

Quantum evolution of disoriented chiral condensates

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The nonequilibrium evolution of the hadronic plasma produced in a high energy heavy ion collision is studied in the $O(4)$ linear σ model to leading order in a large N expansion. Starting from an approximate equilibrium configuration at an initial proper time τ in the disordered phase we study the transition to the ordered broken symmetry phase as the system expands and cools. We give results for the proper time evolution of the effective pion mass, the order parameter $\langle\sigma\rangle$ as well as for the pion two point correlation function. We study the phase space of initial conditions that lead to instabilities (exponentially growing long wavelength modes) which can lead to disoriented chiral condensates. We find that as a consequence of the strong self-coupling instabilities can exist for proper times that are at most 3 fm/c and lead to condensate regions that do not contain large numbers of particles.

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I. INTRODUCTION

In recent years there have been numerous investigations concerning the possibility of forming large correlated regions of misaligned vacuum during highly energetic collisions [1–3]. Those regions, in which the quark condensate $\langle\bar{q}_i q_j\rangle$ is nonzero but points along the wrong direction in isospin space, have been named disoriented chiral condensates (DCC's). If large DCC's are indeed formed, they would produce spectacular events in which one could observe strong correlations between the emitted pions. In fact, the original idea of DCC's was invented [1] to explain rare events observed in cosmic ray experiments [4]: the Centauro events, where there is a deficit of neutral pions. In this approach the explanation of the deficit would be the result of the decay of domains in which the condensate has a vanishing component in the π^0 direction. Anti-Centauro events (whose experimental status is still uncertain), where the emission would be predominantly neutral, would be explained as coming from regions in which the condensate points along the π^0 direction.

If these events are a result of the creation of DCC's, this would be direct evidence for the existence of a chiral phase transition in the plasma formed following an ultrarelativistic collision and would allow us to explore the physics of the chiral phase transition. The possibility of producing DCC's in high energy collisions has originated several experimental proposals [2, 5] and a number of theoretical papers [6–13]. It has been recognized that the possibility of forming large regions of DCC relies on the existence of a substantially large regime in which the hot plasma formed after the collision evolves *out of equilibrium* [3]. In fact, if thermal equilibrium is approximately preserved by the dynamics, the typical correlation length would be determined by the pion mass and therefore would be too small to matter. For this reason, there have been a number of authors making different

attempts to analyze the nonequilibrium aspects of the dynamics of the chiral phase transition. These attempts vary in form and content: some authors performed numerical simulations on classical models [3, 10, 11], others used phenomenological terms, inspired in classical kinetic theory, to model the interaction between the condensate and the quasiparticles [12], while some attempted to incorporate quantum and thermal fluctuations [7, 13] into the theoretical framework.

There are clearly two important questions that any theoretical model should answer. The first one is to determine whether during the evolution that follows the collision there are instabilities affecting the fluctuations. If this happens, then there is a chance for the correlations to grow through a process such as spinodal decomposition. If it does not, then the typical length scale of the problem, the pion Compton wavelength, will be the only correlation length to be imprinted on the final state pions, and no large DCC signal can be expected. If instabilities do occur the second important question is whether the size of correlated domains can grow large enough for many correlated pions to be emitted from each domain.

Without a great deal of detailed calculation it should be clear from the outset that the answer to both of these questions will be dependent on the assumptions one is willing to make about the initial state of the plasma immediately after the collision, as well as its detailed evolution afterwards. For example, the typical way to introduce instability into the problem has been by making a rapid "quench." In a quench, the hot plasma created after the collision is subject to a sudden external action (the expansion) that is assumed to modify the effective potential, turning it "upside down," without changing the instantaneous state of the system. This is clearly an idealization which is appropriate only if the expansion is much more rapid than the typical interaction time in the plasma. Now, the time scale for interactions and instabilities in the plasma is determined by the inter-

action strength and cross sections. Using the linear σ model as the effective field theory to describe the chiral phase transition, the quartic coupling must be rather large ($\lambda_r \approx 10$) in order to model low energy π - π scattering data, even approximately. In this moderately strong coupling regime interaction time scales are *not* long compared to the plasma expansion time scale, and instabilities exist only for short periods of proper time as we shall see below. This makes it difficult to obtain large sizes for the correlated domains. In this regard it is important to realize that intuitions developed in cosmological phase transitions, where the coupling is extremely weak, may not apply at all. In a weakly coupled system the domains can grow for long periods of time, and a totally different picture can emerge. Instabilities and domain growth for weak coupling were studied in Refs. [3, 14, 15].

