, the finite part at $B = 0$ introduces an additional contribution to the moment which must absolutely be taken into account. (However, we shall see that no poles occur at the level investigated in this paper.)

Let us stress that the expressions presented here of the multipole moments are not *manifestly* of compact-supported form (contrarily to say the lowest-order expressions at the 1PN level [1]), but are numerically equal, thanks to the properties of the analytic continuation, to some compact-supported expressions which could be constructed at the price of introducing more complicated potentials in addition to the usual Poisson integrals of the mass and current densities in the source. Such a construction is, however, unnecessary (and is somewhat awkward) in practical computations of the moments for specific sources.

It is likely that the expressions of the moments expressed in this way as integrals extending over the total stress-energy distribution of the matter and gravitational fields and regularized by means of analytic continuation, will admit a generalization to higher nonlinearities in the field, and higher post-Newtonian approximations. Such a generalization, which will be the subject of future work, should permit the resolution, at least in principle, of the problem of the generation of gravitational radiation by a general isolated system up to the high level of approximation required by the observations of coalescing compact binaries.

This paper is organized as follows. In Sec. II we compute the gravitational field both in the near zone of the source (where a direct post-Newtonian expansion of the field equations is performed), and in the external near zone of the source (where we use the multipolar-post-Minkowskian solution of the vacuum equations). In Sec. III we impose that the two fields are isometric in their common domain of validity, namely the external near zone of the source. This yields a matching equation valid up to an appropriate post-Newtonian order, and which is used in Sec. IV to obtain the expressions of both the mass and current multipole moments of the source. The formulas for the asymptotic waveform and the associated energy generated by the source at the 2PN approximation are also given in Sec. IV. The paper ends with three Appendices, and we start with our notation for Einstein's equations.

B. Notation for Einstein's equations

Throughout most of this paper, we shall use Einstein's equations in harmonic coordinates. The field deviation from Minkowski's metric is denoted by $h^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} - \eta^{\alpha\beta}$, where Greek indices α, β, μ, ν... range from 0 to 3, $g_{\alpha\beta}$ is the usual covariant metric, $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$, g is the determinant of $g_{\alpha\beta}$, and $\eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric. The condition of harmonic coordinates reads as

$$\partial_\beta h^{\alpha\beta} = 0 \ , \tag{1.1}$$

where $\partial_\beta = \partial/\partial x^\beta$ is the usual derivation with respect to the harmonic coordinates. Einstein's equations reduced by the condition (1.1) are written in the form

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}\lambda T^{\alpha\beta} + \Lambda^{\alpha\beta}(h) \ . \tag{1.2}$$

We denote by $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ the *flat* d'Alembertian operator, by $T^{\alpha\beta}$ the stress-energy tensor of the nongravitational fields ($T^{\alpha\beta}$ has the dimension of an energy density), and by λ the absolute value of g, i.e. $\lambda = |g| = -g$. In terms of a series expansion in the field variable $h^{\alpha\beta}$, we have

$$\lambda = 1 + h + \frac{1}{2}(h^2 - h_{\mu\nu}h^{\mu\nu}) + O(h^3) \ , \tag{1.3}$$

where $h_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}h^{\alpha\beta}$ and $h = \eta_{\alpha\beta}h^{\alpha\beta}$. The second term in (1.2) is an effective gravitational nonlinear source including all the nonlinearities (quadratic, cubic, ...) of Einstein's equations. We denote

$$\Lambda^{\alpha\beta}(h) = N^{\alpha\beta}(h,h) + M^{\alpha\beta}(h,h,h) + O(h^4) \ , \tag{1.4}$$

where the quadratically nonlinear term is given by

$$\begin{aligned} N^{\alpha\beta}(h,h) = {} & -h^{\mu\nu}\partial_\mu\partial_\nu h^{\alpha\beta} + \frac{1}{2}\partial^\alpha h_{\mu\nu}\partial^\beta h^{\mu\nu} - \frac{1}{4}\partial^\alpha h \partial^\beta h - 2\partial^{(\alpha}h_{\mu\nu}\partial^\mu h^{\beta)\nu} + \partial_\nu h^{\alpha\mu}(\partial^\nu h^\beta_\mu + \partial_\mu h^{\beta\nu}) \\ & + \eta^{\alpha\beta}\left[-\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{1}{8}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho}\right] \ , \end{aligned} \tag{1.5}$$

and where the cubically nonlinear term is given by

$$\begin{aligned} M^{\alpha\beta}(h,h,h) = {} & -h^{\mu\nu}(\partial^\alpha h_{\mu\rho}\partial^\beta h^\rho_\nu + \partial_\rho h^\alpha_\mu \partial^\rho h^\beta_\nu - \partial_\mu h^\alpha_\rho \partial_\nu h^{\beta\rho}) + h^{\alpha\beta}\left[-\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{1}{8}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho}\right] \\ & + \frac{1}{2}h^{\mu\nu}\partial^{(\alpha}h_{\mu\nu}\partial^{\beta)}h + 2h^{\mu\nu}\partial_\rho h^{(\alpha}_\mu\partial^{\beta)}h^\rho_\nu + h^{\mu(\alpha}\left(\partial^{\beta)}h_{\nu\rho}\partial_\mu h^{\nu\rho} - 2\partial_\nu h^{\beta)}_\rho\partial_\mu h^{\nu\rho} - \frac{1}{2}\partial^{\beta)}h\partial_\mu h\right) \\ & + \eta^{\alpha\beta}\left[\frac{1}{8}h^{\mu\nu}\partial_\mu h\partial_\nu h - \frac{1}{4}h^{\mu\nu}\partial_\rho h_{\mu\nu}\partial^\rho h - \frac{1}{4}h^{\rho\sigma}\partial_\rho h_{\mu\nu}\partial_\sigma h^{\mu\nu} - \frac{1}{2}h^{\rho\sigma}\partial_\mu h_{\rho_\nu}\partial^\nu h^\mu_\sigma + \frac{1}{2}h^{\rho\sigma}\partial_\mu h^\nu_\rho\partial^\mu h_{\sigma\nu}\right] \ . \end{aligned} \tag{1.6}$$

In (1.5) and (1.6), we raise and lower all indices with the Minkowski metric, and we denote $t_{(\alpha\beta)} = \frac{1}{2}(t_{\alpha\beta} + t_{\beta\alpha})$. By taking the divergence of (1.5) and (1.6), we obtain the relations

$$\partial_\beta N^{\alpha\beta} = \left(-\partial_\mu h^\alpha_\nu + \frac{1}{2}\partial^\alpha h_{\mu\nu} - \frac{1}{4}\partial^\alpha h\,\eta_{\mu\nu}\right)\Box\, h^{\mu\nu} \ , \tag{1.7}$$

$$\begin{aligned} \partial_\beta M^{\alpha\beta} = {} & \left(\partial_\mu h^\alpha_\nu - \frac{1}{2}\partial^\alpha h_{\mu\nu} + \frac{1}{4}\partial^\alpha h\,\eta_{\mu\nu}\right)N^{\mu\nu} + \left[\frac{1}{2}h^{\alpha\rho}\partial_\rho h_{\mu\nu} - h_{\mu\rho}\partial^\alpha h^\rho_\nu + h_{\mu\rho}\partial_\nu h^{\alpha\rho}\right. \\ & \left. + \frac{1}{4}h_{\mu\nu}\partial^\alpha h + \frac{1}{4}h_{\rho\sigma}\partial^\alpha h^{\rho\sigma}\eta_{\mu\nu} - \frac{1}{4}h^{\alpha\rho}\partial_\rho h\,\eta_{\mu\nu}\right]\Box\, h^{\mu\nu} \ , \end{aligned} \tag{1.8}$$

which are consistent, by Bianchi's identities, with the conservation (in the covariant sense) of the stress-energy tensor of the matter fields.

II. THE INTERNAL AND EXTERNAL GRAVITATIONAL FIELDS

A. Solution of Einstein's equations in the internal near zone

Let us define the internal near zone of the source to be a domain $D_i = \{(\mathbf{x},t)/|\mathbf{x}| < r_i\}$ whose radius r_i is adjusted so that (i) $r_i > a$, where a is the radius of a sphere which totally encloses the source, and (ii) $r_i \ll \lambda\!\!\!^{-}$, where $\lambda\!\!\!^{-} \sim ac/v$ is the (reduced) wavelength of the emitted gravitational radiation, and v is a typical internal velocity in the source.

Defining D_i in such a way assumes in particular that $a \ll \lambda\!\!\!^{-}$, or equivalently $\varepsilon \ll 1$ where $\varepsilon \sim v/c$ is a small "post-Newtonian" parameter appropriate to the description of slowly moving sources. [We shall also assume that the source is self-gravitating so that $GM/(ac^2) \sim \varepsilon^2$ where M is the total mass of the source, and that the internal stresses are such that $T^{ij}/T^{00} \sim \varepsilon^2$.] In D_i, we can solve Einstein's equations (1.1)–(1.2) by formally taking the post-Newtonian limit $\varepsilon \to 0$. The following notation will be used for terms of small order in the post-Newtonian parameter ε. By $A = O(p)$ we mean that A is of order $O(\varepsilon^p)$; by $B^\alpha = O(p,q)$ we mean that the zero component of the vector B^α is $B^0 = O(p)$, and that the spatial components of B^α are $B^i = O(q)$; and similarly by $C^{\alpha\beta} = O(p,q,r)$ we mean that $C^{00} = O(p)$, $C^{0i} = O(q)$, and $C^{ij} = O(r)$. (The latin indices $i, j, \ldots$ range from 1 to 3.)

From the *contravariant* components of the stress-energy tensor $T^{\alpha\beta}$ of the matter fields, we define a mass density σ, a current density σ_i and a stress density σ_{ij} by the formulas

$$\sigma = \frac{T^{00} + T^{ii}}{c^2} , \tag{2.1a}$$

$$\sigma_i = \frac{T^{0i}}{c} , \tag{2.1b}$$

$$\sigma_{ij} = T^{ij} , \tag{2.1c}$$

where in (2.1a) $T^{ii} = \Sigma \delta_{ij} T^{ij}$ denotes the spatial trace of $T^{\alpha\beta}$. In all this paper, we shall assume that the matter densities σ, σ_i, and σ_{ij} are of order ε^0 when $\varepsilon \to 0$, i.e.

$$\sigma,\ \sigma_i,\ \sigma_{ij} = O(0). \tag{2.2}$$

We introduce next some *retarded* potentials generated by the densities σ, σ_i and σ_{ij}. First, V and V_i are the usual retarded scalar and vector potentials of the mass and current densities σ and σ_i, i.e.,

$$V(\mathbf{x},t) = G \int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\, \sigma\left(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right) , \tag{2.3a}$$

$$V_i(\mathbf{x},t) = G \int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\, \sigma_i\left(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right) , \tag{2.3b}$$

satisfying $\Box V = -4\pi G\sigma$ and $\Box V_i = -4\pi G\sigma_i$. Second, W_{ij} is a more complicated retarded tensor potential defined by

$$W_{ij}(\mathbf{x},t) = G \int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \left[\sigma_{ij} + \frac{1}{4\pi G}(\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V)\right] \left(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right) . \tag{2.4}$$

Note that while the potentials V and V_i have a *compact* support limited to that portion of the past null cone issued from the field point $(\mathbf{x}',t)$ which intersects the source, the potential W_{ij} has not a compact support. From (2.3a), we see that $V(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|)$ behaves like $GM/|\mathbf{x}'|$ when $|\mathbf{x}'| \to \infty$, where $M = \int d^3\mathbf{y}\,\sigma(\mathbf{y}, -\infty)$ $+O(2)$ is the initial mass [or Arnowitt-Deser-Misner (ADM) mass] of the source, and we can check from this that the potential W_{ij} is given as a convergent integral. We shall often abbreviate formulas such as (2.3)–(2.4) by denoting the retarded integral of some source $f(\mathbf{x},t)$, having adequate falloff properties along past null cones, by

$$(\Box_R^{-1} f)(\mathbf{x},t) = -\frac{1}{4\pi} \int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f\left(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right) , \tag{2.5}$$

so that, e.g., $V = -4\pi G\, \Box_R^{-1}\, \sigma$.

It is convenient in most of this paper to keep the potentials (2.3)–(2.4) in retarded form, i.e., to *not* expand when $c \to +\infty$ the retardation argument $t - |\mathbf{x}-\mathbf{x}'|/c$. This permits avoiding possible problems of convergence with the expansion of potentials with noncompact support. However, most of the equations of this paper will be valid only up to some remainder in the expansion $\varepsilon \sim c^{-1} \to 0$, and we can replace, when there is no problem of convergence, the retarded potentials by their post-Newtonian expanded forms up to the accuracy of the remainder. For instance, from the Newtonian equations of continuity and of motion,

$$\partial_t \sigma + \partial_i \sigma_i = O(2) , \tag{2.6a}$$

$$\partial_t \sigma_i + \partial_j \sigma_{ij} = \sigma \partial_i V + O(2) , \tag{2.6b}$$

we deduce that the potentials V, V_i, and W_{ij} satisfy the conservation laws

$$\partial_t V + \partial_i V_i = O(2) , \tag{2.7a}$$

$$\partial_t V_i + \partial_j W_{ij} = O(2) . \tag{2.7b}$$

In (2.6b) and (2.7) we can replace the potentials V, V_i, and W_{ij} by corresponding Poisson-type potentials, e.g., $V = U + O(2)$ where U is the Newtonian potential of the mass density σ, satisfying $\Delta U = -4\pi G\sigma$.

We now proceed to solve Einstein's equations (1.1) and (1.2) with an accuracy corresponding to the post-Newtonian order $O(8,7,8)$, by which we mean $O(8)$ in the 00 and ij components of $h^{\alpha\beta}$, and $O(7)$ in its $0i$ components. The insertion of the lowest-order results $h^{00} = -4V/c^2 + O(2)$, $h^{0i} = O(3)$, and $h^{ij} = O(4)$ into the right-hand-side of (1.2) with the explicit expression (1.5), yields first the equations to be solved at order $O(6,5,6)$. We get

$$\Box h^{00} = \frac{16\pi G}{c^4}\left(1 + \frac{4V}{c^2}\right) T^{00} - \frac{14}{c^4}\partial_k V \partial_k V + O(6) , \tag{2.8a}$$

$$\Box h^{0i} = \frac{16\pi G}{c^4} T^{0i} + O(5) , \tag{2.8b}$$

$$\Box h^{ij} = \frac{16\pi G}{c^4} T^{ij} + \frac{4}{c^4}\left\{\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V\right\} + O(6) . \tag{2.8c}$$

These equations can be straightforwardly solved by means of the potentials V, V_i, and W_{ij} we defined in (2.3) and (2.4), and by means of the trace $W \equiv W_{ii} \equiv \Sigma \delta_{ij} W_{ij}$ of the potential W_{ij}. The result is

$$h^{00} = -\frac{4}{c^2} V + \frac{4}{c^4}(W - 2V^2) + O(6) , \tag{2.9a}$$

$$h^{0i} = -\frac{4}{c^3} V_i + O(5) , \tag{2.9b}$$

$$h^{ij} = -\frac{4}{c^4} W_{ij} + O(6) . \tag{2.9c}$$

Since the potentials satisfy the conservation laws (2.7), we see that the field (2.9) satisfies the approximate harmonic gauge condition $\partial_\beta h^{\alpha\beta} = O(5,6)$.

The next step consists in iterating Einstein's equations by using the expression (2.9) of the field. We substitute (2.9) into the right-hand side of (1.2) with the help of (1.3) and of the explicit expressions (1.5) and (1.6) of the quadratic and cubic nonlinearities. In this way, we find the equations to the satisfied by the field up to the order $O(8,7,8)$, which are

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}\overline{\lambda}(V,W)T^{\alpha\beta} + \overline{\Lambda}^{\alpha\beta}(V,V_i,W_{ij}) + O(8,7,8) \ . \tag{2.10}$$

In these equations, we denote by $\overline{\lambda}$ the post-Newtonian expansion of $\lambda = |g| = -g$ when *truncated* at the order $O(6)$, i.e., $\lambda = \overline{\lambda} + O(6)$, and expressed with the potentials V and $W = W_{ii}$. From (1.3) we have

$$\overline{\lambda}(V,W) = 1 + \frac{4}{c^2}V - \frac{8}{c^4}(W - V^2) \ . \tag{2.11}$$

[Note that because the matter stress-energy tensor is $T^{\alpha\beta} = O(-2,-1,0)$, the precision on λ as given by (2.11) is necessary only in the 00 component of the equations (2.10).] Similarly, $\overline{\Lambda}^{\alpha\beta}$ in (2.10) is defined to be the post-Newtonian expansion of the effective nonlinear source in the right-hand-side of Einstein's equations (1.2) when *truncated* at order $O(8,7,8)$, i.e., $\Lambda^{\alpha\beta} = \overline{\Lambda}^{\alpha\beta} + O(8,7,8)$, and expressed in terms of the potentials V, V_i, and W_{ij}. From (1.5) and (1.6), we arrive at the expressions

$$\overline{\Lambda}^{00}(V,V_i,W_{ij}) = -\frac{14}{c^4}\partial_k V \partial_k V + \frac{16}{c^6}\left\{ -V\partial_t^2 V - 2V_k \partial_t \partial_k V - W_{km}\partial^2_{km}V + \frac{5}{8}(\partial_t V)^2 + \frac{1}{2}\partial_k V_m(\partial_k V_m + 3\partial_m V_k) + \partial_k V \partial_t V_k + 2\partial_k V \partial_k W - \frac{7}{2}V\partial_k V \partial_k V \right\} , \tag{2.12a}$$

$$\overline{\Lambda}^{0i}(V,V_i,W_{ij}) = \frac{16}{c^5}\left\{ \partial_k V(\partial_i V_k - \partial_k V_i) + \frac{3}{4}\partial_t V \partial_i V \right\} , \tag{2.12b}$$

$$\overline{\Lambda}^{ij}(V,V_i,W_{ij}) = \frac{4}{c^4}\left\{ \partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V \right\} + \frac{16}{c^6}\left\{ 2\partial_{(i}V\partial_t V_{j)} - \partial_i V_k \partial_j V_k - \partial_k V_i \partial_k V_j + 2\partial_{(i}V_k\partial_k V_{j)} - \frac{3}{8}\delta_{ij}(\partial_t V)^2 - \delta_{ij}\partial_k V \partial_t V_k + \frac{1}{2}\delta_{ij}\partial_k V_m(\partial_k V_m - \partial_m V_k) \right\} . \tag{2.12c}$$

By using the equations of motion and of continuity including ε^2 terms [which are given by (B5) in Appendix B], one can check that the expressions (2.11) and (2.12) imply

$$\partial_\beta \left[\frac{16\pi G}{c^4}\overline{\lambda}T^{\alpha\beta} + \overline{\Lambda}^{\alpha\beta} \right] = O(7,8) \ . \tag{2.13}$$

Note that the condition (2.13) checks all terms in (2.12) except the c^{-6} terms in the 00 component $\overline{\Lambda}^{00}$ of (2.12a) and the c^{-4} term in $\overline{\lambda}$. We have checked these terms by adding in the $0i$ component $\overline{\Lambda}^{0i}$ of (2.12b) the next-order c^{-7} terms and computing the divergence. Note also that the consideration of the next-order c^{-7} terms in $\overline{\Lambda}^{0i}$ would allow the control of the c^{-4} correction terms not only in the mass-type source moments (which is the aim of this paper), but also in the current-type source moments. However, we have chosen here not to include these terms because they are not necessary for solving the 2PN wave generation problem, and because they somewhat complicate the discussion with the need of introducing new potentials besides V, V_i, and W_{ij}, and the necessity of a better control of the external metric in Sec. II B. These terms will be considered in a future work where we investigate how the procedure followed in this paper could be systematically extended to higher post-Newtonian orders.

Finally, a solution of (2.10), with the required precision, can simply be written as

$$h^{\alpha\beta} = \Box_R^{-1}\left[\frac{16\pi G}{c^4}\overline{\lambda}T^{\alpha\beta} + \overline{\Lambda}^{\alpha\beta} \right] + O(8,7,8) \ , \tag{2.14}$$

where $\Box_R^{-1}$ is the retarded integral operator defined in (2.5). By (2.13), this solution satisfies also the approximate harmonic gauge condition

$$\partial_\beta h^{\alpha\beta} = O(7,8) \ . \tag{2.15}$$

Note that we could have added in (2.14) some homogeneous solutions of the wave equation which are *regular* in D_i and satisfy the harmonic gauge condition (2.15). It is simpler to use the solution (2.14) as it stands since we shall show that it directly matches the exterior metric.

B. Solution of Einstein's equations in the external near zone

Let $D_e = \{(\mathbf{x},t) \,/\, r \equiv |\mathbf{x}| > r_e\}$ be an external domain surrounding the source, where r_e is adjusted so that $a < r_e < r_i$, a being the radius of the source and r_i the radius of the inner domain D_i defined in Sec. II A. By our assumption $GM/(ac^2) \sim \varepsilon^2$, gravity is weak everywhere and in particular in D_e, so we can solve Einstein's *vacuum* equations in D_e by means of the multipolar and post-Minkowskian approximation method developed in our previous works [21–24] on foundations laid by Bonnor [16] and Thorne [20]. More specifically, we use the construction of the external field which is defined in Sec. 4.3 of [21] and which is referred there to as the "canonical" external field. Using a formal infinite post-Minkowskian expansion, or expansion in powers of Newton's parameter G, the canonical external field $h^{\mu\nu}_{\rm can} = \sqrt{-g_{\rm can}}\, g^{\mu\nu}_{\rm can} - \eta^{\mu\nu}$ is given as

$$h^{\mu\nu}_{\rm can} = G\, h^{\mu\nu}_{{\rm can}(1)} + G^2 h^{\mu\nu}_{{\rm can}(2)} + \cdots + G^n h^{\mu\nu}_{{\rm can}(n)} + \cdots , \tag{2.16}$$

where the coefficients $h^{\mu\nu}_{{\rm can}(n)}$ of an arbitrary nth power of G are *algorithmically* constructed from the knowledge of the previous coefficients $h^{\mu\nu}_{{\rm can}(m)}$ (with $m < n$) by post-Minkowskian iteration of the vacuum equations. This iteration is made possible by a systematic use of multipole expansions, valid outside the source (in D_e), for all the coefficients $h^{\mu\nu}_{{\rm can}(n)}$ in (2.16).

The whole iteration in (2.16) rests on the first of the coefficients, namely the "linearized" field $h^{\mu\nu}_{{\rm can}(1)}$, which is chosen to be the most general solution, modulo an arbitrary linear gauge transformation, of the linear vacuum equations. This solution reads as

$$G\, h^{00}_{{\rm can}(1)} = -\frac{4}{c^2}\, V^{\rm ext} , \tag{2.17a}$$

$$G\, h^{0i}_{{\rm can}(1)} = -\frac{4}{c^3}\, V^{\rm ext}_i , \tag{2.17b}$$

$$G\, h^{ij}_{{\rm can}(1)} = -\frac{4}{c^4}\, V^{\rm ext}_{ij} , \tag{2.17c}$$

where the external potentials $V^{\rm ext}$, $V^{\rm ext}_i$, $V^{\rm ext}_{ij}$ (different from the inner potentials V, V_i, W_{ij}) are given by some explicit infinite multipole expansions of retarded spherical waves (solutions of the homogeneous wave equation in D_e): namely,

$$V^{\rm ext} = G \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{1}{r}\, M_L\left(t-\frac{r}{c}\right)\right] , \tag{2.18a}$$

$$\begin{aligned} V^{\rm ext}_i = &-G \sum_{\ell=1}^{\infty} \frac{(-)^\ell}{\ell!} \partial_{L-1} \left[\frac{1}{r}\, M^{(1)}_{iL-1}\left(t-\frac{r}{c}\right)\right] \\ &-G \sum_{\ell=1}^{\infty} \frac{(-)^\ell}{\ell!} \frac{\ell}{\ell+1} \varepsilon_{iab} \partial_{aL-1} \left[\frac{1}{r}\, S_{bL-1}\left(t-\frac{r}{c}\right)\right] , \end{aligned} \tag{2.18b}$$

$$\begin{aligned} V^{\rm ext}_{ij} = &\ G \sum_{\ell=2}^{\infty} \frac{(-)^\ell}{\ell!} \partial_{L-2} \left[\frac{1}{r}\, M^{(2)}_{ijL-2}\left(t-\frac{r}{c}\right)\right] \\ &+G \sum_{\ell=2}^{\infty} \frac{(-)^\ell}{\ell!} \frac{2\ell}{\ell+1} \partial_{aL-2} \\ &\times \left[\frac{1}{r} \varepsilon_{ab(i}\, S^{(1)}_{j)bL-2}\left(t-\frac{r}{c}\right)\right] . \end{aligned} \tag{2.18c}$$

See, e.g., (8.12) in Thorne [20]. Our notation is as follows (anticipating also future needs). Upper case latin letters denote multi-indices with the corresponding lower case letters being the number of indices, e.g., $L = i_1 i_2 \cdots i_\ell$. Similarly, $L-1 = i_1 \cdots i_{\ell-1}$, $L-2 = i_1 \cdots i_{\ell-2}$ and $aL = a i_1 \cdots i_\ell$. A product of space derivatives $\partial_i = \partial/\partial x^i$ is denoted by $\partial_L = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}$. Similarly, $x^L = x^{i_1}x^{i_2}\cdots x^{i_\ell}$ and $n^L = n^{i_1}n^{i_2}\cdots n^{i_\ell}$ where $n^i = x^i/|\mathbf{x}| = x^i/r$. The symmetric and trace-free (STF) projection is denoted with a caret, e.g., $\hat\partial_L$ or $\hat x^L$, or sometimes by, e.g., $\partial_{\langle L\rangle}$. $M^{(p)}(t)$ denotes the pth time derivative of $M(t)$, and $T_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$.

The linearized external field (2.17) and (2.18) satisfies the linearized equations $\Box h^{\mu\nu}_{{\rm can}(1)} = 0$ everywhere except at the spatial origin $r = 0$ of the coordinates. The harmonic gauge condition $\partial_\nu h^{\mu\nu}_{{\rm can}(1)} = 0$ follows from the (exact) identities

$$\partial_t V^{\rm ext} + \partial_i V^{\rm ext}_i = 0 , \tag{2.19a}$$

$$\partial_t V^{\rm ext}_i + \partial_j V^{\rm ext}_{ij} = 0 . \tag{2.19b}$$

Note that the potential $V^{\rm ext}_{ij}$ is trace-free:

$$V^{\rm ext}_{ii} = 0 . \tag{2.20}$$

As we see, the potentials (2.18) depend on two infinite sets of functions of time, $M_L(t)$ for $\ell = 0,\ldots,\infty$, and $S_L(t)$ for $\ell = 1,\cdots,\infty$. These functions are STF in their ℓ indices. They can be viewed, respectively, as some canonical mass-type and current-type multipole moments parametrizing the external canonical metric. They are completely arbitrary functions of time except that the lowest multipole moments M (mass monopole), M_i (mass dipole), and S_i (current dipole) are constant: $M^{(1)} = M^{(1)}_i = S^{(1)}_i = 0$. Note that it was assumed in [21] that the moments $M_L(t)$ and $S_L(t)$ are constant before some remote date in the past. We shall admit here that one can cover a more general situation where (for instance) the ℓth time derivatives of $M_L(t)$ and $S_L(t)$ become asymptotically constant when $t \to -\infty$, corresponding to a situation of initial scattering in the infinite past. Furthermore, we shall relax without justification the assumption made in [21] that the multipole expansions are finite; that is, we assume that we really have two infinite sets of moments $M_L(t)$ and $S_L(t)$. These two amendments almost certainly have no incidence on the results derived in this paper.

Starting with the linearized metric (2.17)–(2.18), one constructs iteratively the nth coefficient $h^{\mu\nu}_{{\rm can}(n)}$ of the series (2.16) by the formula

$$h^{\mu\nu}_{{\rm can}(n)} = {\rm FP}_{B=0}\, \Box^{-1}_R \left[r^B \Lambda^{\mu\nu}_{{\rm can}(n)}\right] + q^{\mu\nu}_{{\rm can}(n)} . \tag{2.21}$$

The first term in this formula involves $\Lambda^{\mu\nu}_{{\rm can}(n)}$, which is defined to be the coefficient of G^n in the expansion of the effective gravitational source $\Lambda^{\mu\nu}(h)$ in the right-hand-side of Einstein's equation (1.2) and computed with the canonical field (2.16). That is,

$$\Lambda^{\mu\nu}(h_{\rm can}) = G^2 \Lambda^{\mu\nu}_{{\rm can}(2)} + \cdots + G^n \Lambda^{\mu\nu}_{{\rm can}(n)} + \cdots . \tag{2.22}$$

Since $\Lambda^{\mu\nu}$ is at least quadratic in h, $\Lambda^{\mu\nu}_{{\rm can}(n)}$ for any n is a function only of the previous $n-1$ coefficients

$h_{\text{can}(1)}, \ldots, h_{\text{can}(n-1)}$. For instance, we have (with evident notation)

$$\Lambda^{\mu\nu}_{\text{can}(2)} = N^{\mu\nu}(h_{\text{can}(1)}, h_{\text{can}(1)}) \ , \tag{2.23a}$$

$$\Lambda^{\mu\nu}_{\text{can}(3)} = N^{\mu\nu}(h_{\text{can}(1)}, h_{\text{can}(2)}) + N^{\mu\nu}(h_{\text{can}(2)}, h_{\text{can}(1)}) + M^{\mu\nu}(h_{\text{can}(1)}, h_{\text{can}(1)}, h_{\text{can}(1)}) \ , \tag{2.23b}$$

where $N^{\mu\nu}$ and $M^{\mu\nu}$ are given by (1.4)–(1.6). The retarded integral operator $\Box_R^{-1}$, which is defined in (2.5), acts on the source $\Lambda^{\mu\nu}_{\text{can}(n)}$ but multiplied by an analytic continuation factor r^B, where $r = |\mathbf{x}|$ and B is a complex number. The introduction of this factor is required because the linearized metric $h_{\text{can}(1)}$ of (2.17) and (2.18), and all subsequent metrics $h_{\text{can}(n)}$ and sources $\Lambda_{\text{can}(n)}$, are valid only in the exterior of the source (in D_e) and are singular at the spatial origin of the coordinates, $r = 0$, located within the source. It has been shown in [21] that for B a complex number the retarded integral $f(B) = \Box_R^{-1}(r^B \Lambda_{\text{can}(n)})$ defines an analytic function of B all over the complex plane except in general at integer values of B. Near the value $B = 0$, $f(B)$ admits a Laurent expansion of the type $f(B) = \Sigma a_p B^p$, where $p \in \mathbb{Z}$. The coefficient of B^0 in this expansion, i.e., a_0, is what we call the *finite part at* $B = 0$ (or $\text{FP}_{B=0}$) of the retarded integral $f(B)$. This is the first term in (2.21); it satisfies $\Box a_0 = \Lambda_{\text{can}(n)}$ (and is also singular at the origin). Thus the introduction of the analytic continuation factor r^B is a mean (and a convenient one) to obtain a solution of the Einstein equation (1.2) in D_e — this is the only thing we need (see [21] for more details about this way of proceeding). The second term $q_{\text{can}(n)}$ in (2.21) is a particular retarded solution of the wave equation, i.e., $\Box q_{\text{can}(n)} = 0$, whose divergence is the opposite of the divergence of the first term, and thus which permits ensuring the satisfaction of the harmonic coordinate condition $\partial_\nu h^{\mu\nu}_{\text{can}(n)} = 0$. The precise definition of the term $q_{\text{can}(n)}$ is reported in Appendix A where we control its order of magnitude in the post-Newtonian expansion.

The external field (2.16), in which we have (2.17), (2.18), and (2.21), represents the most general solution of Einstein's equations in D_e, and is parametrized by the arbitrary canonical multipole moments $M_L(t)$ and $S_L(t)$. Now the point is that one knows (see [22,1,3]) how to relate the observable "radiative" multipole moments $U_L(t)$ and $V_L(t)$, parametrizing the field in the distant wave zone of the source (where the detector is located), to the moments $M_L(t)$ and $S_L(t)$ (see Sec. IV below). Therefore, what is only needed in order to compute the distant wave field is to consider the near-zone expansion, or expansion when $c \to +\infty$ or $\varepsilon \to 0$, of the external field (2.16) up to some suitable order, so as to give by matching to the inner metric constructed in Sec. II A a suitably accurate physical meaning to the moments $M_L(t)$ and $S_L(t)$ in terms of the source's parameters. Equivalently, this means that we must consider the external field (2.16) in that part of the external domain D_e which belongs to the near zone, i.e., $D_e \cap D_i = \{(\mathbf{x}, t)/r_e < r < r_i\}$, in which we can simultaneously expand the external field when $\varepsilon \to 0$ and keep its multipole moment structure. First of all, from the fact that an arbitrary nonlinear coefficient $h^{\mu\nu}_{\text{can}(n)}$ in (2.16) is of order $O(2n, 2n+1, 2n)$ when $\varepsilon \to 0$ [see (5.5) in [21]], we find that the neglect of all nonlinear iterations with $n \geq 4$ permits the computation of the field up to the order $O(8,9,8)$ when $\varepsilon \to 0$, i.e.,

$$h^{\mu\nu}_{\text{can}} = G h^{\mu\nu}_{\text{can}(1)} + G^2 h^{\mu\nu}_{\text{can}(2)} + G^3 h^{\mu\nu}_{\text{can}(3)} + O(8,9,8) \ . \tag{2.24}$$

Second, it is shown in Appendix A that the second terms $q^{\mu\nu}_{\text{can}(2)}$ and $q^{\mu\nu}_{\text{can}(3)}$ in the definitions of the quadratic and cubic coefficients (2.21) have (at least) the following orders of magnitude when $\varepsilon \to 0$:

$$q^{\mu\nu}_{\text{can}(2)} = O(7,7,7) \ , \tag{2.25a}$$

$$q^{\mu\nu}_{\text{can}(3)} = O(8,7,8) \ . \tag{2.25b}$$

Therefore, from (2.21), we can write the expansion when $\varepsilon \to 0$ of the canonical external field (2.24) in the form

$$h^{\mu\nu}_{\text{can}} = G h^{\mu\nu}_{\text{can}(1)} + \text{FP}_{B=0}\, \Box_R^{-1} \left[r^B (G^2 \Lambda^{\mu\nu}_{\text{can}(2)} + G^3 \Lambda^{\mu\nu}_{\text{can}(3)}) \right] + O(7,7,7) \ , \tag{2.26}$$

where the remainder term $O(7,7,7)$ is dominated by the contribution (2.25a) of $q^{\mu\nu}_{\text{can}(2)}$.

The source terms $\Lambda_{\text{can}(2)}$ and $\Lambda_{\text{can}(3)}$ in (2.26) are given more explicitly by (2.23). To compute their sum with an accuracy consistent with the remainder in (2.26), we must know the quadratic metric $h_{\text{can}(2)}$ up to the order $O(6,5,6)$. This is of course analogous to the computation we have done in Sec. II A, where we had first to solve Einstein's equations up to the order $O(6,5,6)$ before reaching the looked-for order $O(8,7,8)$. The quadratic source at the order $O(6,5,6)$ is readily obtained by substituting (2.17) and (2.18) into (2.23a) and discarding $O(6,5,6)$ terms. We obtain

$$G^2 \Lambda^{00}_{\text{can}(2)} = -\frac{14}{c^4} \partial_k V^{\text{ext}} \partial_k V^{\text{ext}} + O(6) \ , \tag{2.27a}$$

$$G^2 \Lambda^{0i}_{\text{can}(2)} = O(5) \ , \tag{2.27b}$$

$$G^2 \Lambda^{ij}_{\text{can}(2)} = \frac{4}{c^4} \left\{ \partial_i V^{\text{ext}} \partial_j V^{\text{ext}} - \frac{1}{2} \delta_{ij} \partial_k V^{\text{ext}} \partial_k V^{\text{ext}} \right\} + O(6) \ , \tag{2.27c}$$

from which we deduce

$$G^2 h^{00}_{\text{can}(2)} = -\frac{7}{c^4} (V^{\text{ext}})^2 + O(6) \ , \tag{2.28a}$$

$$G^2 h^{0i}_{\text{can}(2)} = O(5) \ , \tag{2.28b}$$

$$G^2 h^{ij}_{\text{can}(2)} = -\frac{4}{c^4} Z^{\text{ext}}_{ij} + O(6) \ , \tag{2.28c}$$

where we have introduced the new external potential

$$Z^{\text{ext}}_{ij} = \text{FP}_{B=0}\, \Box_R^{-1} \left[r^B \left(-\partial_i V^{\text{ext}} \partial_j V^{\text{ext}} + \frac{1}{2} \delta_{ij} \partial_k V^{\text{ext}} \partial_k V^{\text{ext}} \right) \right] \ . \tag{2.29}$$

The justification of (2.28) is as follows. We know that the post-Newtonian expansion of the regularized retarded operator $\mathrm{FP}\Box_R^{-1}$ acting on terms belonging to the quadratic source $\Lambda^{\mu\nu}_{\mathrm{can}(2)}$ is equal to the expansion obtained from the action of the regularized "instantaneous" operator $\mathrm{FP}I^{-1} := \mathrm{FP}\Sigma_{k=0}^{\infty}(\partial/c\partial t)^{2k}\Delta^{-k-1}$, where Δ^{-k-1} is the $(k+1)$th iteration of the Poisson operator Δ^{-1}, modulo negligible terms of order $O(10,9,8)$ [see, e.g., (3.7) and (3.21), with $n = 2$, in [24]]. Thus, $\mathrm{FP}\Box_R^{-1}$ acting on the remainder $O(6,5,6)$ in (2.27) is itself of order $O(6,5,6)$. Furthermore, we know that $\mathrm{FP}\Delta^{-1}$ (namely the first term in $\mathrm{FP}I^{-1}$) gives simply $(U^{\mathrm{ext}})^2$ when acting on $\Delta[(U^{\mathrm{ext}})^2]$, where U^{ext} is the "Newtonian" potential associated with V^{ext} and given by $U^{\mathrm{ext}} = G\Sigma\frac{(-)^\ell}{\ell!}(\partial_L r^{-1})M_L(t)$ (see, e.g., [1], p. 395). Since $V^{\mathrm{ext}} = U^{\mathrm{ext}} + O(2)$, we have $\mathrm{FP}\,\Box_R^{-1}(\Delta[(V^{\mathrm{ext}})^2]) = (V^{\mathrm{ext}})^2 + O(2)$ as has been used in (2.28a). Note that this last fact implies that the trace $Z^{\mathrm{ext}} \equiv Z^{\mathrm{ext}}_{ii}$ of the potential (2.29) satisfies

$$Z^{\mathrm{ext}} = \frac{1}{4}(V^{\mathrm{ext}})^2 + O(2) \ . \tag{2.30}$$

Using now (2.30) and (2.20), we can write the canonical field up to the order $O(6,5,6)$ in a form which is formally identical to that of the inner field (2.9), i.e.,

$$h^{00}_{\mathrm{can}} = -\frac{4}{c^2}\ V^{\mathrm{ext}} + \frac{4}{c^4}\ [W^{\mathrm{ext}} - 2(V^{\mathrm{ext}})^2] + O(6) \ , \tag{2.31a}$$

$$h^{0i}_{\mathrm{can}} = -\frac{4}{c^3}\ V^{\mathrm{ext}}_i + O(5) \ , \tag{2.31b}$$

$$h^{ij}_{\mathrm{can}} = -\frac{4}{c^4}\ W^{\mathrm{ext}}_{ij} + O(6) \ , \tag{2.31c}$$

where we have defined

$$W^{\mathrm{ext}}_{ij} = V^{\mathrm{ext}}_{ij} + Z^{\mathrm{ext}}_{ij} \ , \tag{2.32a}$$

whose trace is

$$W^{\mathrm{ext}} \equiv W^{\mathrm{ext}}_{ii} = Z^{\mathrm{ext}} = \frac{1}{4}(V^{\mathrm{ext}})^2 + O(2) \ . \tag{2.32b}$$

Finally, since the canonical field (2.31) has the same form in terms of the external potentials as the inner field (2.9) has in terms of the inner potentials, it is clear that the source term $G^2\Lambda^{\mu\nu}_{\mathrm{can}(2)} + G^3\Lambda^{\mu\nu}_{\mathrm{can}(3)}$ in the right-hand-side of (2.26) will be equal modulo the same order $O(8,7,8)$ as in (2.10) to the truncated source $\overline{\Lambda}^{\mu\nu}$ defined in (2.12), but computed with the external potentials V^{ext}, V^{ext}_i, and W^{ext}_{ij} instead of the inner potentials V, V_i, W_{ij}. (Indeed, never a Laplace or d'Alembertian operator enters in the effective gravitational source of Einstein's equations, which would give a different result when acting on an external potential or on an inner one.) Thus, we can write

$$G^2\Lambda^{\mu\nu}_{\mathrm{can}(2)} + G^3\Lambda^{\mu\nu}_{\mathrm{can}(3)} = \overline{\Lambda}^{\mu\nu}(V^{\mathrm{ext}}, V^{\mathrm{ext}}_i, W^{\mathrm{ext}}_{ij}) + O(8,7,8) \ , \tag{2.33}$$

where the right-hand-side is obtained from (2.12) by the simple replacement V, V_i, $W_{ij} \to V^{\mathrm{ext}}, V^{\mathrm{ext}}_i, W^{\mathrm{ext}}_{ij}$. Inserting (2.33) into (2.26), and recalling that $\mathrm{FP}\Box_R^{-1}$ acting on $O(8,7,8)$ is also $O(8,7,8)$, yields our looked-for expression of the external field, namely

$$h^{\mu\nu}_{\mathrm{can}} = Gh^{\mu\nu}_{\mathrm{can}(1)} + \mathrm{FP}_{B=0}\,\Box_R^{-1}\left[r^B\overline{\Lambda}^{\mu\nu}(V^{\mathrm{ext}}, V^{\mathrm{ext}}_i, W^{\mathrm{ext}}_{ij})\right] + O(7,7,7) \ . \tag{2.34}$$

It is to be noticed that the remainder in (2.34) is $O(7,7,7)$ instead of the remainder $O(8,7,8)$ in the inner field (2.14) because it involves the not controlled contribution $O(7,7,7)$ of $q^{\mu\nu}_{\mathrm{can}(2)}$ in (2.25a). In the next section we match the external field (2.34) to the corresponding inner field (2.14).