In this paper our aim is to study the evolution of DCC's without imposing the quench approximation or any other *ad hoc* way of forcing the appearance of an instability. Instead, we shall specify initial conditions for the evolution in the symmetric phase (where all particle masses are positive) by choosing a thermal distribution of particles above the critical temperature. This initial condition is not necessary, but is one way of ensuring that the initial system is in the disordered phase, and parametrizing the initial state with a single parameter, the temperature. More complicated initial conditions may be studied as well. Then, we model the cooling of the plasma by the boost invariant kinematics appropriate for the central rapidity region of an ultrarelativistic heavy-ion collision as introduced originally by Landau, and discussed extensively by Bjorken [16]. This picture is consistent with various hydrodynamic approaches to hadronic as well as heavy-ion collisions. Boost invariance is equivalent to the assumption of flat rapidity distributions and implies that mean field expectation values should depend only on the fluid proper time $\tau = \sqrt{t^2 - x^2}$, where x is the distance along the collision axis in the center of mass frame. Then the system expands and cools as proper time evolves, which leads automatically to an axially symmetric (*not* radially symmetric) form of the “baked alaska” scenario described by several authors [2, 3]. Whether or not instabilities develop becomes a dynamical problem which may be studied numerically by solving the updated equations for the quantum modes as well as for the proper time evolving expectation values. The technical means of incorporating thermal and nonequilibrium quantum effects into the evolution problem is the large N expansion. The lowest order in the large N approximation is related to the Hartree approximation and has been widely discussed in the literature [18] and [19]. How to calculate systematic corrections to the lowest order results is discussed in detail in a recent publication [17].

In this way, we are able to study the evolution of the plasma in a self-consistent way without imposing any instability by hand. We analyze various reasonable initial conditions on the fields and determine whether they lead to instabilities by examining the effective pion mass as a function of proper time τ . When the effective pion mass becomes negative instabilities ensue. We also determine the time evolving order parameter $\langle \sigma \rangle$ and the adiabatic

phase space number density and pair density which determine the spatial correlation function for the pion field. These number densities are related to physical measurable (such as the rapidity distribution of final particles) at later times. The initial conditions which lead to the largest instabilities have initial velocities in either the σ or π directions. We compare the nonequilibrium results for the number density with those that would have resulted from an expansion with local thermal equilibrium in the comoving frame. When instabilities arise the distributions tend to narrow in momentum space, especially in the transverse direction.

In the investigations done so far, simple phenomenological models have been used hoping that they describe the fundamental physics involved in the dynamics of the phase transition. We will employ the linear σ model, the most popular one in this context, which seems to have the essential attributes of being simple but realistic enough: it appropriately describes the low energy phenomenology of pions and has also the correct chiral symmetry properties. The initial conditions we will impose are motivated by matching it to the situation one expects to attain in a highly energetic collision. We will assume that the quantum state of the system is a thermal density matrix at an initial instant of proper time τ_0 . We will choose the initial temperature T to be slightly above the critical temperature for being in the disordered phase. As shown in the Appendix the critical temperature is given by $T_c^2 = 3f_\pi^2$. We will choose the parameters of the model to give reasonable values for three experimentally determined quantities: the mass of the pion m_π , the pion decay constant f_π , and the s wave π - π phase shifts above threshold. These three measurements completely determine the parameters of the model.

One important constraint on the linear σ model is its triviality as the cutoff is removed. The theory only makes sense at cutoffs below the Landau pole which occurs at a value of the cutoff Λ when the bare coupling constant first becomes negative for positive renormalized coupling constant. This limits the size of the renormalized coupling constant. The maximum renormalized coupling constant as a function of Λ obeys, for large Λ ,

$$\lambda_r^{\max} \approx \frac{2\pi^2}{\ln\left(\frac{2\Lambda}{m}\right)}. \quad (1.1)$$

Since the mass difference between the σ and π is directly proportional to λ_r , this leads to an upper bound for the mass of the σ resonance as a function of Λ . Therefore, unlike at tree level, the mass of the σ in the fully quantized theory is constrained in this model. Λ is also constrained from the physical consideration that we want the mass of the σ to be less than the cutoff. However, the mass of the σ resonance increases as we decrease the cutoff since then the renormalized coupling increases.