III. MATCHING OF THE INTERNAL AND EXTERNAL FIELDS

A. Coordinate transformation between the internal and external fields

We require that the internal field $h^{\alpha\beta}$ constructed in Sec. II A and the external field $h^{\mu\nu}_{\mathrm{can}}$ constructed in Sec. II B are *isometric* in their common domain of validity, which is the exterior near-zone domain $D_i \cap D_e = \{(\mathbf{x},t)/\, r_e < r < r_i\}$ of the source. Denoting by x^α the harmonic coordinates used in the inner domain D_i (see Sec. II A), and by x^μ_{can} the canonical harmonic coordinates used in the exterior domain D_e (Sec. II B), we thus look for a compatible coordinate transformation

$$x^\mu_{\mathrm{can}}(x) = x^\mu + \varphi^\mu(x) \ , \tag{3.1}$$

where the vector $\varphi^\mu(x)$ is assumed to be in the form of a multipolar and post-Newtonian expansion appropriate in $D_i \cap D_e$. [In Sec. II B, we have for notational convenience abusively denoted by x^μ what really are the canonical harmonic coordinates x^μ_{can}.] Since the two coordinate systems x^μ and x^μ_{can} are harmonic, the vector φ^μ satisfies the (exact) relation

$$\Box\varphi^\mu + h^{\alpha\beta}(x)\partial^2_{\alpha\beta}\varphi^\mu = 0 \ , \tag{3.2}$$

where $\Box = \eta^{\alpha\beta}\partial^2_{\alpha\beta}$. The (also exact) transformation law of the field deviation $h^{\alpha\beta}$ under the change of coordinates (3.1) is given by

$$\eta^{\mu\nu} + h^{\mu\nu}_{\mathrm{can}}(x_{\mathrm{can}}) = \frac{1}{|J|}(\delta^\mu_\alpha + \partial_\alpha\varphi^\mu)(\delta^\nu_\beta + \partial_\beta\varphi^\nu) \times[\eta^{\alpha\beta} + h^{\alpha\beta}(x)] \ , \tag{3.3}$$

where $J = \det(\partial x_{\mathrm{can}}/\partial x)$ denotes the Jacobian determinant of the coordinate transformation.

We now expand the transformation law (3.3) when $\varepsilon \to 0$. The field deviation $h^{\alpha\beta}$ in the right-hand side of (3.3) is by (2.9) of order $O(2,3,4)$ when $\varepsilon \to 0$. Furthermore, let us assume that the vector φ^μ in the coordinate transformation (3.1) is of order

$$\varphi^\mu = O(3,4) \ . \tag{3.4}$$

This assumption (which has already been made in previous works [1,2]) is proved below, when we show that it leads to a consistent matching. We can then easily see that the transformation law (3.3) reduces up to the order $O(6,7,8)$ to a linear transformation,

$$h^{\mu\nu}_{\rm can}(x) = h^{\mu\nu}(x) + \partial\varphi^{\mu\nu}(x) + O(6,7,8) \ , \tag{3.5}$$

where we have expressed both sides of the equation in terms of the inner coordinates x^μ, and where $\partial\varphi^{\mu\nu}$ denotes the linear part of the coordinate transformation given by

$$\partial\varphi^{\mu\nu} = \partial^\mu\varphi^\nu + \partial^\nu\varphi^\mu - \eta^{\mu\nu}\partial_\lambda\varphi^\lambda \ . \tag{3.6}$$

By taking the divergence of (3.6) and using (3.2) we obtain

$$\partial_\nu\partial\varphi^{\mu\nu} = \Box\varphi^\mu = -h^{\rho\sigma}\partial^2_{\rho\sigma}\varphi^\mu \ , \tag{3.7a}$$

from which we deduce the order of magnitude

$$\partial_\nu\partial\varphi^{\mu\nu} = O(7,8) \ . \tag{3.7b}$$

Note that one could have *a priori* expected some nonlinear terms to appear at the order ε^6 in the ij component of Eq. (3.5) — for instance, terms such as $h^{00}\partial^{(i}\varphi^{j)}$ or $h^{0(i}\partial^{j)}\varphi^0$. Such nonlinear terms are, however, absent at this order (as was also found in [2]). On the contrary, some nonlinear terms arise at the order ε^6 in the 00 component of (3.5). These terms, which will be needed in the following, are obtained by a short computation showing that the 00 component of (3.5), now valid up to the order $O(8)$, is given by

$$\begin{aligned} h^{00}_{\rm can}(x) = h^{00}(x) + \partial\varphi^{00} + 2h^{0\mu}\partial_\mu\varphi^0 - \partial_\mu(h^{00}\varphi^\mu) \\ +\partial_i\varphi^0\partial_i\varphi^0 + O(8) \ . \end{aligned} \tag{3.8}$$

Here also we have expressed both sides of the equation in terms of the inner coordinate system x^μ.

B. Matching of the compact-supported potentials V and V_i

A requisite in the matching procedure is to find the relations linking the *multipole expansions* outside the source of the inner potentials V, V_i, and W_{ij} and the external potentials $V^{\rm ext}$, $V_i^{\rm ext}$, and $W_{ij}^{\rm ext}$. We deal in this subsection with the case of the compact-supported potentials V and V_i. The more complicated case of the noncompact-supported potential W_{ij} is reported in the next subsection.

The inner and outer fields have been shown in Sec. II to take, up to the order $O(6,5,6)$, the same functional forms (2.9) and (2.31) in terms of their respective potentials. On the other hand, the coordinate transformation between them takes, up to this order (and even up to a higher order), the linear form (3.5) with (3.6). We first substitute into the sum of the 00 component and of the spatial trace ii of (3.5) the inner and outer fields (2.9) and (2.31). This leads to an equation whose solution is easily seen to be

$$V^{\rm ext} = V + c\,\partial_t\varphi^0 + O(4) \ . \tag{3.9}$$

[Indeed, we have $\varphi^0 = O(3)$.] Similarly, by inserting (2.9) and (2.31) into the $0i$ component of (3.5), we obtain

$$V_i^{\rm ext} = V_i - \frac{c^3}{4}\,\partial_i\varphi^0 + O(2) \ . \tag{3.10}$$

Note that (3.9) is valid with the inclusion of relativistic corrections ε^2, while (3.10) is valid at the nonrelativistic level only.

The relations (3.9) and (3.10) are *numerically* true in the region $D_i \cap D_e$. We now transform them into *matching equations*, i.e., equations relating mathematical expressions of the same nature. To do this, we need only to replace the inner potentials in the right-hand-sides of (3.9) and (3.10) by their multipole expansions valid *outside* the source. This is simple because V and V_i are the retarded integrals of the compact-supported mass and current densities σ and σ_i [see (2.3)]. Thus the multipole expansions $\mathcal{M}(V)$ and $\mathcal{M}(V_i)$ of V and V_i are given by

$$\mathcal{M}(V) = G\sum_{\ell=0}^{\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathcal{V}^L\left(t-\frac{r}{c}\right)\right] \ , \tag{3.11a}$$

$$\mathcal{M}(V_i) = G\sum_{\ell=0}^{\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathcal{V}_i^L\left(t-\frac{r}{c}\right)\right] \ , \tag{3.11b}$$

where the multipole moments $\mathcal{V}^L(t)$ and $\mathcal{V}_i^L(t)$ are given by explicit integrals extending over the mass and current densities in the source: namely,

$$\mathcal{V}^L(t) = \int d^3\mathbf{y}\,\hat{y}_L\int_{-1}^{1}dz\,\delta_\ell(z)\sigma\left(\mathbf{y}, t + z\frac{|\mathbf{y}|}{c}\right) \ , \tag{3.12a}$$

$$\mathcal{V}_i^L(t) = \int d^3\mathbf{y}\,\hat{y}_L\int_{-1}^{1}dz\,\delta_\ell(z)\sigma_i\left(\mathbf{y}, t + z\frac{|\mathbf{y}|}{c}\right) \ . \tag{3.12b}$$

In (3.12), $\hat{y}_L$ denotes the trace-free part of $y_L = y_{i_1}y_{i_2}\cdots y_{i_\ell}$, and $\delta_\ell(z)$ is given by

$$\delta_\ell(z) = \frac{(2\ell+1)!!}{2^{\ell+1}\ell!}(1-z^2)^\ell \quad , \qquad \int_{-1}^{1}dz\,\delta_\ell(z) = 1 \ . \tag{3.13}$$

The formulas (3.11)–(3.13) have been proved in the appendix B of [1]. Now, we have $V = \mathcal{M}(V)$ and $V_i = \mathcal{M}(V_i)$ in $D_i \cap D_e$; hence, we can write

$$V^{\rm ext} = \mathcal{M}(V) + c\,\partial_t\varphi^0 + O(4) \ , \tag{3.14}$$

$$V_i^{\rm ext} = \mathcal{M}(V_i) - \frac{c^3}{4}\,\partial_i\varphi^0 + O(2) \ . \tag{3.15}$$

These matching equations relate the multipole expanded external potentials $V^{\rm ext}$, $V_i^{\rm ext}$ given by (2.18) to the multipole expansions $\mathcal{M}(V)$, $\mathcal{M}(V_i)$ of the inner potentials, and can be used to obtain the expressions of the canonical moments $M_L(t)$ and $S_L(t)$ entering (2.18) in terms of the source's parameters, with first relativistic accuracy ε^2 in $M_L(t)$ and nonrelativistic accuracy in $S_L(t)$. One can also obtain, in this way, the coordinate change φ^0 (in the form of a multipole expansion). This was the method followed in [1], and we can check again that the results of [1] are indeed equivalent to (3.14)–(3.15). (We shall in particular recover below the results of [1] by a more general method.)

C. Matching of the noncompact-supported potential W_{ij}

By inserting in the spatial components ij of the transformation law (3.5) the inner and outer fields (2.9) and (2.31), we obtain

$$W_{ij}^{\rm ext} = W_{ij} - \frac{c^4}{4}[\partial_i\varphi^j + \partial_j\varphi^i - \delta_{ij}(\partial_0\varphi^0 + \partial_k\varphi^k)] + O(2) \ . \tag{3.16}$$

This equation is valid at the nonrelativistic level only. Like (3.9) and (3.10), it is numerically true in the region $D_i \cap D_e$. To transform (3.16) into a matching equation, we must first compute the multipole expansion, valid outside the source, of the inner potential W_{ij} which is, contrarily to the potentials V and V_i, of noncompact support.

Recall that $W_{ij}^{\rm ext}$ in the left-hand-side of (3.16) is the sum of the potential $V_{ij}^{\rm ext}$ parametrizing the linear metric (2.17c), and of the nonlinear potential $Z_{ij}^{\rm ext} = Z_{ij}(V^{\rm ext})$ defined in (2.29). Now $V^{\rm ext}$ has been matched to the multipole expansion of V in the previous subsection; $V^{\rm ext} = \mathcal{M}(V) + O(2)$ as deduced from (3.14). Thus we can write the external potential $W_{ij}^{\rm ext}$ as

$$W_{ij}^{\rm ext} = V_{ij}^{\rm ext} + Z_{ij}(\mathcal{M}(V)) + O(2) \ , \tag{3.17}$$

where $Z_{ij}(\mathcal{M}(V))$ is given by

$$Z_{ij}(\mathcal{M}(V)) = {\rm FP}_{B=0} \ \Box_R^{-1}\left[r^B\left(-\partial_i\mathcal{M}(V)\partial_j\mathcal{M}(V) + \frac{1}{2}\,\delta_{ij}\partial_k\mathcal{M}(V)\partial_k\mathcal{M}(V)\right)\right] \ . \tag{3.18}$$

Note that it is crucial to replace $V^{\rm ext}$ in $Z_{ij}(V^{\rm ext})$ not by V, but by the *multipole* expansion $\mathcal{M}(V)$ of V [modulo $O(2)$]. Indeed, contrarily to $V^{\rm ext} = V + O(2)$ which is true only in the region $D_i \cap D_e$, the matching equation $V^{\rm ext} = \mathcal{M}(V) + O(2)$ is an identity which is valid "everywhere" and in particular on the whole past null cone, issued from the considered field point, on which depends the retarded integral in (3.18).

As for the inner potential W_{ij}, we recall that it is given by

$$W_{ij} = \Box_R^{-1}\left[-4\pi G\,\sigma_{ij} - \partial_i V\partial_j V + \frac{1}{2}\delta_{ij}\partial_k V\partial_k V\right] \ . \tag{3.19}$$

We shall now prove that the multipole expansion $\mathcal{M}(W_{ij})$, valid outside of source, of the (noncompact-supported) potential (3.19) reads as

$$\mathcal{M}(W_{ij}) = Z_{ij}[\mathcal{M}(V)] + G\sum_{\ell=0}^{\infty}\frac{(-)^\ell}{\ell!}\partial_L \times\left[\frac{1}{r}\mathcal{W}_{ij}^L\left(t-\frac{r}{c}\right)\right] \ , \tag{3.20}$$

where the first term is given by (3.18), and where the multipole moments $\mathcal{W}_{ij}^L(t)$ in the second term are given by

$$\mathcal{W}_{ij}^L(t) = {\rm FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\int_{-1}^1 dz\,\delta_\ell(z) \times\left[\sigma_{ij} + \frac{1}{4\pi G}\left(\partial_i V\partial_j V - \frac{1}{2}\delta_{ij}\partial_k V\partial_k V\right)\right] \times\left(\mathbf{y}, t + z\frac{|\mathbf{y}|}{c}\right) \ , \tag{3.21}$$

where $\delta_\ell(z)$ is defined in (3.13).

The proof of (3.20) and (3.21) goes as follows. The first term in W_{ij}, involving the compact-supported matter stresses σ_{ij}, can be treated by formulas identical to (3.11)–(3.13). We thus focus the proof on the second term involving $\partial_i V\partial_j V$, from which we easily deduce the last term. Let us consider the difference between (minus) this term and the corresponding term in (3.18) involving the finite part at $B = 0$ of the retarded integral: i.e.,

$$X_{ij} = \Box_R^{-1}[\partial_i V\partial_j V] - {\rm FP}_{B=0}\,\Box_R^{-1}\,[r^B\partial_i\mathcal{M}(V)\partial_j\mathcal{M}(V)]. \tag{3.22}$$

The analytic continuation factor r^B in the second retarded integral deals with the singular behavior of the multipole expansions near the spatial origin $r = 0$. The first integral does not need any regularization factor because the integrand is perfectly regular at $r = 0$. However, in order to make a better comparison between the two integrals, let us introduce the same factor r^B, and the "finite part" prescription, into the first integral. Since this integral is convergent, the finite part prescription simply gives back the value of the integral. Hence we can rewrite (3.22) as

$$X_{ij} = {\rm FP}_{B=0}\,\Box_R^{-1}\,\{r^B[\partial_i V\partial_j V - \partial_i\mathcal{M}(V)\partial_j\mathcal{M}(V)]\}. \tag{3.23}$$

The important point is that, under the form (3.23), we see that X_{ij} *does* have a compact support limited to the distribution of matter in the source. Indeed, outside the compact support of the source (i.e., in the domain D_e), the potential V numerically agrees with its multipole expansion $\mathcal{M}(V)$, and therefore the integrand in (3.23) is identically zero. (It is at this point that we need to assume that the multipole expansions in the canonical construction of the exterior metric in D_e involve an infinite number of multipoles.) We can thus compute the multipole expansion $\mathcal{M}(X_{ij})$ of (3.23) in D_e by exactly the same formulas (3.11)–(3.13) as was used for the compact-supported potentials V and V_i. The only difference is that the analytic continuation factor r^B must be kept inside the integral, and that one must apply to the formulas the finite part prescription. One is thus led to the multipole expansion

$$\mathcal{M}(X_{ij}) = \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!}\, \partial_L \left[\frac{1}{r}\mathcal{X}^L_{ij}\left(t-\frac{r}{c}\right)\right] , \qquad (3.24)$$

where the multipole moments $\mathcal{X}^L_{ij}(t)$ are given by

$$\mathcal{X}^L_{ij}(t) = \frac{-1}{4\pi}\,\mathrm{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B \hat{y}_L \int_{-1}^{1} dz\,\delta_\ell(z)[\partial_i V\partial_j V - \partial_i\mathcal{M}(V)\partial_j\mathcal{M}(V)]\left(\mathbf{y}, t+z\frac{|\mathbf{y}|}{c}\right) . \qquad (3.25)$$

[The finite part commutes with the derivation operator ∂_L in (3.24).] We now prove that the second term in (3.25), which involves the multipole expansion $\mathcal{M}(V)$ in the integrand, is in fact *zero* by analytic continuation. Indeed, the multipole expansion $\mathcal{M}(V)(\mathbf{y},t)$ is by (3.11a) an expansion of the type $\Sigma\hat{n}_{L'}|\mathbf{y}|^{-p}\mathcal{F}(t-|\mathbf{y}|/c)$, where L' is some multi-index, p is some integer, and $\mathcal{F}$ some function of time (all indices suppressed). Thus, $\mathcal{M}(V)(\mathbf{y},t+z|\mathbf{y}|/c)$ is of the type $\Sigma\hat{n}_{L'}|\mathbf{y}|^{-p}\mathcal{F}[t+(z-1)|\mathbf{y}|/c]$. By differentiating and squaring the latter expansion, we see that the product $\partial_i\mathcal{M}(V)\partial_j\mathcal{M}(V)(\mathbf{y},t+z|\mathbf{y}|/c)$ entering (3.25) is an expansion of the same type $\Sigma\hat{n}_{L''}|\mathbf{y}|^{-q}\mathcal{G}(t+(z-1)|\mathbf{y}|/c)$, with L'' a multi-index, q an integer, and $\mathcal{G}$ a function of time. We then expand by Taylor's formula each function $\mathcal{G}$ when $c\to+\infty$. This introduces many powers of $|\mathbf{y}|$ and of $(z-1)$. Multiplying by $\delta_\ell(z)$ and integrating over z, we find a multipole expansion of the type $\Sigma\hat{n}_{L''}|\mathbf{y}|^k\mathcal{H}(t)$, where k is an integer and $\mathcal{H}$ another function of time; then multiplying by $\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L$ and performing the angular integration (using $d^3\mathbf{y}=d|\mathbf{y}||\mathbf{y}|^2d\Omega$ and $\int d\Omega\hat{n}_{L''}\hat{n}_L = \text{const}\times\ \delta_{\ell,\ell''}$), we arrive at a multipole expansion of the type $\Sigma\hat{n}_L\mathcal{H}(t)\int_0^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+m}$, where m is an integer. Finally, each of the latter integrals $\int_0^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+m}$ is zero by analytic continuation. [Indeed, we cut the integrals into two pieces, $I_1=\int_0^Y d|\mathbf{y}||\mathbf{y}|^{B+m}$ and $I_2=\int_Y^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+m}$, where Y is some constant >0. By choosing the real part of B to be such that $\mathrm{Re}\,B+m>-1$, we compute $I_1=Y^{B+m+1}/(B+m+1)$; and by choosing the real part of B to be such that $\mathrm{Re}\,B+m<-1$, we compute $I_2=-Y^{B+m+1}/(B+m+1)$. Then both I_1 and I_2 can be analytically continued for all complex values of B except the single value $-m-1$. The integral $\int_0^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+m}$ is the sum of the analytic continuations of I_1 and I_2, and is thus identically zero on the whole complex plane (including $-m-1$).] Note that the proof that the second term in (3.25) is zero can be easily extended to the case where we have, instead of $\partial_i\mathcal{M}(V)\partial_j\mathcal{M}(V)$, an arbitrary nonlinear multipolar product of the type $\partial_\mu\mathcal{M}_1\partial_\nu\mathcal{M}_2\mathcal{M}_3\cdots$, where $\mathcal{M}_1(\mathbf{y},t)$, $\mathcal{M}_2(\mathbf{y},t)$, $\mathcal{M}_3(\mathbf{y},t)\cdots$ denote formal multipolar and post-Newtonian expansions of the type $\Sigma\hat{n}_{L'}|\mathbf{y}|^p(\ln|\mathbf{y}|)^q\mathcal{K}(t)$, where p,q are integers (such expansions are known to arise in higher-order nonlinear approximations of the external field [21]). Thus, the multipole moments $\mathcal{X}^L_{ij}(t)$ in (3.25) can be simply written as

$$\mathcal{X}^L_{ij}(t) = \frac{-1}{4\pi}\,\mathrm{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\int_{-1}^{1}dz\delta_\ell(z)(\partial_iV\partial_jV) \times\left(\mathbf{y},t+z\frac{|\mathbf{y}|}{c}\right) , \qquad (3.26)$$

and the expressions (3.20) and (3.21) are then easily deduced. Indeed, we write W_{ij} as the sum of $Z_{ij}[\mathcal{M}(V)]$, of the compact-support integral $\Box_R^{-1}[-4\pi G\,\sigma_{ij}]$ and of the term $-X_{ij}+\frac{1}{2}\,\delta_{ij}X_{kk}$. The multipole expansion $\mathcal{M}(W_{ij})$ is thus the sum of $Z_{ij}[\mathcal{M}(V)]$, which is already a multipole expansion, of the multipole expansion of $\Box_R^{-1}[-4\pi G\sigma_{ij}]$ computed by formulas such as (3.11) and (3.12), and of $-\mathcal{M}(X_{ij})+\frac{1}{2}\delta_{ij}\mathcal{M}(X_{kk})$ as deduced from (3.24) and (3.26). We add the factor $|\mathbf{y}|^B$ and the finite part prescription into the moments of the expansion of $\Box_R^{-1}[-4\pi G\sigma_{ij}]$ (which are convergent anyway), and the result is $\mathcal{M}(W_{ij})$ given by (3.20) and (3.21).

One should note the remarkable role of the analytic continuation factor $|\mathbf{y}|^B$ in (3.25) and (3.26). The integral (3.25) is of compact support and is thus perfectly well defined at infinity $|\mathbf{y}|\to+\infty$. The role of the factor $|\mathbf{y}|^B$ in (3.25) is to deal with the singular behavior at the *origin* $|\mathbf{y}|=0$ of the second term in the integrand, involving the multipole expansion $\mathcal{M}(V)$. That is, one can compute (3.25) by choosing $\mathrm{Re}\,B$ to be a large *positive* number so as to compensate the bad behavior of the integral at the origin, then analytically continuing the integral near $B=0$ and deducing the finite part at $B=0$. On the contrary, the integral (3.26) is perfectly well defined near the origin $|\mathbf{y}|=0$, but it is not apparently of compact support. The factor $|\mathbf{y}|^B$ in (3.26) then deals with the *a priori* bad behavior of the integral at *infinity* $|\mathbf{y}|\to\infty$, where $\hat{y}_L\partial_iV\partial_jV$ behaves for large ℓ like a large power $\sim|\mathbf{y}|^{\ell-4}$ blowing up at infinity. That is, one can compute (3.26) by choosing $\mathrm{Re}\,B$ to be a large *negative* number so as to make the integral convergent when $|\mathbf{y}|\to\infty$. These two different procedures, $\mathrm{Re}\,B$ a large positive number in (3.25) and a large negative number in (3.26), give the same numerical result, as we have just proved. In conclusion, the form (3.26) although not apparently of compact support is however, thanks to the properties of analytic continuation, numerically equal to the compact-supported form (3.25). It is evident that the form (3.26), where the analytic continuation factor deals with the behavior of the integral at *infinity* from the source, is the one which should be used in applications.

Equation (3.16) can finally be replaced by the matching equation

$$W^{\mathrm{ext}}_{ij} = \mathcal{M}(W_{ij}) - \frac{c^4}{4}[\partial_i\varphi^j+\partial_j\varphi^i-\delta_{ij}(\partial_0\varphi^0+\partial_k\varphi^k)] +O(2) , \qquad (3.27)$$

where the multipole expansion $\mathcal{M}(W_{ij})$ is given by (3.20) and (3.21). This equation can be equivalently rewritten as

$$V_{ij}^{\rm ext} = G\sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!}\,\partial_L\left[\frac{1}{r}\mathcal{W}_{ij}^L\left(t-\frac{r}{c}\right)\right] -\frac{c^4}{4}[\partial_i\varphi^j + \partial_j\varphi^i - \delta_{ij}(\partial_0\varphi^0+\partial_k\varphi^k)] + O(2)\ , \tag{3.28}$$

and in this form can be used to compute, if desired, the vector φ^i [since φ^0 is known from (3.15)]. To this end, one needs to decompose the moments $\mathcal{W}_{ij}^L$ into irreducible STF tensors with respect to the indices ij and L.

D. Matching equation at the post-Newtonian order ε^6

We now have in hand all the material needed to match the inner metric (2.14), namely

$$h^{\alpha\beta}(x) = \Box_R^{-1}\left[\frac{16\pi G}{c^4}\,\overline{\lambda}(V,W)T^{\alpha\beta} + \overline{\Lambda}^{\alpha\beta}(V,V_i,W_{ij})\right] +O(8,7,8)\ , \tag{3.29}$$

where $\overline{\lambda}$ and $\overline{\Lambda}^{\alpha\beta}$ are defined in (2.11) and (2.12), to the corresponding outer metric (2.34), namely

$$h^{\mu\nu}_{\rm can}(x_{\rm can}) = Gh^{\mu\nu}_{{\rm can}(1)}(x_{\rm can}) + {\rm FP}_{B=0}\,\Box_R^{-1} \times\left[r^B\overline{\Lambda}^{\mu\nu}(V^{\rm ext},V_i^{\rm ext},W_{ij}^{\rm ext})\right] + O(7,7,7)\ , \tag{3.30}$$

where $h^{\mu\nu}_{{\rm can}(1)}$ is the linear metric (2.17) and (2.18), and $\overline{\Lambda}^{\mu\nu}$ is the same expression as in (3.29) but expressed in terms of the external potentials $V^{\rm ext}$, $V_i^{\rm ext}$, and $W_{ij}^{\rm ext}$. The metrics (3.29) and (3.30) are given in their respective coordinate systems x^μ and $x^\mu_{\rm can}$, and are valid, respectively, in D_i and $D_i\cap D_e$.

In the two previous subsections, we have related the external potentials $V^{\rm ext}$, $V_i^{\rm ext}$, and $W_{ij}^{\rm ext}$ to the multipole expansions $\mathcal{M}(V)$, $\mathcal{M}(V_i)$, and $\mathcal{M}(W_{ij})$ of the internal potentials V, V_i, and W_{ij}. These relations, which are (3.14), (3.15), and (3.27), allow us to compute the effective nonlinear source $\overline{\Lambda}^{\mu\nu}(V^{\rm ext},V_i^{\rm ext},W_{ij}^{\rm ext})$ in the right-hand side of the outer metric (3.30) in terms of the inner potentials. Indeed, by using (3.14), (3.15), and (3.27) into the expression (2.12) of $\overline{\Lambda}^{\mu\nu}$, and by using $\Box\mathcal{M}(V)=\Box\mathcal{M}(V_i)=0$ and $\Box\varphi^\mu = O(7,8)$ [see (3.7b)], we find

$$\overline{\Lambda}^{\mu\nu}(V^{\rm ext},V_i^{\rm ext},W_{ij}^{\rm ext}) = \overline{\Lambda}^{\mu\nu}(\mathcal{M}(V),\mathcal{M}(V_i),\mathcal{M}(W_{ij})) +\Box\,\Omega^{\mu\nu} + O(8,7,8)\ , \tag{3.31}$$

where in the right-hand-side extra terms appear which are due to the coordinate transformation between the inner and outer metrics, and which can be written as a d'Alembertian operator $\Box$ acting on the tensor:

$$\Omega^{00} = -\frac{8}{c^3}\left[\mathcal{M}(V)\partial_t\varphi^0 + \mathcal{M}(V_i)\partial_i\varphi^0 -\frac{c}{2}\partial_\mu(\mathcal{M}(V)\varphi^\mu)\right] + \partial_i\varphi^0\partial_i\varphi^0\ , \tag{3.32a}$$

$$\Omega^{0i} = 0\ , \tag{3.32b}$$

$$\Omega^{ij} = 0\ . \tag{3.32c}$$

[The only nonzero component is Ω^{00} which is of order $O(6)$.] Now, to evaluate (3.30) we must apply on both sides of (3.31) the regularized inverse d'Alembertian operator ${\rm FP}\Box_R^{-1}$. One easily checks that ${\rm FP}\Box_R^{-1}(\Box\Omega^{00}) = \Omega^{00}+O(7)$. Indeed, this follows from the structure of Ω^{00} which is made at leading level ε^6 of terms of the type $\Sigma\hat{n}_L r^{-(\ell+2+2k)}$, where k is a positive integer (see [1], p. 395). [$\mathcal{M}(V)$, $\mathcal{M}(V_i)$ and φ^0, φ^i all have the structure $\Sigma\hat{n}_{L'}r^{-(\ell'+1)}$ at leading level.] Hence, we find that the outer metric (3.30) reads as

$$h^{\mu\nu}_{\rm can}(x_{\rm can}) = Gh^{\mu\nu}_{{\rm can}(1)}(x_{\rm can}) + {\rm FP}_{B=0}\,\Box_R^{-1} \times\left[r^B\overline{\Lambda}^{\mu\nu}(\mathcal{M}(V),\mathcal{M}(V_i),\mathcal{M}(W_{ij}))\right] + \Omega^{\mu\nu} +O(7,7,7)\ . \tag{3.33}$$

On the other hand, we have shown in Sec. III A that the inner and outer metrics should be linked by a coordinate transformation involving nonlinear terms at order ε^6 in its 00 component, see (3.8). Using $h^{00}=-\frac{4}{c^2}\mathcal{M}(V)+\cdots$ and $h^{0i}=-\frac{4}{c^3}\mathcal{M}(V_i)+\cdots$ as is appropriate in $D_i\cap D_e$, it is easy to see that the nonlinear terms in (3.8) are precisely equal to Ω^{00} given by (3.32a), modulo $O(8)$. Hence, the coordinate transformation (3.5)–(3.8) is in fact given by

$$h^{\mu\nu}_{\rm can}(x) = h^{\mu\nu}(x) + \partial\varphi^{\mu\nu} + \Omega^{\mu\nu} + O(8,7,8)\ , \tag{3.34}$$

[where the linear part $\partial\varphi^{\mu\nu}$ of the coordinate transformation is defined by (3.6), and where both sides are expressed in terms of the inner coordinates x^μ]. Substituting in the left-hand side of (3.34) the outer metric (3.33), and in the right-hand side the inner metric (3.29), we arrive at the following equation for the external linear metric (in the coordinates x^μ):

$$Gh^{\mu\nu}_{{\rm can}(1)}(x) = \Box_R^{-1}\left[\frac{16\pi G}{c^4}\overline{\lambda}(V,W)T^{\mu\nu} + \overline{\Lambda}^{\mu\nu}(V,V_i,W_{ij})\right] -{\rm FP}_{B=0}\,\Box_R^{-1}\left[r^B\overline{\Lambda}^{\mu\nu}(\mathcal{M}(V),\mathcal{M}(V_i),\mathcal{M}(W_{ij}))\right] + \partial\varphi^{\mu\nu} + O(7,7,7)\ . \tag{3.35}$$

The nonlinear part $\Omega^{\mu\nu}$ of the coordinate transformation has been canceled, and it remains only the linear coordinate transformation $\partial\varphi^{\mu\nu}$.

Equation (3.35) is numerically valid in $D_i \cap D_e$, but must now be transformed into a matching equation. The reasoning is exactly the same as the one we followed in dealing with the noncompact-supported potential W_{ij} in Sec. III C, and we simply repeat here the arguments. First, the term in (3.35) involving the compact-supported matter stress-energy tensor $T^{\mu\nu}$ is treated by formulas such as (3.11)–(3.13). Then consider, analogously to (3.22), the difference between the two terms involving $\overline{\Lambda}^{\mu\nu}$ in (3.35). By adding the factor r^B and the finite part prescription in the first of these terms (whose integrand is regular at $r = 0$) we can write this difference, analogously to (3.23), in a manifestly compact-supported form. [Indeed, the potentials V, V_i and W_{ij} are numerically equal in D_e to their multipole expansions $\mathcal{M}(V)$, $\mathcal{M}(V_i)$, and $\mathcal{M}(W_{ij})$.] Hence, we can also apply to this difference the formulas (3.11)–(3.13) and obtain, analogously to (3.24) and (3.25), some expressions for the multipole moments allowing for the analytic continuation factor $|\mathbf{y}|^B$ and the finite part at $B = 0$. Finally the contribution associated with the multipole expanded source $\overline{\Lambda}^{\mu\nu}(\mathcal{M}(V), \mathcal{M}(V_i), \mathcal{M}(W_{ij}))$ in these multipole moments is shown, analogously to (3.26), to be zero by analytic continuation. This follows from the fact that $\overline{\Lambda}^{\mu\nu}(\mathcal{M}(V), \mathcal{M}(V_i), \mathcal{M}(W_{ij}))$ is a sum of quadratic or cubic terms $\partial_\rho\mathcal{M}_1\partial_\sigma\mathcal{M}_2$ or $\partial_\rho\mathcal{M}_1\partial_\sigma\mathcal{M}_2\mathcal{M}_3$ where the $\mathcal{M}_k$'s admit formal multipolar and post-Newtonian expansions of the type $\Sigma\hat{n}_{L'}|\mathbf{y}|^p(\ln|\mathbf{y}|)^q\mathcal{K}(t)$, with p, q some integers [$q = 0$ in the case of $\mathcal{M}(V)$ and $\mathcal{M}(V_i)$, but $q = 0$ or 1 in the case of $\mathcal{M}(W_{ij})$]; see the discussion above (3.26). An important result of this paper can now be written down. We introduce an effective (truncated) *total stress-energy tensor* of the matter fields and of the gravitational field,

$$\overline{\tau}^{\mu\nu}(V, V_i, W_{ij}) = \overline{\lambda}(V, W)T^{\mu\nu} + \frac{c^4}{16\pi G}\overline{\Lambda}^{\mu\nu}(V, V_i, W_{ij}) \ , \tag{3.36}$$

where $\overline{\lambda}$ and $\overline{\Lambda}^{\mu\nu}$ are given by (2.11) and (2.12). By (2.13) this tensor satisfies

$$\partial_\nu\overline{\tau}^{\mu\nu} = O(3,4) \ . \tag{3.37}$$

Then we can transform (3.35) into the matching equation

$$Gh^{\mu\nu}_{\mathrm{can}(1)}[M_L, S_L] = -\frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\overline{\mathcal{T}}^{\mu\nu}_L\left(t-\frac{r}{c}\right)\right] + \partial\varphi^{\mu\nu} + O(7,7,7) \ , \tag{3.38}$$

relating the exterior linear metric (2.17) and (2.18), that we recall is a functional of the canonical moments M_L and S_L, to the total truncated stress-energy tensor (3.36) via the multipole moments

$$\overline{\mathcal{T}}^{\mu\nu}_L(t) = \mathrm{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\int_{-1}^{1}dz\,\delta_\ell(z)\overline{\tau}^{\mu\nu} \times\left(\mathbf{y}, t+z\frac{|\mathbf{y}|}{c}\right) \ . \tag{3.39}$$

The result (3.38) and (3.39) is especially simple. It says that the *linear* metric $h^{\mu\nu}_{\mathrm{can}(1)}$ is equal, modulo the *linear* coordinate transformation $\partial\varphi^{\mu\nu}$ [and modulo $O(7,7,7)$ terms], to the multipole expansion outside the source we would obtain *if* the effective total stress-energy tensor $\overline{\tau}^{\mu\nu}$ had a compact support limited to the material source only. The difference is that the moments (3.39) carry the analytic continuation factor $|\mathbf{y}|^B$ and the finite part at $B = 0$ to deal with the poles at $B = 0$ coming from the behavior of the integral at its upper bound $|\mathbf{y}| \to +\infty$. When no poles arise as will be the case at the 2PN approximation (see Sec. IV), we can say that (3.38) and (3.39) justifies the formal procedure followed by Epstein and Wagoner [30] and Thorne [20] to compute the multipole moments. (However recall that in the works [30] and [20] these multipole moments are assumed to be the moments which are radiated at infinity, while one has still to add to these moments all tails and nonlinear contributions in the radiation field, see Sec. IV below.) Let us emphasize again that the expression (3.39) is not manifestly of compact-supported form, but is numerically equal to the expression obtained by multipole expanding the compact-supported right-hand-side of (3.35). In particular, this shows that the multipole moments (3.39) are *retarded* functionals of the source's parameters, depending on the source at times $t' \le t$ only [or $t' \le t - r/c$ in (3.38)], contrary to what is apparent on their expressions (3.39) but in accordance with what they must be.

IV. GENERATION OF GRAVITATIONAL WAVES

A. Relations between the canonical moments and the source moments

In order to find from the matching equations (3.38) and (3.39) the expressions of the canonical moments M_L, S_L as functions of the source's parameters, it suffices simply to decompose the reducible moments $\overline{\mathcal{T}}^{\mu\nu}_L(t)$ into irreducible STF multipole moments. This has already been done by Damour and Iyer [31] in the case of *linearized* gravity, i.e., in the case where the total stress-energy tensor $\overline{\tau}^{\mu\nu}$ in (3.38) is replaced by the usual compact-supported stress-energy tensor $T^{\mu\nu}$ of the matter fields, supposed to be exactly conserved: $\partial_\nu T^{\mu\nu} = 0$, and where of course there are no analytic continuation factors in the expressions of the moments.