The rest of the paper is organized as follows. In Sec. II we describe the linear σ model in the leading order in the $1/N$ expansion. In Sec. III we discuss the baked alaska scenario, and we derive renormalized update equations

for the proper time evolution of the field theory. In Sec. IV we describe the initial conditions we use to study the development of disoriented chiral condensates. In Sec. V we discuss the results of extensive numerical simulations. In the Appendix we present the properties of the linear σ model in the large N approximation, for completeness.

II. MODEL AND APPROXIMATIONS

We will use the O(4) linear σ model to describe the evolution of the pions. We are well aware of the limitations of this approach, which provides a reasonable phenomenological model only for a limited range of energies (typically smaller than 1 GeV). In spite of its shortcomings, this model captures some of the essential physics involved in the dynamics of the phase transition that may produce disoriented chiral condensates. In particular the chiral phase transition takes place at a reasonable temperature ($T_c = \sqrt{3}f_\pi$), and the low energy π - π scattering amplitudes are reasonable in this model. The mesons are organized in an O(4) vector $\Phi = (\sigma, \boldsymbol{\pi})$, and the action is (in natural units $\hbar = c = 1$)

$$S = \int d^4x \left\{ \frac{1}{2} \partial\Phi \cdot \partial\Phi - \frac{1}{4} \lambda (\Phi \cdot \Phi - v^2)^2 + H\sigma \right\}, \quad (2.1)$$

where we have used the Bjorken and Drell metric: (1, -1, -1, -1). We will describe the evolution of the mean value $\bar{\Phi} \equiv \langle \Phi \rangle$ and the two-point correlation functions including the effects of quantum and thermal fluctuations. The expectation value of the equations of motion obtained from (2.1) in a given initial density matrix defined at $\tau = \tau_0$, defines the initial value problem for the correlation function of the quantum problem. Perturbation theory is useless for our purpose [14, 19] and a scheme which is nonperturbative in λ must be adopted. In this context, one approach that allows for a real time analysis is the Hartree (or Gaussian) ansatz [18]. However the large N expansion of the O(N) sigma model offers several advantages [20]. On the one hand, the expansion is systematic and allows us to study higher order corrections (work is in progress in this direction, and results will be presented elsewhere [21]). On the other hand, when using the Hartree ansatz in the context of the study of DCC's one is forced to take also the large N limit (see Ref. [13]). This is due to the well-known fact that the Gaussian approximation violates the Goldstone theorem giving an unphysical (and not necessarily small) mass to the pions in the $H = 0$ limit. Of course, the approximation we adopt here is not expected to capture all the features of the phase transition (as is well known, mean field theory fails to predict the correct critical exponents but allows us to explore the strong coupling regime). Because of the triviality of the O(N) σ model as one takes away the cutoff, an aspect of the exact theory that is preserved in the large N approximation, we have to seriously take into account the cutoff and its ramifications. One of these ramifications is that the renormalized coupling constant has a modest upper bound of the order of ten at a cutoff of one GeV. This gives an upper bound to the mass of the σ resonance whose value depends on our choice of cutoff.

The large N effective equations can be obtained in a variety of ways, which are extensively discussed in the literature [20]. A very convenient method is to use an effective action, which is a functional of the mean values of the original fields Φ and of an auxiliary constrained field χ [22]. We start with a classical action $S[\Phi, \chi]$ constructed from (2.1) by replacing $\chi = \lambda(\Phi^2/2N - v^2)$. As this action is now quadratic in Φ we can perform the functional integral over those fields and are left with a functional integration over χ which, to leading order in $1/N$, can be calculated by the stationary phase method. In the Appendix we review the details of this calculation and calculate all the propagators and the π - π scattering amplitude in the leading order in the large N expansion. Higher order corrections can be systematically computed in this way [22] and an expansion of the effective action $\Gamma[\bar{\Phi}, \bar{\chi}]$ in powers of $1/N$ can be obtained [20]. We will consider here only the leading order terms which give the equations (for notational convenience we drop the overbars and denote the expectation values Φ and χ)