Let us prove that the computation done in [31] basically applies to our case, which involves both analytic continuation factors and a stress-energy tensor $\overline{\tau}^{\mu\nu}$ which is only approximately conserved: $\partial_\nu\overline{\tau}^{\mu\nu} = O(3,4)$ by (3.37). To this end, we need only to check that the multipole expansion in the right-hand side of (3.38) is divergenceless up to $O(3,4)$ as a consequence of the approximate conservation of $\overline{\tau}^{\mu\nu}$. We take the time derivative of the 0μ components of the multipole moments (3.39), use $c^{-1}\partial_t\overline{\tau}^{0\mu} = -\partial_j\overline{\tau}^{j\mu} + O(3,4)$, perform some integrations by parts both with respect to $\mathbf{y}$ and to z, and finally insert the identity $(d/dz)^2\delta_{\ell+1}(z) = (2\ell+1)(2\ell+3)[\delta_{\ell-1}(z) - \delta_\ell(z)]$. These operations result in

$$\frac{d}{cdt}\overline{\mathcal{T}}^{0\mu}_L(t) = \ell\, \overline{\mathcal{T}}^{\mu\langle i_\ell}_{L-1\rangle}(t) + \frac{1}{2\ell+3}\left(\frac{d}{cdt}\right)^2 \overline{\mathcal{T}}^{j\mu}_{jL}(t)$$
$$+\mathrm{FP}_{B=0}\left\{B\int d^3\mathbf{y}|\mathbf{y}|^{B-2}y_j\hat{y}_L\int_{-1}^{1}dz\delta_\ell(z)\overline{\tau}^{j\mu}(\mathbf{y},t+z|\mathbf{y}|/c)\right\} + O(3,4)\ . \qquad (4.1)$$

The first two terms in the right-hand side of (4.1) are the ones which would ensure the exact zero divergency of the multipole expansion in (3.38). ($\overline{\mathcal{T}}^{\mu\langle i_\ell}_{L-1\rangle}$ means the STF part of $\overline{\mathcal{T}}^{\mu i_\ell}_{L-1}$.) The third term is the finite part at $B=0$ of an integral having explicitly B as a factor. This term arises because of the derivation of the factor $|\mathbf{y}|^B$ during the integration by parts. (Note that we have discarded a surface term which is easily seen to be zero by analytic continuation.) Finally, the remainder $O(3,4)$ comes from the only approximate conservation of $\overline{\tau}^{\mu\nu}$. We now show that the third term in (4.1) in fact vanishes up to order $O(3,4)$, which means, thanks to the explicit factor B, that the integral has at this order no simple pole at $B=0$. This follows from a general lemma which we now state and which will turn out to be very useful in the remaining of the paper. Let $\mathbf{y}$ be a "source" point, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n "field" points, and $\alpha_0, \alpha_1, \ldots, \alpha_n$ be $n+1$ real numbers. Then we can state that

$$\mathrm{FP}_{B=0}\left\{B\int d^3\mathbf{y}|\mathbf{y}|^{B+\alpha_0}\hat{y}_L|\mathbf{y}-\mathbf{x}_1|^{\alpha_1}\cdots|\mathbf{y}-\mathbf{x}_n|^{\alpha_n}\right\} = 0 \qquad (4.2)$$

if the sum $\alpha_0+\alpha_1+\cdots+\alpha_n$ is *not* an odd (positive or negative) integer. To prove that this is true, we need to investigate the behavior of the integrand at the bound $|\mathbf{y}|\to+\infty$ of the integral, where each $|\mathbf{y}-\mathbf{x}_1|^{\alpha_1},\ldots$ admits a multipole expansion of the type $\Sigma x^{L_1-1}\partial_{L_1}|\mathbf{y}|^{\alpha_1}\cdots$ (coefficients suppressed). Thus, one is led to consider the behavior of integrals of the type $\int d^3\mathbf{y}|\mathbf{y}|^{B+\alpha_0}\hat{y}_L\partial_{L_1}|\mathbf{y}|^{\alpha_1}\cdots\partial_{L_n}|\mathbf{y}|^{\alpha_n}$. By performing the integration over the angles (using $d^3\mathbf{y}=|\mathbf{y}|^2d|\mathbf{y}|d\Omega$), we find that these integrals are zero if $\ell+\ell_1+\cdots+\ell_n$ is an odd integer. When $\ell+\ell_1+\cdots+\ell_n$ is an even integer, a single type of radial integral remains, which is $\int d|\mathbf{y}||\mathbf{y}|^{B+\beta}$ where $\beta = 2+\alpha_0+\ell+(\alpha_1-\ell_1)+\cdots+(\alpha_n-\ell_n)$. A pole at $B=0$ will arise only if $\beta=-1$, and thus, since $\ell+\ell_1+\cdots+\ell_n$ is an even integer, only if $\alpha_0+\alpha_1+\cdots+\alpha_n$ is an odd integer. If $\alpha_0+\alpha_1+\cdots+\alpha_n$ is not an odd integer, there is no pole and (4.2) is true by virtue of the explicit factor B in front of the integral. [Note that this condition for (4.2) to hold is sufficient but by no means necessary.] Now, a nonzero contribution to the third term in (4.1) can come only from the noncompact supported part of $\overline{\tau}^{j\mu}$ in the integrand, i.e., the part involving $\overline{\Lambda}^{j\mu}$ [see (3.36)]. We first expand the argument $t+z|\mathbf{y}|/c$ when $c\to+\infty$ up to an order consistent with the remainder $O(3,4)$ in (4.1), and integrate over z [using (4.8) below]. Thus we have to consider integrals involving $\overline{\Lambda}^{j\mu}$ and its time-derivatives, and multiplied by some *even* powers of $|\mathbf{y}|$. The structure of $\overline{\Lambda}^{j\mu}$ as given by (2.12b) and (2.12c) is of the type $\partial V\partial V$, where V represents a retarded potential (2.3a) or (2.3b), and ∂ is some time or space derivation (all indices suppressed). By expanding the retardation argument in the V's up to the same order $O(3,4)$, we see that the structure of $\overline{\Lambda}^{j\mu}$ is of the type $\partial U\partial U$ or $\partial U\partial X$, where U and X are instantaneous (Poisson-type) potentials of the type $\int d^3\mathbf{x}\,\sigma(x)|\mathbf{y}-\mathbf{x}|^\alpha$ with $\alpha=\pm1$ [see (4.14) and (4.15) below]. Thus, $\overline{\Lambda}^{j\mu}$ is composed of terms of the type $\int d^3\mathbf{x}_1 d^3\mathbf{x}_2\partial_{x_1}\partial_{x_2}\sigma(x_1)\sigma(x_2)|\mathbf{y}-\mathbf{x}_1|^{\alpha_1}|\mathbf{y}-\mathbf{x}_2|^{\alpha_2}$ where $\alpha_1+\alpha_2=-2$ or 0, and where the derivatives act on the source points $x_1=(\mathbf{x}_1,t)$ and $x_2=(\mathbf{x}_2,t)$ (we use the fact that $\partial_\mathbf{y}|\mathbf{y}-\mathbf{x}|^\alpha = -\partial_\mathbf{x}|\mathbf{y}-\mathbf{x}|^\alpha$). Finally, it remains to commute the integration over $\mathbf{y}$ with the integrations over $\mathbf{x}_1$ and $\mathbf{x}_2$ to arrive at a series of integrals of the type $\int d^3\mathbf{y}|\mathbf{y}|^{B+\alpha_0}\hat{y}_{L'}|\mathbf{y}-\mathbf{x}_1|^{\alpha_1}|\mathbf{y}-\mathbf{x}_2|^{\alpha_2}$, where α_0 is an *even* integer and $\alpha_1+\alpha_2=-2$ or 0 (we have used $y_i\hat{y}_L\sim\hat{y}_{iL}+|\mathbf{y}|^2\delta_{i<i_\ell}\hat{y}_{L-1>}$). These integrals have no poles by the lemma (4.2). Therefore, we have proved that (4.1) reads in fact as

$$\frac{d}{cdt}\overline{\mathcal{T}}^{0\mu}_L(t) = \ell\,\overline{\mathcal{T}}^{\mu\langle i_\ell}_{L-1\rangle}(t) + \frac{1}{2\ell+3}\left(\frac{d}{cdt}\right)^2\overline{\mathcal{T}}^{j\mu}_{jL}(t) + O(3,4)\ . \qquad (4.3)$$

This relation shows that the multipole expansion in the right-hand side of (3.38) is divergenceless modulo small terms of order $O(7,8)$. This is consistent with the fact that the "q part" of the external metric is found to be zero at order $O(7,7,7)$, see (2.25) and Appendix A. Because of the uncontrolled remainder in (4.3), we find that the four relations (5.27) in [31] are modified by small post-Newtonian terms: $\mathcal{C}_L$ is modified by a term $O(7)$, and $\mathcal{G}_L$, $\mathcal{T}_L$, $\mathcal{J}_L$ are modified by terms $O(8)$. Thus the $0i$ components (5.9b) in [31] involve some uncontrolled terms $O(7)$, while the ij components involve uncontrolled terms $O(8)$ (the 00 component not being affected). All these terms fall into the remainder term $O(7,7,7)$ in (3.38), showing that the end formulas (5.33) and (5.35) of [31] giving the expressions of the mass-type and current-type multipole moments in linearized gravity can be applied to our case with the replacement of the matter stress-energy tensor $T^{\mu\nu}$ by the total stress-energy tensor $\overline{\tau}^{\mu\nu}$, and with the addition of analytic continuation factors $|\mathbf{y}|^B$ and of the finite part prescription.

The matching equation (3.38) can thus be written in the form

$$Gh^{\mu\nu}_{\mathrm{can}(1)}[M_L,S_L] = Gh^{\mu\nu}_{\mathrm{can}(1)}[I_L,J_L] + \partial\omega^{\mu\nu} + \partial\varphi^{\mu\nu} + O(7,7,7)\ , \qquad (4.4)$$

where $\partial\omega^{\mu\nu}$ is a certain harmonic linear gauge transformation ($\Box\omega^\mu = 0$), and where the mass-type and current-type *source moments* I_L and J_L are given by

$$I_L(t) = \mathrm{FP}_{B=0} \int d^3\mathbf{y}|\mathbf{y}|^B \int_{-1}^{1} dz \left[\delta_\ell(z)\hat{y}_L\overline{\Sigma} - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\,\delta_{\ell+1}(z)\hat{y}_{iL}\partial_t\overline{\Sigma}_i + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\delta_{\ell+2}(z)\hat{y}_{ijL}\partial_t^2\overline{\Sigma}_{ij}\right]\left(\mathbf{y}, t + z\frac{|\mathbf{y}|}{c}\right) , \tag{4.5a}$$

$$J_L(t) = \mathrm{FP}_{B=0} \int d^3\mathbf{y}|\mathbf{y}|^B \int_{-1}^{1} dz \left[\delta_\ell(z)\varepsilon_{ab<i_\ell}\hat{y}_{L-1>a}\overline{\Sigma}_b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\delta_{\ell+1}(z)\varepsilon_{ab<i_\ell}\hat{y}_{L-1>ac}\partial_t\overline{\Sigma}_{bc}\right]\left(\mathbf{y}, t + z\frac{|\mathbf{y}|}{c}\right) . \tag{4.5b}$$

The mass, current, and stress densities in (4.5) are defined by

$$\overline{\Sigma} = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} , \tag{4.6a}$$

$$\overline{\Sigma}_i = \frac{\overline{\tau}^{0i}}{c} , \tag{4.6b}$$

$$\overline{\Sigma}_{ij} = \overline{\tau}^{ij} , \tag{4.6c}$$

where the stress-energy tensor $\overline{\tau}^{\mu\nu}$ is given by (3.36). The weighting function $\delta_\ell(z)$ is defined in (3.13). The matching equation (4.4) then tells us that the gauge transformation $\partial\omega^{\mu\nu}$ necessarily satisfies $\partial\omega^{\mu\nu} = -\partial\varphi^{\mu\nu} + O(7,8)$ [indeed remember that $\Box\varphi^\mu = O(7,8)$], *and* that the canonical moments M_L, S_L in the left-hand side of (4.4) are related to the source moments I_L, J_L by

$$M_L(t) = I_L(t) + O(5) , \tag{4.7a}$$

$$S_L(t) = J_L(t) + O(4) . \tag{4.7b}$$

[The relation (4.7a) for the mass moment comes from the 00 component of the matching equation (4.4), while the relation (4.7b) for the current moment comes from the $0i$ components of (4.4).] The relations (4.7a) and (4.7b) are exactly the ones needed to solve the wave generation problem at the second-post-Newtonian approximation.

B. Expressions of the mass-type and current-type source moments

Since the relations (4.7) are only valid up to some post-Newtonian order, it is sufficient to consider the post-Newtonian expansion $c \to +\infty$ of the source moments I_L, J_L defined by (4.5). This expansion is achieved by means of a formula derived in Appendix B of [1], and giving the expansion when $c \to +\infty$ of terms involving the average over z appearing in (4.5). This formula reads as

$$\int_{-1}^{1} dz\delta_\ell(z)\overline{\Sigma}\left(\mathbf{y}, t + z\frac{|\mathbf{y}|}{c}\right) = \overline{\Sigma}(\mathbf{y},t) + \frac{|\mathbf{y}|^2}{2c^2(2\ell+3)}\partial_t^2\overline{\Sigma}(\mathbf{y},t) + \frac{|\mathbf{y}|^4}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\overline{\Sigma}(\mathbf{y},t) + O(6) . \tag{4.8}$$

Using (4.8), and retaining consistently the powers of c^{-1}, we obtain

$$I_L(t) = \mathrm{FP}_{B=0} \int d^3\mathbf{y}|\mathbf{y}|^B \left[\hat{y}_L\overline{\Sigma} + \frac{|\mathbf{y}|^2\hat{y}_L}{2c^2(2\ell+3)}\partial_t^2\overline{\Sigma} + \frac{|\mathbf{y}|^4\hat{y}_L}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\overline{\Sigma} - \frac{4(2\ell+1)\hat{y}_{iL}}{c^2(\ell+1)(2\ell+3)}\partial_t\overline{\Sigma}_i - \frac{2(2\ell+1)|\mathbf{y}|^2\hat{y}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)}\partial_t^3\overline{\Sigma}_i + \frac{2(2\ell+1)\hat{y}_{ijL}}{c^4(\ell+1)(\ell+2)(2\ell+5)}\partial_t^2\overline{\Sigma}_{ij}\right](\mathbf{y},t) + O(6) , \tag{4.9}$$

$$J_L(t) = \mathrm{FP}_{B=0}\,\varepsilon_{ab<i_\ell} \int d^3\mathbf{y}|\mathbf{y}|^B \left[\hat{y}_{L-1>a}\overline{\Sigma}_b + \frac{|\mathbf{y}|^2\hat{y}_{L-1>a}}{2c^2(2\ell+3)}\partial_t^2\overline{\Sigma}_b - \frac{(2\ell+1)\hat{y}_{L-1>ac}}{c^2(\ell+2)(2\ell+3)}\partial_t\overline{\Sigma}_{bc}\right](\mathbf{y},t) + O(4) . \tag{4.10}$$

We must then insert into (4.9) and (4.10) explicit formulas of $\overline{\Sigma}$, $\overline{\Sigma}_i$, and $\overline{\Sigma}_{ij}$ which are easily computed from (3.36) where we use the expression (2.11) of $\overline{\lambda} = |g| + O(6)$ and the expressions (2.12) of the components of the effective nonlinear source $\overline{\Lambda}^{\mu\nu}$. We find the formulas

$$\overline{\Sigma} = \left[1 + \frac{4V}{c^2} - \frac{8}{c^4}(W - V^2)\right]\sigma - \frac{1}{\pi G c^2}\,\partial_i V\partial_i V + \frac{1}{\pi G c^4}\left\{-V\partial_t^2 V - 2V_i\partial_t\partial_i V - W_{ij}\partial_{ij}^2 V - \frac{1}{2}(\partial_t V)^2 + 2\partial_i V_j\partial_j V_i + 2\partial_i V\partial_i W - \frac{7}{2}V\partial_i V\partial_i V\right\} , \tag{4.11a}$$

$$\overline{\Sigma}_i = \left[1 + \frac{4V}{c^2}\right]\sigma_i + \frac{1}{\pi G c^2}\left\{\partial_k V(\partial_i V_k - \partial_k V_i) + \frac{3}{4}\partial_t V\partial_i V\right\} + O(4) , \tag{4.11b}$$

$$\overline{\Sigma}_{ij} = \sigma_{ij} + \frac{1}{4\pi G}\left\{\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V\right\} + O(2) \ . \tag{4.11c}$$

Again retaining the powers of c^{-1} consistently with the accuracy indicated in (4.9) and (4.10), we can now write formulas for the source moments in raw form:

$$\begin{aligned} I_L(t) = \mathrm{FP}_{B=0} \int d^3\mathbf{y}|\mathbf{y}|^B \Bigg\{ & \left[1 + \frac{4}{c^2}V - \frac{8}{c^4}(W - V^2)\right]\hat{y}_L\sigma - \frac{1}{\pi G c^2}\hat{y}_L \partial_i V \partial_i V \\ & + \frac{1}{\pi G c^4}\hat{y}_L\left[-V\partial_t^2 V - 2V_i \partial_t \partial_i V - W_{ij}\partial^2_{ij}V - \frac{1}{2}(\partial_t V)^2 + 2\partial_i V_j \partial_j V_i + 2\partial_i V \partial_i W - \frac{7}{2}V\partial_i V \partial_i V\right] \\ & + \frac{|\mathbf{y}|^2 \hat{y}_L}{2c^2(2\ell+3)}\partial_t^2\left[\left(1+\frac{4V}{c^2}\right)\sigma - \frac{1}{\pi G c^2}\partial_i V \partial_i V\right] \\ & + \frac{|\mathbf{y}|^4\hat{y}_L}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\sigma - \frac{2(2\ell+1)|\mathbf{y}|^2\hat{y}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)}\partial_t^3\sigma_i \\ & - \frac{4(2\ell+1)\hat{y}_{iL}}{c^2(\ell+1)(2\ell+3)}\partial_t\left[\left(1+\frac{4V}{c^2}\right)\sigma_i + \frac{1}{\pi G c^2}\left\{\partial_k V(\partial_i V_k - \partial_k V_i) + \frac{3}{4}\partial_t V \partial_i V\right\}\right] \\ & + \frac{2(2\ell+1)\hat{y}_{ijL}}{c^4(\ell+1)(\ell+2)(2\ell+5)}\partial_t^2\left[\sigma_{ij} + \frac{1}{4\pi G}\partial_i V \partial_j V\right]\Bigg\} + O(6) \ , \end{aligned} \tag{4.12}$$

$$\begin{aligned} J_L(t) = \mathrm{FP}_{B=0}\, \varepsilon_{ab<i_\ell} \int d^3\mathbf{y}|\mathbf{y}|^B \Bigg\{ & \hat{y}_{L-1>a}\left(1 + \frac{4}{c^2}V\right)\sigma_b + \frac{|\mathbf{y}|^2\hat{y}_{L-1>a}}{2c^2(2\ell+3)}\partial_t^2\sigma_b \\ & + \frac{1}{\pi G c^2}\hat{y}_{L-1>a}\left[\partial_k V(\partial_b V_k - \partial_k V_b) + \frac{3}{4}\partial_t V \partial_b V\right] - \frac{(2\ell+1)\hat{y}_{L-1>ac}}{c^2(\ell+2)(2\ell+3)}\partial_t\left[\sigma_{bc} + \frac{1}{4\pi G}\partial_b V \partial_c V\right]\Bigg\} + O(4) \ . \end{aligned} \tag{4.13}$$

Recall that the analytic continuation procedure in (4.12) and (4.13) is needed only for handling the (apparently) noncompact supported terms such as the term $\hat{y}_L \partial_i V \partial_i V$ in (4.12), and could be removed from manifestly compact supported terms such as, e.g., $\hat{y}_L \sigma V^2$.