$$[\square_x + \chi(x)]\Phi_i(x) = H\delta_{i1}, \quad (2.2)$$

$$\chi(x) = \lambda(-v^2 + \Phi^2(x) + NG_0(x, x)), \quad (2.3)$$

$$G_0^{-1}(x, y) = i[\square_x + \chi(x)]\delta(x - y). \quad (2.4)$$

The structure of these equations is indeed very simple. The field χ plays the role of the effective mass for the mean values Φ_i and satisfies the ‘‘gap equation’’ (2.3). The function $G_0(x, x)$ that appears in (2.3) is the coincidence limit of the propagator $G_0(x, y)$ that inverts the operator G_0^{-1} defined in (2.4). We can use an auxiliary quantum field $\phi(x)$ where $\langle \phi(x) \rangle = 0$ and

$$[\square_x + \chi(x)]\phi(x) = 0 \quad (2.5)$$

to determine $G_0(x, y)$. We construct the propagator as

$$G_0(x, y) = \langle T\phi(x)\phi(y) \rangle.$$

$\delta_{ij}G_0(x, y)$ is the pion propagator when $\langle \pi_i(x) \rangle = 0$. The initial value problem associated with Eqs. (2.2)–(2.4) will be solved in the next section. Here, we would like to address the issue of how to use this model to make contact with the phenomenology we want to describe. Thus, we must fix the values of the parameters appearing in the above equations so that they describe low energy pion physics. The measurable quantities we want to reproduce are the pion mass $m_\pi = 135$ MeV, the pion decay constant $f_\pi = 92.5$ MeV, as well as the s wave, $I = 0$ phase shift in the energy range 300–420 MeV. To fit these physical quantities we analyze our equations in the ‘‘true vacuum’’ state (i.e., in equilibrium at zero temperature). In such a state the derivatives of the expectation values vanish, and we have $\Phi = (\sigma_v, \mathbf{0})$, $\chi = \chi_v$, where σ_v and χ_v are some constants whose values will be determined below. The physical masses can be related to the parameters of the theory by computing the inverse propagators of the pion and sigma fields. The s -wave phase shift is determined from the π - π scattering amplitude obtained in this approximation which is given by the exchange of the composite field χ propagator in all three channels.

A complete review of the vacuum and finite temperature properties of this model in leading order in large N is found in the Appendix, and we summarize the results here. The vacuum expectation value of σ is determined by f_π :

$$\sigma_v = f_\pi. \quad (2.6)$$

On the other hand, in the vacuum we have $\pi = 0$, and the pion inverse propagator is $G_{ij}^{-1}{}_{\pi,\pi}(x,y) = G_0^{-1}(x,y)\delta_{ij}$. Therefore, the vacuum expectation value of χ is

$$\chi_v = m_\pi^2 \equiv m^2. \quad (2.7)$$

The σ mass can be approximately determined in terms of the inverse σ propagator as the zero in the real part of the inverse propagator (a more precise determination which gives a slightly different result is from the peak in the $I = 0$, $l = 0$ scattering amplitude). This leads to the equation

$$m_\sigma^2 = m^2 + \sigma^2 \text{Re}[\hat{G}_{0\chi\chi}(m_\sigma^2)], \quad (2.8)$$

where $\hat{G}_{0\chi\chi}$ is the composite field propagator in the absence of symmetry breaking, i.e.,

$$\hat{G}_{0\chi\chi}^{-1}(p^2) = \frac{1}{2\lambda} + \frac{N}{2}\Pi(p^2), \quad (2.9)$$

and the polarization $\Pi = iG_0^2$ is given by

$$\Pi(p^2) = -i \int [d^4q] (\chi - q^2)^{-1} (\chi - (p+q)^2)^{-1} \quad (2.10)$$

and is explicitly given in the Appendix. We use the notation that $[d^nq] = d^nq/(2\pi)^n$. In the absence of symmetry breaking the composite field propagator is the geometric sum of the bubble chains. In the presence of symmetry breakdown one is also summing the graphs where the bubble is replaced by a single propagator and two tadpoles (this includes the σ exchange graphs). One can determine the bare mass of the pion in terms of the physical pion mass using the gap equation, since $\chi_v = m^2$