Let us now derive an equivalent but somewhat simpler form for the mass-type multipole source moment (4.12). First, we replace the retarded potentials V, V_i, W_{ij} used in (4.12) by "instantaneous" potentials U, X, U_i, P_{ij} (and $P = P_{ii}$) defined by the post-Newtonian expansions

$$V = U + \frac{1}{2c^2}\partial_t^2 X + O(3) \ , \tag{4.14a}$$

$$V_i = U_i + O(2) \ , \tag{4.14b}$$

$$W_{ij} = P_{ij} + O(1) \ . \tag{4.14c}$$

These instantaneous potentials are

$$U(\mathbf{x},t) = G\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma(\mathbf{x}',t) \ , \tag{4.15a}$$

$$X(\mathbf{x},t) = G\int d^3\mathbf{x}'|\mathbf{x}-\mathbf{x}'|\sigma(\mathbf{x}',t) \ , \tag{4.15b}$$

$$U_i(\mathbf{x},t) = G\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma_i(\mathbf{x}',t) \ , \tag{4.15c}$$

$$P_{ij}(\mathbf{x},t) = G\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ij} + \frac{1}{4\pi G}\left(\partial_i U \partial_j U - \frac{1}{2}\delta_{ij}\partial_k U \partial_k U\right)\right](\mathbf{x}',t) \ , \tag{4.15d}$$

$$P(\mathbf{x},t) = G\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ii} - \frac{1}{2}\sigma U\right](\mathbf{x}',t) + \frac{U^2}{4} \ . \tag{4.15e}$$

We can then rewrite (4.12) as

$$
\begin{aligned}
I_L(t) = \mathrm{FP}_{B=0} \int d^3\mathbf{y}|\mathbf{y}|^B \Bigg\{ & \left[1 + \frac{4}{c^2}U + \frac{1}{c^4}(2\partial_t^2 X - 8P + 8U^2)\right]\hat{y}_L\sigma \\
& - \frac{1}{\pi G c^2}\hat{y}_L\partial_i U\partial_i U + \frac{1}{\pi G c^4}\hat{y}_L\Big[-\partial_i U\partial_i\partial_t^2 X - U\partial_t^2 U \\
& -2U_i\partial_t\partial_i U - P_{ij}\partial_{ij}^2 U - \frac{1}{2}(\partial_t U)^2 + 2\partial_i U_j\partial_j U_i + 2\partial_i U\partial_i P - \frac{7}{2}U\partial_i U\partial_i U\Big] \\
& + \frac{|\mathbf{y}|^2\hat{y}_L}{2c^2(2\ell+3)}\partial_t^2\left[\left(1+\frac{4U}{c^2}\right)\sigma - \frac{1}{\pi G c^2}\partial_i U\partial_i U\right] \\
& + \frac{|\mathbf{y}|^4\hat{y}_L}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\sigma - \frac{2(2\ell+1)|\mathbf{y}|^2\hat{y}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)}\partial_t^3\sigma_i \\
& - \frac{4(2\ell+1)\hat{y}_{iL}}{c^2(\ell+1)(2\ell+3)}\partial_t\left[\left(1+\frac{4U}{c^2}\right)\sigma_i + \frac{1}{\pi G c^2}\left\{\partial_k U(\partial_i U_k - \partial_k U_i) + \frac{3}{4}\partial_t U\partial_i U\right\}\right] \\
& + \frac{2(2\ell+1)\hat{y}_{ijL}}{c^4(\ell+1)(\ell+2)(2\ell+5)}\partial_t^2\left[\sigma_{ij} + \frac{1}{4\pi G}\partial_i U\partial_j U\right]\Bigg\} + O(5) \ . \qquad (4.16)
\end{aligned}
$$

Several reductions of this expression can be done by integrating various terms by parts. For instance we can operate by part the term $\sim \hat{y}_L\partial_i U\partial_i U$ using $\Delta\hat{y}_L = 0$ and $\Delta U = -4\pi G\sigma$. This yields

$$\mathrm{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\partial_i U\partial_i U = \mathrm{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\left\{4\pi G\hat{y}_L\sigma U + \frac{1}{2}\partial_i\left[\hat{y}_L\partial_i U^2 - \partial_i\hat{y}_L U^2\right]\right\} \ . \qquad (4.17)$$

The second term in the right-hand side of (4.17), which is made of the product of $|\mathbf{y}|^B$ and of a pure divergence, is easily shown to be zero thanks to the lemma (4.2). Indeed, the differentiation of the factor $|\mathbf{y}|^B$ yields $\alpha_0 = -2$, and since the term involves two potentials U it has $\alpha_1 + \alpha_2 = -2$. Hence we can write

$$\mathrm{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\partial_i U\partial_i U = 4\pi G\int d^3\mathbf{y}\hat{y}_L\sigma U \ , \qquad (4.18)$$

where we have removed the factor $|\mathbf{y}|^B$ in the right-hand side since the term is compact-supported. Note that the use of the lemma (4.2) permits freely integrating by parts all noncompact-supported terms in (4.16) except the term involving the product of *three* U's, i.e., the term $\sim \hat{y}_L U\partial_i U\partial_i U$, which must be treated separately. For this term we write, like in (4.17),

$$\mathrm{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L U\partial_i U\partial_i U = \mathrm{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\left\{2\pi G\hat{y}_L\sigma U^2 + \frac{1}{6}\partial_i\left[\hat{y}_L\partial_i U^3 - \partial_i\hat{y}_L U^3\right]\right\} \ . \qquad (4.19)$$

To cancel the second term in the right-hand side, we have to show that the integral $\int d^3\mathbf{y}|\mathbf{y}|^{B-2}\hat{y}_{L'}|\mathbf{y}-\mathbf{x}_1|^{-1}$ $|\mathbf{y}-\mathbf{x}_2|^{-1}$ $|\mathbf{y}-\mathbf{x}_3|^{-1}$ has no pole at $B = 0$ [this case is not covered by (4.2)]. When $|\mathbf{y}| \to +\infty$ the integrand behaves like $|\mathbf{y}|^{B+\ell'-\ell_1-\ell_2-\ell_3-5}\hat{n}_{L'}\hat{n}_{L_1}\hat{n}_{L_2}\hat{n}_{L_3}$ (where $n_i = y_i/|\mathbf{y}|$). The angular integration shows that $\ell_1+\ell_2+\ell_3 = \ell' + 2p$ where p is a *positive* integer, so that the remaining radial integral is $\int d|\mathbf{y}||\mathbf{y}|^{B-2p-3}$ which cannot have a pole (this would mean $p = -1$). Thus we can conclude, as in (4.18), that

$$\mathrm{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L U\partial_i U\partial_i U = 2\pi G\int d^3\mathbf{y}\hat{y}_L\sigma U^2 \ . \qquad (4.20)$$

Incidentally, the above proof shows that in higher post-Newtonian approximations, terms can arise which cannot be integrated by parts without generating poles. This is, for instance, the case of the term $\sim \hat{y}_L U\partial_i U\partial_i\partial_t^2 X$ which is expected to arise at the $O(6)$ level.

The identity (4.18) shows that the term $\sim \hat{y}_L\partial_i U\partial_i U$ in (4.16) exactly cancels a previous term $\sim \hat{y}_L\sigma U$, so that we recover, at the first post-Newtonian approximation, the expression obtained in [1], i.e., (2.27) in [1]. We use also the identity (4.20), and then perform several other manipulations such as, for instance, one showing that the two terms involving the potential X in (4.16) can be advantageously replaced by a single term $\sim \hat{y}_L U\partial_t^2 U$—all these manipulations are justified by the lemma (4.2). They yield the expression

$$\begin{aligned} I_L(t) = \mathrm{FP}_{B=0} \int d^3\mathbf{y}|\mathbf{y}|^B \Bigg\{ & \hat{y}_L \left[\sigma + \frac{4}{c^4}(\sigma_{ii}U - \sigma P) \right] + \frac{|\mathbf{y}|^2 \hat{y}_L}{2c^2(2\ell+3)} \partial_t^2 \sigma \\ & - \frac{4(2\ell+1)\hat{y}_{iL}}{c^2(\ell+1)(2\ell+3)} \partial_t \left[\left(1 + \frac{4U}{c^2}\right) \sigma_i + \frac{1}{\pi G c^2} \left(\partial_k U [\partial_i U_k - \partial_k U_i] + \frac{3}{4} \partial_t U \partial_i U \right) \right] \\ & + \frac{|\mathbf{y}|^4 \hat{y}_L}{8c^4(2\ell+3)(2\ell+5)} \partial_t^4 \sigma - \frac{2(2\ell+1)|\mathbf{y}|^2 \hat{y}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)} \partial_t^3 \sigma_i \\ & + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \hat{y}_{ijL} \partial_t^2 \left[\sigma_{ij} + \frac{1}{4\pi G} \partial_i U \partial_j U \right] \\ & + \frac{1}{\pi G c^4} \hat{y}_L \left[-P_{ij} \partial_{ij}^2 U - 2U_i \partial_t \partial_i U + 2\partial_i U_j \partial_j U_i - \frac{3}{2} (\partial_t U)^2 - U \partial_t^2 U \right] \Bigg\} + O(5) \, . \end{aligned} \tag{4.21}$$

The expression (4.21) will be applied, in a forthcoming paper [4], to the problem of the generation of waves by a coalescing compact binary at the 2PN approximation. Note that many other transformations of the expression (4.21) could be done using the equations of motion and of conservation of mass. In Appendix B we show that in the case $\ell = 0$ the expression (4.21) reduces to the expression of the conserved total mass M at 2PN which is known from the equation of conservation of mass at 2PN.

Finally, we prove that the expression (4.13) of the current-type source moment is equivalent to the expression obtained in [2]. We make the comparison with the expression (5.18) in [2], which involves a STF tensor $Y^L(\mathbf{y}_1, \mathbf{y}_2)$ depending on two source points $\mathbf{y}_1$, $\mathbf{y}_2$, and defined by

$$Y^L(\mathbf{y}_1, \mathbf{y}_2) = |\mathbf{y}_1 - \mathbf{y}_2| \int_0^1 d\alpha\, y_\alpha^{\langle L \rangle} \, , \tag{4.22a}$$

where $y_\alpha^i = \alpha y_1^i + (1-\alpha) y_2^i$, and where $y_\alpha^{\langle L \rangle}$ denotes the STF part of $y_\alpha^L = y_\alpha^{i_1} \cdots y_\alpha^{i_\ell}$. An alternative form of Y^L can be obtained by explicitly performing the integration over α. It reads

$$Y^L(\mathbf{y}_1, \mathbf{y}_2) = \frac{|\mathbf{y}_1 - \mathbf{y}_2|}{\ell+1} \sum_{p=0}^{\ell} y_1^{\langle L-P} y_2^{P\rangle} \, , \tag{4.22b}$$

where we sum over the number p of indices present on $y_2^P = y_2^{i_1} \cdots y_2^{i_p}$ (in which case $y_1^{L-P} = y_1^{i_{p+1}} \cdots y_1^{i_\ell}$) and without p-dependent coefficient in the sum, e.g., $Y^{ij} = \frac{1}{3}|\mathbf{y}_1 - \mathbf{y}_2| \times (y_1^{\langle i} y_1^{j\rangle} + y_1^{\langle i} y_2^{j\rangle} + y_2^{\langle i} y_2^{j\rangle})$. Then the formula which permits relating our work with the formalism used in [2] is

$$\mathrm{FP}_{B=0} \int \frac{d^3\mathbf{y}|\mathbf{y}|^B \hat{y}_L}{|\mathbf{y} - \mathbf{y}_1||\mathbf{y} - \mathbf{y}_2|} = -2\pi Y^L(\mathbf{y}_1, \mathbf{y}_2) \, . \tag{4.23}$$

The proof of this formula is as follows. We know from (3.9c) in [2] that $(|\mathbf{y} - \mathbf{y}_1||\mathbf{y} - \mathbf{y}_2|)^{-1} = \Delta_{\mathbf{y}} k - 2\pi\delta_{12}$ where $\delta_{12} = |\mathbf{y}_1 - \mathbf{y}_2| \int_0^1 d\alpha \delta(\mathbf{y} - \mathbf{y}_\alpha)$ represents a Dirac distribution on the segment joining $\mathbf{y}_1$ and $\mathbf{y}_2$ (δ is the usual three-dimensional Dirac distribution), and where k is some kernel given by $k = \frac{1}{2}\ln[(|\mathbf{y} - \mathbf{y}_1| + |\mathbf{y} - \mathbf{y}_2|)^2 - |\mathbf{y}_1 - \mathbf{y}_2|^2]$. By substitution into the left-hand-side of (4.23) one gets two terms. The first one is the finite part at $B = 0$ of the integral $\int d^3\mathbf{y}|\mathbf{y}|^B \hat{y}_L \Delta_{\mathbf{y}} k$, which is also equal to $B \int d^3\mathbf{y}|\mathbf{y}|^{B-2} y_i [\partial_i \hat{y}_L k - \hat{y}_L \partial_i k]$. We replace into the latter integral the convergent Taylor expansion when $\mathbf{y}_1$, $\mathbf{y}_2 \to 0$ of the kernel k [which is analytic in $\mathbf{y}_1$, $\mathbf{y}_2$; see (3.14) in [2]], and find that the only remaining radial integrals are of the type $B \int d^3\mathbf{y}|\mathbf{y}|^{B-2k}$ or $B \int d^3\mathbf{y}|\mathbf{y}|^{B-2} \ln|\mathbf{y}|$, where k is an integer. These radial integrals are zero at $B = 0$ (no poles). Thus it remains only the second term which is the finite part at $B = 0$ of $\int d^3\mathbf{y}|\mathbf{y}|^B \hat{y}_L(-2\pi\delta_{12})$, and readily yields the result (4.23). By multiplying (4.23) by some densities $\sigma(\mathbf{y}_1, t)$ and $\sigma(\mathbf{y}_2, t)$ and integrating over $d^3\mathbf{y}_1$ and $d^3\mathbf{y}_2$ we obtain such relations as

$$\begin{aligned} \mathrm{FP}_{B=0} \int d^3\mathbf{y}|\mathbf{y}|^B \hat{y}_L \partial_i U \partial_j U = -2\pi \iint & d^3\mathbf{y}_1 d^3\mathbf{y}_2 \sigma(y_1) \\ & \times \sigma(y_2) Y^L_{,ij}(\mathbf{y}_1, \mathbf{y}_2) \, , \end{aligned} \tag{4.24}$$

where $Y^L_{,ij} = \partial^2 Y^L / \partial y_1^i \partial y_2^j$, showing the complete equivalence between our expression (4.13) above (where the V's can be replaced by the corresponding U's) and the expression (5.18) in [2].

C. The asymptotic waveform at the 2PN approximation

It has been shown in [22] (see also [32]) that the canonical external field (2.16), which satisfies all over D_e the harmonic-coordinates Einstein's equations (1.1) and (1.2) (in the sense of formal nonlinear expansions), can be rewritten in a so-called *radiative* coordinate system $X^\mu = (cT, \mathbf{X})$ in which it is of the Bondi type at large distances from the source. It is sufficient to consider the transverse-traceless (TT) projection of the leading-order $1/R$ part of the spatial metric (where $R = |\mathbf{X}|$ is the distance to the source). Denoting by $h_{km}^{\mathrm{TT}}(X^\mu) = (g_{km}(X^\mu) - \delta_{km})^{\mathrm{TT}}$ this TT projection of the spatial metric (where g_{km} is the usual covariant metric), we can then *uniquely* decompose the $1/R$ part of h_{km}^{TT} into the infinite multipole moment series [20]

$$h^{\rm TT}_{km}(\mathbf{X},T) = \frac{4G}{c^2R}\mathcal{P}_{ijkm}(\mathbf{N})\sum_{\ell=2}^{\infty}\frac{1}{c^\ell \ell!}\left\{N_{L-2}U_{ijL-2}(T-R/c) - \frac{2\ell}{(\ell+1)c}N_{aL-2}\varepsilon_{ab(i}V_{j)bL-2}(T-R/c)\right\} + O\left(\frac{1}{R^2}\right) , \tag{4.25}$$

where the *radiative* moments U_L and V_L represent two infinite sets of functions of the retarded time $T - R/c$, which are STF in their indices $L = i_1 \cdots i_\ell$ (ℓ goes from 2 up to infinity). These functions *by definition* parametrize the asymptotic waveform. The coefficients in (4.25) have been chosen so that the moments U_L and V_L reduce, in the nonrelativistic limit $c \to +\infty$ to the ℓth time derivatives of the usual "Newtonian" mass-type and current-type moments of the source [20]. Our notation in (4.25) is $N_{L-2} = N_{i_1} \cdots N_{i_{\ell-2}}$ with $N_i = X^i/R$, $N_{aL-2} = N_a N_{L-2}$, $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$, and

$$\mathcal{P}_{ijkm}(\mathbf{N}) = (\delta_{ik} - N_iN_k)(\delta_{jm} - N_jN_m) - \frac{1}{2}(\delta_{ij} - N_iN_j)(\delta_{km} - N_kN_m) . \tag{4.26}$$

At the 2PN approximation, including all terms up to the level $\varepsilon^4 \sim c^{-4}$, the waveform (4.25) reads as

$$h^{\rm TT}_{km} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}\left\{U_{ij} + \frac{1}{c}\left[\frac{1}{3}N_aU_{ija} + \frac{4}{3}\varepsilon_{ab(i}V_{j)a}N_b\right] + \frac{1}{c^2}\left[\frac{1}{12}N_{ab}U_{ijab} + \frac{1}{2}\varepsilon_{ab(i}V_{j)ac}N_{bc}\right] + \frac{1}{c^3}\left[\frac{1}{60}N_{abc}U_{ijabc} + \frac{2}{15}\varepsilon_{ab(i}V_{j)acd}N_{bcd}\right] + \frac{1}{c^4}\left[\frac{1}{360}N_{abcd}U_{ijabcd} + \frac{1}{36}\varepsilon_{ab(i}V_{j)acde}N_{bcde}\right] + O(5)\right\} . \tag{4.27}$$

By differentiating, squaring, and averaging over angles this expression, one obtains the energy loss formula at the 2PN approximation, giving the rate of decrease of the Bondi energy E_B:

$$\frac{dE_B}{dT} = -\frac{G}{c^5}\left\{\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}U^{(1)}_{ijk}U^{(1)}_{ijk} + \frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right] + \frac{1}{c^4}\left[\frac{1}{9072}U^{(1)}_{ijkm}U^{(1)}_{ijkm} + \frac{1}{84}V^{(1)}_{ijk}V^{(1)}_{ijk}\right] + O(6)\right\} . \tag{4.28}$$

Now the point is that the external field (2.16) is algorithmically constructed in [21] from the linearized metric (2.17) and (2.18), which is parametrized by the canonical multipole moments M_L, S_L, and that the coordinate transformation between the harmonic coordinates and the radiative ones can also be algorithmically implemented [22,32]. Therefore, the radiative moments U_L and V_L parametrizing the multipole expansion of the asymptotic waveform (4.25) are necessarily given as some algorithmically computable functionals of the canonical moments M_L and S_L. It has been shown in previous papers [1,3] that U_L and V_L are given by some nonlinear infinite expansions in G (consistently with our whole approach) of the type

$$U_L(T) = M^{(\ell)}_L(T) + \sum_{n\geq 2}\frac{G^{n-1}}{c^{3(n-1)+\Sigma\underline{\ell}_i-\ell}}X_{nL}(T), \tag{4.29a}$$

$$\varepsilon_{ai_\ell i_{\ell-1}}V_{aL-2}(T) = \varepsilon_{ai_\ell i_{\ell-1}}S^{(\ell-1)}_{aL-2}(T) + \sum_{n\geq 2}\frac{G^{n-1}}{c^{3(n-1)+\Sigma\underline{\ell}_i-\ell}}Y_{nL}(T) , \tag{4.29b}$$

where $M^{(\ell)}_L(T)$ and $S^{(\ell-1)}_{L-1}(T)$ denote the ℓth and $(\ell-1)$th time derivatives of M_L and S_{L-1} computed at the radiative time T, and where $X_{nL}(T)$ and $Y_{nL}(T)$ are nonlinear functionals of order n of the moments M_L and S_L and their time derivatives. The general structure of X_{nL} and Y_{nL} is

$$X_{nL}(T), Y_{nL}(T) = \sum\int_{-\infty}^{T}dU_1\cdots\int_{-\infty}^{T}dU_n\mathcal{K}_{L\underline{L}_1\cdots\underline{L}_n}(T,U_1,\cdots,U_n)\mathcal{P}^{(a_1)}_{\underline{L}_1}(U_1)\cdots\mathcal{P}^{(a_n)}_{\underline{L}_n}(U_n) , \tag{4.30}$$

where $\mathcal{P}^{(a)}_{\underline{L}}$ denotes the ath time derivative of either a mass moment M_L (in which case $\underline{\ell} = \ell$) or a current moment $\varepsilon_{ai_{\ell+1}i_\ell}S_{aL-1}$ endowed with a Levi-Civita symbol (in which case $\underline{\ell} = \ell + 1$). The tensor $\mathcal{K}_{L\underline{L}_1\cdots\underline{L}_n}$ denotes some dimensionless kernel whose index structure is made out only of Kronecker symbols, and which depends only on variables having the dimension of time. The powers of G and c in (4.29a) are obtained by a simple dimensional argument, namely that the mass and current moments M_L and S_L have the usual dimensions of multipole moments. The notation $\Sigma\underline{\ell}_i$ is for $\Sigma^n_{i=1}\underline{\ell}_i = \Sigma^n_{i=1}\ell_i + s$, where $\Sigma^n_{i=1}\ell_i$ is the total number of indices present on the n moments $\mathcal{P}_{\underline{L}}$ in (4.30), and where s is the number of current moments among these n moments. As the tensor $\mathcal{K}_{L\underline{L}_1\cdots\underline{L}_n}$ represents an operation of complete contraction between the indices $L, \underline{L}_1, \ldots, \underline{L}_n$, we have necessarily the equality

$$\sum_{i=1}^{n}\underline{\ell}_i = \ell + 2k , \tag{4.31}$$

where k is the number of contractions among the indices $\underline{L}_1, \ldots \underline{L}_n$.

In view of the explicit powers of c^{-1} in front of the multipole moment contributions present in (4.27), we need to compute the relations (4.29) linking the radiative and canonical moments only up to some definite order in c^{-1}. Namely, U_{ij} is to be computed up to c^{-4} inclusively, U_{ijk} and V_{ij} are to be computed up to c^{-3}, U_{ijkm} and V_{ijk} up to c^{-2}, and so on. Now the equality (4.31) shows that the nonlinear terms in (4.29) having $n \geq 3$, and thus coming from the cubic and higher nonlinearities of Einstein's equations, are at least of order $O(c^{-6}) = O(6)$ and can be neglected for our purpose. Furthermore, the terms with $n = 2$ must have $k = 0$ in (4.31) since for $k \geq 1$ the corresponding order is $O(5)$, also negligible for our purpose. Finally, the remaining nonlinear terms with $n = 2$ and $k = 0$, which represent corrections $\sim c^{-3}$ in the radiative moments, are to be computed only in the quadrupole and octupole mass moments U_{ij} and U_{ijk}, and in the quadrupole current moment V_{ij}. We then easily arrive at the only possibilities $\ell = \ell_1 + \ell_2$ [see (4.31)] with $\ell = 2$ (case of U_{ij}) and $\ell_1 = 0$, $\ell_2 = 2$, or $\ell = 3$ (cases of U_{ijk} and V_{ij}) and $\ell_1 = 0$, $\ell_2 = 3$, or $\ell_1 = 1$, $\ell_2 = 2$. (Indeed, one of the moments in (4.30) is necessarily nonstatic, $\ell_2 \geq 2$ say.) This corresponds to the interaction of the mass monopole M and of the mass quadrupole M_{ij} (case of U_{ij}), of M and of the mass octupole M_{ijk}, or of the mass dipole M_i and of M_{ij} (case of U_{ijk}), and of M and the current quadrupole S_{ij} (case of V_{ij}). Let us combine this information with the results of [3] showing that two (and only two) types of "hereditary" contributions arise in the radiative moments U_L, V_L at the quadratic nonlinear approximation, namely the "tail" contributions involving the interaction between M and nonstatic moments M_L or S_L, and the "memory" contribution involving the interaction between two nonstatic moments M_L. By the previous reasoning, the latter memory contribution can be neglected, and the former tail contributions need to be included only in the radiative moments U_{ij}, U_{ijk}, and V_{ij}. Hence we can write, from (2.42) and (3.4) in [3],

$$U_{ij}(T) = M^{(2)}_{ij}(T) + \frac{2GM}{c^3}\int_0^{+\infty} dV \left[\ln\left(\frac{V}{2b}\right) + \kappa_2\right] \times M^{(4)}_{ij}(T-V) + O(5) \ , \quad (4.32a)$$

$$U_{ijk}(T) = M^{(3)}_{ijk}(T) + \frac{2GM}{c^3}\int_0^{+\infty} dV \left[\ln\left(\frac{V}{2b}\right) + \kappa_3\right] \times M^{(5)}_{ijk}(T-V) + O(5) \ , \quad (4.32b)$$

$$V_{ij}(T) = S^{(2)}_{ij}(T) + \frac{2GM}{c^3}\int_0^{+\infty} dV \left[\ln\left(\frac{V}{2b}\right) + \kappa'_2\right] \times S^{(4)}_{ij}(T-V) + O(5) \ . \quad (4.32c)$$

The other radiative moments $U_{ijkm}, \ldots, V_{ijkmn}$ in (4.27) and (4.28) are equal, with the required precision, to the corresponding $M^{(4)}_{ijkm}, \ldots, S^{(5)}_{ijkmn}$. Three purely numerical constants κ_2, κ_3, and κ'_2 appear in (4.32), which are in factor of "instantaneous" (nonhereditary) contributions. Note that there is *a priori* also a contribution involving the interaction between the mass dipole M_i and the mass quadrupole $M_{jk}(T)$ in the octupole moment $U_{ijk}(T)$ of (4.32b). This contribution, which is necessarily instantaneous and of the type $(\sigma_3 G/c^3) M_{\langle i} M^{(4)}_{jk\rangle}(T)$, where σ_3 is some numerical constant, has been set to zero in (4.32b) by requiring that the (harmonic) exterior coordinate system is mass centered, i.e., $M_i = 0$. The computation of κ_2, κ_3, and κ'_2 necessitates the implementation of the algorithm for the construction of the external metric. This was already done in [3] for the computation of κ_2. The computation done in Appendix C yields the values

$$\kappa_2 = \frac{11}{12} \ , \qquad \kappa_3 = \frac{97}{60} \ , \qquad \kappa'_2 = \frac{7}{6} \ . \quad (4.33)$$

The constant b entering the tail contributions in (4.32) is a constant (with dimension of a time) which parametrizes the relation between the radiative coordinate system (T, R) in which the metric is of the Bondi type and the harmonic coordinate system $(t_{\rm can}, r_{\rm can})$ of Sec. II B. It is such that

$$T - \frac{R}{c} = t_{\rm can} - \frac{r_{\rm can}}{c} - \frac{2GM}{c^3}\ln\left(\frac{r_{\rm can}}{cb}\right) + O(5) \quad (4.34)$$

[where terms of order $O(1/r^2_{\rm can})$ in the distance to the source are neglected]. A possibly convenient choice for the constant b is $b \sim 1/\omega_0$, where ω_0 is a typical frequency at which some detector at large distances from the source is operating [11].

The relations (4.32) are still not expressed in terms of the source's parameters, and the last step obviously consists in using the relations (4.7) linking the canonical moments M_L, S_L to the real source moments I_L, J_L. We can thus rewrite (4.32) as

$$U_{ij}(T) = I^{(2)}_{ij}(T) + \frac{2GM}{c^3}\int_0^{+\infty} dV \left[\ln\left(\frac{V}{2b}\right) + \frac{11}{12}\right] \times I^{(4)}_{ij}(T-V) + O(5) \ , \quad (4.35a)$$

$$U_{ijk}(T) = I^{(3)}_{ijk}(T) + \frac{2GM}{c^3}\int_0^{+\infty} dV \left[\ln\left(\frac{V}{2b}\right) + \frac{97}{60}\right] \times I^{(5)}_{ijk}(T-V) + O(5) \ , \quad (4.35b)$$

$$V_{ij}(T) = J^{(2)}_{ij}(T) + \frac{2GM}{c^3}\int_0^{+\infty} dV \left[\ln\left(\frac{V}{2b}\right) + \frac{7}{6}\right] \times J^{(4)}_{ij}(T-V) + O(4) \ , \quad (4.35c)$$

(with relations limited to the first term in the right-hand side for the higher-order moments $U_{ijkm}, \ldots$). One must insert these relations, together with the explicit expressions (4.21) and (4.13) of the source moments, into the waveform (4.27) and/or the energy-loss formula (4.28). [Note that the only tail contribution in the energy-loss formula (4.28) comes from the "mass-quadrupole" tail associated with the moment U_{ij} in (4.35a).] This solves the problem of the generation of gravitational waves by a general isolated system at the 2PN approximation.

ACKNOWLEDGMENTS

It is a pleasure to thank Kip S. Thorne and Clifford M. Will for discussions raising the author's interest in this problem.

APPENDIX A: PN EXPANSION OF PART OF THE EXTERNAL FIELD

The "q-part" of the external metric is the second term in the definition of the nonlinear canonical field (2.21). We shall denote the first term in (2.21) by

$$p^{\mu\nu}_{\mathrm{can}(n)} = \mathrm{FP}_{B=0}\,\Box_R^{-1}\,[r^B \Lambda^{\mu\nu}_{\mathrm{can}(n)}] \ . \tag{A1}$$

The q-part of the metric is computed from the "p-part" (A1) as follows [21]. We compute the divergence of (A1), namely $r^\mu_{\mathrm{can}(n)} = \partial_\nu p^{\mu\nu}_{\mathrm{can}(n)}$, and obtain

$$r^\mu_{\mathrm{can}(n)} = \mathrm{FP}_{B=0}\,\Box_R^{-1}\,[B r^{B-1} n_i \Lambda^{\mu i}_{\mathrm{can}(n)}] \ , \tag{A2}$$

where $n_i = x^i/r$ and where the factor B comes from the derivation of the analytic continuation factor r^B. This divergence is known to be a retarded solution of the wave equation, and can thus be decomposed, in a unique manner, as

$$r^0_{\mathrm{can}(n)} = \sum_{\ell\geq 0} \partial_L \left[\frac{1}{r} A_L\left(t-\frac{r}{c}\right)\right] \ , \tag{A3a}$$

$$r^i_{\mathrm{can}(n)} = \sum_{\ell\geq 0} \partial_{iL} \left[\frac{1}{r} B_L\left(t-\frac{r}{c}\right)\right] + \sum_{\ell\geq 1}\left\{\partial_{L-1}\left[\frac{1}{r} C_{iL-1}\left(t-\frac{r}{c}\right)\right] + \varepsilon_{iab}\partial_{aL-1}\left[\frac{1}{r} D_{bL-1}\left(t-\frac{r}{c}\right)\right]\right\} \ , \tag{A3b}$$

where A_L, B_L, C_L, and D_L are some STF tensorial functions of the retarded time. Then the q-part of the external metric is defined by its components as

$$q^{00}_{\mathrm{can}(n)} = -\frac{c}{r}A^{(-1)} - c\partial_a\left(\frac{1}{r}A_a^{(-1)}\right) + c^2\partial_a\left(\frac{1}{r}C_a^{(-2)}\right) \ , \tag{A4a}$$

$$q^{0i}_{\mathrm{can}(n)} = -\frac{c}{r}C_i^{(-1)} - c\varepsilon_{iab}\partial_a\left(\frac{1}{r}D_b^{(-1)}\right) - \sum_{\ell\geq 2}\partial_{L-1}\left(\frac{1}{r}A_{iL-1}\right) \ , \tag{A4b}$$

$$q^{ij}_{\mathrm{can}(n)} = -\delta_{ij}\left[\frac{1}{r}B + \partial_a\left(\frac{1}{r}B_a\right)\right] + \sum_{\ell\geq 2}\left\{\partial_{L-2}\left(\frac{1}{rc}A^{(1)}_{ijL-2} + \frac{3}{rc^2}B^{(2)}_{ijL-2} - \frac{1}{r}C_{ijL-2}\right)\right.$$
$$\left. +2\delta_{ij}\partial_L\left(\frac{1}{r}B_L\right) - 6\partial_{L-1(i}\left(\frac{1}{r}B_{j)L-1}\right) - 2\partial_{aL-2}\left(\varepsilon_{ab(i}\frac{1}{r}D_{j)bL-2}\right)\right\} \tag{A4c}$$

[where, e.g., $A^{(-1)}(t) = \int_{-\infty}^t dt' A(t')$].

The main task is to deal with the quadratic case $n = 2$. We first control the post-Newtonian expansion of the divergence $r^\mu_{\mathrm{can}(2)}$ of (A2). The quadratic source $\Lambda^{\mu\nu}_{\mathrm{can}(2)} = N^{\mu\nu}(h_{\mathrm{can}(1)})$ is computed by inserting the linear metric (2.17) and (2.18) into (1.5). Its post-Newtonian expansion starts at $O(4,5,4)$, with a next term at $O(6,7,6)$. Let us write

$$\Lambda^{\mu\nu}_{\mathrm{can}(2)} = \frac{1}{c^{4+\omega}}F^{\mu\nu} + \frac{1}{c^{6+\omega}}G^{\mu\nu} + O(7,8,7) \ , \tag{A5}$$

where $F^{\mu\nu}$ and $G^{\mu\nu}$ are the coefficients of the leading-order and next-order terms in the post-Newtonian expansion, and where we use the notation $\omega = 0$ when $\mu\nu = 00$ or ij and $\omega = 1$ when $\mu\nu = 0i$. Then it is easy to show that the structures of $F^{\mu\nu}$ and $G^{\mu\nu}$, as concerns their spatial dependence, are

$$F^{\mu\nu} = \sum_{p+q\geq 2}\partial_P\left(\frac{1}{r}\right)\partial_Q\left(\frac{1}{r}\right) \ , \tag{A6a}$$

$$G^{\mu\nu} = \sum_{p+q\geq 1}\partial_P\left(\frac{1}{r}\right)\partial_Q\left(\frac{1}{r}\right) + \sum_{p+q\geq 2}\partial_P\left(\frac{1}{r}\right)\hat{\partial}_Q(r) \ , \tag{A6b}$$

where P and Q are multi-indices with p and q indices. Important for our purpose is the fact that the number of spatial derivatives is $p+q \geq 2$ in $F^{\mu\nu}$ and in the second term in $G^{\mu\nu}$, and is $p+q\geq 1$ in the first term in $G^{\mu\nu}$. Knowing the expansion (A5) we can write the corresponding expansion of the divergence (A2). We have

$$r^\mu_{\mathrm{can}(2)} = \frac{1}{c^{4+\omega}}\mathrm{FP}_{B=0}\Delta^{-1}[Br^{B-1}n_iF^{\mu i}]$$
$$+\frac{1}{c^{6+\omega}}\mathrm{FP}_{B=0}\left\{\Delta^{-1}[Br^{B-1}n_iG^{\mu i}] + \left(\frac{\partial}{\partial t}\right)^2\right.$$
$$\left.\times\Delta^{-2}[Br^{B-1}n_iF^{\mu i}]\right\} + O(8,7) \ , \tag{A7}$$

where Δ^{-1} is the Poisson operator and $\Delta^{-2} = (\Delta^{-1})^2$, and where $\omega = 1$ when $\mu = 0$ and $\omega = 0$ when $\mu = i$. The justification of Eq. (A7) can be found in our previous papers [see, e.g., (3.25) in [24]]. Using the structures (A6) of $F^{\mu\nu}$ and $G^{\mu\nu}$ into (A7) shows that all explicit terms in (A7) are zero. Indeed, by multiplying by $r^{B-1}n_i$ the term $\partial_P(r^{-1})\partial_Q(r^{-1})$ in (A6a) or (A6b) and projecting on STF tensors, we get a series of terms of the type $r^{B-\ell-2k-2}\hat{n}_L$ where k is a positive or zero integer and $\ell = p+q+1-2k$. By applying Δ^{-1} and Δ^{-2} we obtain $r^{B-\ell-2k}\hat{n}_L/D_{1,2}(B)$ where the denominators are, respectively, $D_1(B) = (B-2\ell-2k)(B-2k+1)$ and $D_2(B) = (B-2\ell-2k+2)(B-2\ell-2k)(B-2k+1)(B-2k+3)$. When $p+q\geq 1$ (which implies $\ell+k\geq 1$) neither $D_1(B)$ nor $D_2(B)$ vanish at $B=0$, and when $p+q\geq 2$

(which implies $\ell + k \geq 2$) $D_2(B)$ does not vanish either. Furthermore, by multiplying by $r^{B-1}n_i$ the second term $\partial_P(r^{-1})\hat{\partial}_Q(r)$ in (A6b), projecting on STF tensors and applying Δ^{-1}, we get $r^{B-\ell-2k+2}\hat{n}_L/D_3(B)$ where the denominator is $D_3(B) = (B-2\ell-2k+2)(B-2k+3)$ which does not vanish at $B = 0$ when $p + q \geq 2$. Since all denominators D_1, D_2, D_3 take nonzero values at $B = 0$, we conclude that all terms will be zero at $B = 0$ thanks to the explicit factors B present in (A7). Thus,

$$r^{\mu}_{\text{can}(2)} = O(8,7) \; . \tag{A8}$$

Equation (A3) then shows that for $n = 2$ the function A_L is $O(8)$ while B_L, C_L, and D_L are $O(7)$. Hence we can write, from (A4),

$$q^{00}_{\text{can}(2)} = c^2\partial_a\left(\frac{1}{r}C^{(-2)}_a\right) + O(7) \; , \tag{A9a}$$

$$q^{0i}_{\text{can}(2)} = -\frac{c}{r}C^{(-1)}_i - c\varepsilon_{iab}\partial_a\left(\frac{1}{r}D^{(-1)}_b\right) + O(8) \; , \tag{A9b}$$

$$q^{ij}_{\text{can}(2)} = O(7) \; . \tag{A9c}$$

Thus it remains to control three terms involving antiderivatives of vectors C_i and D_i and having low multipolarities $\ell = 0, 1$. We know that the dependence on c^{-1} of a term with multipolarity ℓ in $q^{\mu\nu}_{\text{can}(2)}$ is $O(5+\underline{\ell}_1+\underline{\ell}_2-\ell)$ [see, e.g., (3.23) in [24]], where $\underline{\ell}_1$ and $\underline{\ell}_2$ are the number of indices on the two moments $\mathcal{P}_{\underline{L}_1}$ and $\mathcal{P}_{\underline{L}_2}$ composing the term (notation of Sec. IV C). Now one of the two moments is necessarily nonstatic since for stationary metrics the q-part of the metric is zero (Appendix C in [21]), thus $\underline{\ell}_1 \geq 2$ say. On the other hand, to form a vector C_i or D_i one needs $\underline{\ell}_2 = \underline{\ell}_1 \pm 1$ thus $\underline{\ell}_1 + \underline{\ell}_2 \geq 3$. This, together with the fact that $\ell \leq 1$, shows that the remaining terms in (A9) are $O(7)$ at least. Thus we have proved

$$q^{\mu\nu}_{\text{can}(2)} = O(7,7,7) \; . \tag{A10}$$

We finally deal with the cubic case $n = 3$. In this case we know that the post-Newtonian expansion of the source $\Lambda^{\mu\nu}_{\text{can}(3)}$ starts at $O(6,7,8)$. [The fact that the spatial components ij of the cubic source are $O(8)$ instead of the expected $O(6)$ is not obvious but has been proved in [21]—see the proof that $\tilde{A} = 0$ on p. 424 in [21]; indeed a possible term $O(6)$ would be made of three mass-type multipoles M_L.] Thus the divergence (A2) with $n = 3$ is at least $r^{\mu}_{\text{can}(3)} = O(7,8)$, from which we deduce $q^{\mu\nu}_{\text{can}(3)} = O(6,7,8)$. On the other hand the dependence in c^{-1} of a term in $q^{\mu\nu}_{\text{can}(3)}$ is $O(8 + \Sigma\underline{\ell}_i - \ell)$ [see (3.23) in [24]], and we know that $\Sigma\underline{\ell}_i \geq \ell - s$ by the law of addition of angular momenta, from which we deduce also $q^{\mu\nu}_{\text{can}(3)} = O(8 - s) = O(8,7,6)$. These two results imply

$$q^{\mu\nu}_{\text{can}(3)} = O(8,7,8) \; . \tag{A11}$$

Equations (A10) and (A11) are the ones which are used in the text.

APPENDIX B: THE CONSERVED 2PN TOTAL MASS

We first obtain the expression of the total conserved mass at the 2PN approximation. The equation of continuity at this level of approximation reads as

$$\begin{aligned}\partial_t\left[\sigma\left(1-4P/c^4\right)\right] + \partial_j\left[\sigma_j\left(1-4P/c^4\right)\right] &= \frac{1}{c^2}(\partial_t\sigma_{jj} - \sigma\partial_t V) - \frac{4}{c^4}(\sigma U_j\partial_j U + \sigma_{jk}\partial_j U_k)\\ &\quad + O(5) \; ,\end{aligned} \tag{B1}$$

where our notation can be found in (2.1)–(2.4) and (4.14) and (4.15). By integrating this equation over the three-dimensional space we obtain

$$\begin{aligned}\frac{d}{dt}\left[\int d^3\mathbf{y}\left\{\sigma - \frac{1}{c^2}\sigma_{jj} - \frac{4}{c^4}\sigma P\right\}\right] &= \int d^3\mathbf{y}\left\{-\frac{1}{c^2}\sigma\partial_t V - \frac{4}{c^4}(\sigma U_j\partial_j U + \sigma_{jk}\partial_j U_k)\right\}\\ &\quad + O(5) \; .\end{aligned} \tag{B2}$$

Several transformations of both sides of this equation yield the equation of conservation of mass at the 2PN approximation: namely,

$$\frac{d}{dt}\left[\int d^3\mathbf{y}\left\{\sigma + \frac{1}{c^2}\left(-\sigma_{jj} + \frac{1}{2}\sigma V\right) + \frac{1}{c^4}\left(\sigma U^2 + 2\sigma_i U_i - 4\sigma_{jj}U - \frac{1}{4}\partial_t\sigma\partial_t X\right)\right\}\right] = O(5) \; . \tag{B3}$$

We now show that the expression (4.21) we obtained in the text for the general mass-type source moment I_L reduces when $\ell = 0$ to the conserved mass appearing in the square brackets of (B3). When $\ell = 0$ the expression (4.21) becomes

$$\begin{aligned}I = \text{FP}_{B=0}\int d^3\mathbf{y}|\mathbf{y}|^B\Big\{&\sigma + \frac{4}{c^4}(\sigma_{ii}U - \sigma P) + \frac{1}{6c^2}|\mathbf{y}|^2\partial_t^2\sigma\\ &-\frac{4}{3c^2}y_i\partial_t\left[\left(1+\frac{4U}{c^2}\right)\sigma_i + \frac{1}{\pi Gc^2}\left(\partial_k U[\partial_i U_k - \partial_k U_i] + \frac{3}{4}\partial_t U\partial_i U\right)\right]\\ &+\frac{1}{120c^4}|\mathbf{y}|^4\partial_t^4\sigma - \frac{2}{15c^4}|\mathbf{y}|^2 y_i\partial_t^3\sigma_i + \frac{1}{5c^4}\hat{y}_{ij}\partial_t^2\left[\sigma_{ij} + \frac{1}{4\pi G}\partial_i U\partial_j U\right]\\ &+\frac{1}{\pi Gc^4}\left[-P_{ij}\partial^2_{ij}U - 2U_i\partial_t\partial_i U + 2\partial_i U_j\partial_j U_i - \frac{3}{2}(\partial_t U)^2 - U\partial_t^2 U\right]\Big\} + O(5) \; .\end{aligned} \tag{B4}$$

We shall not write down the rather long calculation but only indicate its main steps. One must use the equation of continuity (B1) at the 1PN approximation, i.e.,

$$\partial_t\sigma + \partial_j\sigma_j = \frac{1}{c^2}(\partial_t\sigma_{jj} - \sigma\partial_t U) + O(4) \ , \tag{B5a}$$

together with the corresponding equation of motion

$$\partial_t\left[\sigma_i\left(1+4U/c^2\right)\right] + \partial_j\left[\sigma_{ij}\left(1+4U/c^2\right)\right] = \sigma\partial_i V + \frac{4}{c^2}[\sigma\partial_t U_i + \sigma_j(\partial_j U_i - \partial_i U_j)] + O(4) \ . \tag{B5b}$$

Thanks to the lemma (4.2) we know that the non-compact supported terms in (B4) can be freely integrated by parts as if the analytic continuation factor $|\mathbf{y}|^B$ and the finite part prescription were absent, and as if all surface terms were zero. Let us quote here a list of identities, valid up to the required precision and up to the addition of a total divergence, which are used in the reduction of (B4):

$$\frac{4}{c^4}(\sigma_{ii}U - \sigma P) = \frac{1}{c^4}\sigma U^2 \ , \tag{B6a}$$

$$-\frac{1}{\pi G c^4}P_{ij}\partial^2_{ij}U = -\frac{1}{\pi G c^4}U\partial_t^2 U \ , \tag{B6b}$$

$$\frac{1}{120c^4}|\mathbf{y}|^4\partial_t^4\sigma = \frac{1}{30c^4}|\mathbf{y}|^2 y_i\partial_t^3\sigma_i \ , \tag{B6c}$$

$$\frac{1}{5c^4}\hat{y}_{ij}\partial_t^2\left[\sigma_{ij} + \frac{1}{4\pi G}\partial_i U\partial_j U\right] = \frac{1}{10c^4}|\mathbf{y}|^2 y_i\partial_t^3\sigma_i - \frac{1}{6c^4}|\mathbf{y}|^2\partial_t^2\left[\sigma_{ii} - \frac{1}{8\pi G}\partial_i U\partial_i U\right] \ , \tag{B6d}$$

$$-\frac{4}{3\pi G c^4}y_i\partial_t[\partial_k U(\partial_i U_k - \partial_k U_i)] = \frac{16}{3c^4}y_i\partial_t\left[U\sigma_i - \frac{1}{4\pi G}U\partial_i\partial_t U\right] \ . \tag{B6e}$$

Using these and other identities, we arrive finally at a manifestly compact-supported expression (on which we can remove the analytic continuation factors) which reads as

$$I = \int d^3\mathbf{y}\left\{\sigma + \frac{1}{c^2}\left(-\sigma_{jj} + \frac{1}{2}\sigma V\right) + \frac{1}{c^4}\left(\sigma U^2 + 2\sigma_i U_i - 4\sigma_{jj}U - \frac{1}{4}\partial_t\sigma\partial_t X\right)\right\} + O(5) \ , \tag{B7}$$

in perfect agreement with (B3).

APPENDIX C: COMPUTATION OF THREE CONSTANTS

To compute the three constants κ_2, κ_3, and κ'_2 appearing in (4.32) one needs to implement the construction of the external metric for the interacting multipoles $M\times M_{ij}$ (case of κ_2), $M\times M_{ijk}$ (case of κ_3), and $M\times S_{ij}$ (κ'_2). The more general cases of interacting multipoles $M\times M_L(t)$ and $M\times S_L(t)$ are in fact not more difficult to handle and probably will be useful in future work, so we shall compute κ_ℓ and κ'_ℓ for any $\ell \geq 2$.

In Appendix B of [3], where $\kappa_2 = 11/12$ was obtained, we computed the complete $M\times M_{ij}$ metric valid all over D_e. Here we shall only compute the terms $1/r$ (and $\ln r/r$) in the $M\times M_L$ and $M\times S_L$ metrics at large distances from the source, since κ_ℓ and κ'_ℓ are contained in these terms. The quadratic source (1.5) of Einstein's equations, computed with the linearized metric (2.17) and (2.18) and in which we retain only the products of multipoles $M\times M_L(t)$ and $M\times S_L(t)$, is made of a series of terms of the type $\partial_P(r^{-1})\hat\partial_Q(r^{-1}F(t-r/c))$, where the function $F(t)$ is some time derivative of a moment $M_L(t)$ or $S_L(t)$ and where the number of space derivatives acting on r^{-1} is at most two, i.e. $p=0,1$ or 2 (and where q is arbitrary). Thus we need only to compute the leading term when $r\to+\infty$ of the (finite part of the) retarded integral of $\partial_P(r^{-1})\hat\partial_Q(r^{-1}F(t-r/c))$ when $p=0,1$ or 2. When $p=0$ we know from our previous papers that this retarded integral involves a tail. A computation using (2.26) in [3] and (4.24) in [23] leads to

$$\mathrm{FP}_{B=0}\,\Box_R^{-1}\left[r^{B-1}\hat\partial_Q\left(r^{-1}F\left(t-\frac{r}{c}\right)\right)\right] = \frac{(-)^q\hat n_Q}{2rc^q}\int_0^\infty dx F^{(q)}\left(t-\frac{r}{c}-\frac{x}{c}\right)\left[\ln\left(\frac{x}{2r}\right)+\sum_{k=1}^q\frac{1}{k}\right] + O\left(\frac{\ln r}{r^2}\right) \ . \tag{C1}$$

[Note that the sum $\Sigma_{k=1}^q(1/k)$ is multiplied by a factor 1 in the present formula (C1) and by a factor 2 in the formula (2.26) in [3].] When $p=1$ or 2 there are no tails, but the calculation is in fact somewhat more complicated. One must use (4.26) in [23] to get the polar part at $B=0$ of some integrals. The results are

$$\mathrm{FP}_{B=0}\,\Box_R^{-1}\left[r^B\partial_i(r^{-1})\hat\partial_Q\left(r^{-1}F\left(t-\frac{r}{c}\right)\right)\right]=\frac{(-)^q}{2(q+1)}(n_i\hat n_Q-\delta_{i\langle a_q}n_{Q-1\rangle})\frac{F^{(q)}\left(t-\frac{r}{c}\right)}{rc^q}+O\left(\frac{1}{r^2}\right)\,, \tag{C2}$$

$$\begin{aligned}\mathrm{FP}_{B=0}\,\Box_R^{-1}\left[r^B\partial_{ij}(r^{-1})\hat\partial_Q\left(r^{-1}F\left(t-\frac{r}{c}\right)\right)\right]=&\;\frac{(-)^{q+1}}{2(q+1)(q+2)}\Big\{(n_{ij}+\delta_{ij})\hat n_Q-2[\delta_{i\langle a_q}n_{Q-1\rangle}n_j+\delta_{j\langle a_q}n_{Q-1\rangle}n_i]\\&+2\delta_{i\langle a_q}\delta_{\underline{j}a_{q-1}}n_{Q-2\rangle}\Big\}\frac{F^{(q+1)}\left(t-\frac{r}{c}\right)}{rc^{q+1}}+O\left(\frac{1}{r^2}\right)\,.\end{aligned} \tag{C3}$$

[We denote $Q=a_1\cdots a_q$, and in the last term of (C3), $\underline{j}$ means that j has to be excluded from the STF operation $\langle\,\rangle$.] It is then straightforward to obtain the needed nonlinear sources with the help of (1.5) and to apply the formulas (C1)–(C3) on each terms of these sources. The p-part of the metric obtained in this way [see (A1)] is found to be divergenceless up to order $O(\ln r/r^2)$ and thus, as a short reasoning shows, the q-part of the metric [see (A4)] vanishes at this order. In the case of the interacting multipoles $M\times M_L$, we find the metric

$$h^{00}_{\mathrm{can}(2)}=\frac{8}{\ell!(\ell-1)}\frac{n_LMM_L^{(\ell+1)}}{r}-\frac{8}{\ell!}\frac{n_LM}{r}\int_0^\infty dxM_L^{(\ell+2)}\left[\ln\left(\frac{x}{2r}\right)+\sum_{k=1}^{\ell}\frac{1}{k}\right]+O\left(\frac{\ln r}{r^2}\right)\,, \tag{C4a}$$

$$\begin{aligned}h^{0i}_{\mathrm{can}(2)}=&\;\frac{-2}{\ell!(\ell+1)}\frac{n_{iL}MM_L^{(\ell+1)}}{r}+\frac{2}{\ell!}\frac{\ell^2+3\ell+4}{\ell(\ell+1)(\ell-1)}\frac{n_{L-1}MM_{iL-1}^{(\ell+1)}}{r}\\&-\frac{8}{\ell!}\frac{n_{L-1}M}{r}\int_0^\infty dxM_{iL-1}^{(\ell+2)}\left[\ln\left(\frac{x}{2r}\right)+\sum_{k=1}^{\ell-1}\frac{1}{k}\right]+O\left(\frac{\ln r}{r^2}\right)\,,\end{aligned} \tag{C4b}$$

$$\begin{aligned}h^{ij}_{\mathrm{can}(2)}=&-\frac{4}{\ell!(\ell+2)}\frac{n_{ijL}MM_L^{(\ell+1)}}{r}+\frac{4}{\ell!}\frac{5\ell^2+10\ell+8}{\ell(\ell+1)(\ell+2)}\frac{n_{L-1(i}MM_{j)L-1}^{(\ell+1)}}{r}\\&-\frac{4}{\ell!}\frac{2\ell^2+5\ell+4}{\ell(\ell+1)(\ell+2)}\frac{\delta_{ij}n_LMM_L^{(\ell+1)}}{r}-\frac{8}{\ell!}\frac{2\ell^2+5\ell+4}{\ell(\ell+1)(\ell+2)}\frac{n_{L-2}MM_{ijL-2}^{(\ell+1)}}{r}\\&-\frac{8}{\ell!}\frac{n_{L-2}M}{r}\int_0^\infty dxM_{ijL-2}^{(\ell+2)}\left[\ln\left(\frac{x}{2r}\right)+\sum_{k=1}^{\ell-2}\frac{1}{k}\right]+O\left(\frac{\ln r}{r^2}\right)\end{aligned} \tag{C4c}$$

(with $c=1$). When $\ell=2$ this metric reduces to (B4) in Appendix B of [3]. In the case of the interacting multipoles $M\times S_L$, we find

$$h^{00}_{\mathrm{can}(2)}=O\left(\frac{\ln r}{r^2}\right)\,, \tag{C5a}$$

$$\begin{aligned}h^{0i}_{\mathrm{can}(2)}=&-\frac{8(\ell+2)}{(\ell+1)!(\ell+1)}\frac{n_{aL-1}\varepsilon_{iab}MS_{bL-1}^{(\ell+1)}}{r}\\&+\frac{8\ell}{(\ell+1)!}\frac{n_{aL-1}\varepsilon_{iab}M}{r}\int_0^\infty dxS_{bL-1}^{(\ell+2)}\left[\ln\left(\frac{x}{2r}\right)+\sum_{k=1}^{\ell}\frac{1}{k}\right]+O\left(\frac{\ln r}{r^2}\right)\,,\end{aligned} \tag{C5b}$$

$$\begin{aligned}h^{ij}_{\mathrm{can}(2)}=&-\frac{16\ell}{(\ell+1)!(\ell+1)}\frac{n_{aL-1(i}\varepsilon_{j)ab}MS_{bL-1}^{(\ell+1)}}{r}+\frac{16(\ell-1)}{(\ell+1)!(\ell+1)}\frac{n_{aL-2}\varepsilon_{ab(i}MS_{j)bL-2}^{(\ell+1)}}{r}\\&+\frac{16\ell}{(\ell+1)!}\frac{n_{aL-2}\varepsilon_{ab(i}M}{r}\int_0^\infty dxS_{j)bL-2}^{(\ell+2)}\left[\ln\left(\frac{x}{2r}\right)+\sum_{k=1}^{\ell-1}\frac{1}{k}\right]+O\left(\frac{\ln r}{r^2}\right)\,.\end{aligned} \tag{C5c}$$

In (C4) and (C5) the moments are evaluated at $t-r/c$ in the instantaneous terms, and at $t-r/c-x/c$ in the tail terms. From the metrics (C4) and (C5) we immediately deduce the values of κ_ℓ and κ'_ℓ entering the tail terms in the radiative moments. These are

$$\kappa_\ell=\frac{2\ell^2+5\ell+4}{\ell(\ell+1)(\ell+2)}+\sum_{k=1}^{\ell-2}\frac{1}{k}\,, \tag{C6}$$

$$\kappa'_\ell=\frac{\ell-1}{\ell(\ell+1)}+\sum_{k=1}^{\ell-1}\frac{1}{k}\,. \tag{C7}$$

We thus find the values quoted in (4.33). Note that the constants κ_ℓ and κ'_ℓ depend on the coordinate system which is used, namely the harmonic coordinate system. For instance, the constants would be $\kappa_\ell+\frac{1}{2}$ and $\kappa'_\ell+\frac{1}{2}$

in a (perturbed) Schwarzschild coordinate system. Note also that

$$\lim_{\ell\to+\infty}[\kappa_\ell - \ln\ell] = C \; , \qquad \text{(C8a)}$$

$$\lim_{\ell\to+\infty}[\kappa'_\ell - \ln\ell] = C \; , \qquad \text{(C8b)}$$

where $C = 0.577\ldots$ is Euler's constant. This can be of interest since the combinations $\kappa_\ell - \ln\ell - C$ and $\kappa'_\ell - \ln\ell - C$ arise in the phase of the Fourier transform of the waveform [see (3.5) in [33] for the mass-quadrupole case $\ell = 2$].

[1] L. Blanchet and T. Damour, Ann. Inst. Henri Poincaré A **50**, 377 (1989).

[2] T. Damour and B.R. Iyer, Ann. Inst. Henri Poincaré A **54**, 115 (1991).

[3] L. Blanchet and T. Damour, Phys. Rev. D **46**, 4304 (1992).

[4] L. Blanchet, T. Damour, and B.R. Iyer (in preparation).

[5] A. Krolak and B.F. Schutz, Gen. Relativ. Gravit. **19**, 1163 (1987).

[6] A. Krolak, in *Gravitational Wave Data Analysis*, edited by B.F. Schutz (Kluwer Academic, Dordrecht, 1989).

[7] C.W. Lincoln and C.M. Will, Phys. Rev. D **42**, 1123 (1990).

[8] C. Cutler, T.A. Apostolatos, L. Bildsten, L.S. Finn, E.E. Flanagan, D. Kennefick, D.M. Markovic, A. Ori, E. Poisson, G.J. Sussman, and K.S. Thorne, Phys. Rev. Lett. **70**, 2984 (1993).

[9] L.S. Finn and D.F. Chernoff, Phys. Rev. D **47**, 2198 (1993).

[10] C. Cutler and E. Flanagan, Phys. Rev. D **49**, 2658 (1994).

[11] L. Blanchet and B.S. Sathyaprakash, Phys. Rev. Lett. **74**, 1067 (1995).

[12] E. Poisson, Phys. Rev. D **47**, 1497 (1993).

[13] C. Cutler, L.S. Finn, E. Poisson, and G.J. Sussmann, Phys. Rev. D **47**, 1511 (1993).

[14] H. Tagoshi and T. Nakamura, Phys. Rev. D **49**, 4016 (1994).

[15] H. Tagoshi and M. Sasaki, Prog. Theor. Phys. **99**, 745 (1994).

[16] W.B. Bonnor, Philos. Trans. R. Soc. London **A251**, 233 (1959).

[17] W.B. Bonnor and M.A. Rotenberg, Proc. R. Soc. London **A265**, 109 (1961).

[18] W.B. Bonnor and M.A. Rotenberg, Proc. R. Soc. London **A289**, 247 (1966).

[19] A.J. Hunter and M.A. Rotenberg, J. Phys. A **2**, 34 (1969).

[20] K.S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).

[21] L. Blanchet and T. Damour, Philos. Trans. R. Soc. London **A320**, 379 (1986).

[22] L. Blanchet, Proc. R. Soc. London **A409**, 383 (1987).

[23] L. Blanchet and T. Damour, Phys. Rev. D **37**, 1410 (1988).

[24] L. Blanchet, Phys. Rev. D **47**, 4392 (1993).

[25] P.A. Lagerström, L.N. Howard, and C.S. Liu, *Fluid Mechanics and Singular Perturbations; A Collection of Papers by Saul Kaplun* (Academic, New York, 1967).

[26] K.S. Thorne, Astrophys. J. **158**, 997 (1969).

[27] W.L. Burke, J. Math. Phys. **12**, 401 (1971).

[28] R.E. Kates, Phys. Rev. D **22**, 1853 (1980).

[29] L. Blanchet and T. Damour, Phys. Lett. **104A**, 82 (1984).

[30] R. Epstein and R.V. Wagoner, Astrophys. J. **197**, 717 (1975).

[31] T. Damour and B.R. Iyer, Phys. Rev. D **43**, 3259 (1991).

[32] L. Blanchet, in *Proceedings of the 4th Marcel Grossmann Meeting on General Relativity*, Rome, Italy, 1985, edited by R. Ruffini (North-Holland, Amsterdam, 1986), p. 895.

[33] L. Blanchet and G. Schäfer, Class. Quantum Grav. **10**, 2699 (1993).