$$m^2 = -\lambda v^2 + \lambda f_\pi^2 + \lambda N G_0(x, x), \quad (2.11)$$

where

$$G_0(x, x) = \int_0^\Lambda [d^3k] \frac{1}{2\sqrt{k^2 + m^2}}. \quad (2.12)$$

This relationship tells us that the bare mass goes to negative infinity quadratically with the cutoff, as is true for the exact lattice theory. (That is the reason why in strong coupling the theory is related to the Ising model). This relationship also allows us to eliminate the bare mass from the theory as well as the quadratic dependence on the cutoff. The logarithmic dependence on the cutoff is removed by coupling constant renormalization. We obtain a value of the bare coupling λ at a fixed cutoff Λ (or equivalently the renormalized coupling λ_r) by comparing our large- N result with the Padé fit of Basdevant and Lee [25] to the $I = 0$ s -wave scattering amplitude. This is discussed in detail in the Appendix. Once λ is determined both v and m_σ are fixed. Therefore all the

bare parameters of this theory are fitted in terms of the three experiments which determine m , f_π , and λ . The external field H is determined from the equilibrium time-independent solution to the field equation for the σ field: $\chi\sigma = m_\pi^2 f_\pi = H$. Our numerical simulations automatically have a cutoff since we are performing numerical simulations in a box. Thus one can use either the bare or renormalized parameters to describe the problem. The renormalized parameters, however, are determined from physical measurements and are more fundamental. However since the σ model is not an asymptotically free field theory, it is a trivial theory when we take the cutoff away. Thus one is not allowed to take the continuum limit and must consider the theory as an effective field theory which is not valid for energies in excess of a few GeV's. Our model is a theory which makes physical sense (such as having a well-defined ground state) only with a physical cutoff Λ_p (see [23, 24] for discussions on Φ^4 as a cutoff theory). On physical grounds we want $2m_\pi < m_\sigma < \Lambda$. This relation is quite a constraint on Λ since when we lower Λ we increase the value of the mass of the σ . We certainly do not expect the theory to be valid for energies above 1 GeV since in that regime the correct dynamics is described by QCD. In fact if we raise Λ above 1 GeV, the maximum value of the renormalized coupling goes down and the σ mass becomes unacceptably low. Reasonable values for the mass of the σ constrain Λ to be in the range $0.7 \text{ GeV} < \Lambda \leq 1 \text{ GeV}$. In that range the best value of the renormalized coupling λ_r is between 7 and 10. For this range of values for λ_r this model agrees qualitatively with low energy scattering data. With this choice of parameters, it is difficult to obtain values of the σ mass higher than 450 MeV. Based on the perturbative calculations of Basdevant and Lee [25] which were then subjected to a Padé improvement, and the connection between $1/N$ expansions and resummations we expect that at next order in $1/N$ this upper bound on the σ mass will be raised slightly.

As will be clear from our discussion below, when solving the equations we can use a different value of the cutoff provided we scale the bare couplings appropriately (so that we keep the physical quantities unchanged). As the theory we are using is renormalizable, this scaling is well defined and known. However, the cutoff cannot be taken too large since the theory only makes sense for positive values of the bare coupling. This is the Landau pole problem and is related to the triviality of the theory [27, 28]. As is discussed in detail in the Appendix, the bare and renormalized couplings are related by the equation

$$\lambda = \frac{\lambda_r}{1 - N\lambda_r\Pi(0)}. \quad (2.13)$$

We see that λ has a pole when $1 = N\lambda_r\Pi(0)$. At large values of the cutoff this Landau pole occurs when

$$\ln(2\Lambda/m_\pi) \approx \frac{8\pi^2}{N\lambda_r}. \quad (2.14)$$

For fixed λ_r we notice that for values of the cutoff larger than that given by (2.14), the bare coupling becomes negative which makes the theory undefinable as a Euclidean

lattice field theory. This relationship also means that at a fixed cutoff, there is a maximum λ_r that we can consider. That is the inverse relationship

$$\lambda_r = \frac{\lambda}{1 + N\lambda\Pi(0)}. \quad (2.15)$$

The maximum of λ_r is reached at infinite λ and one obtains

$$\lambda_r^{\max} = \frac{1}{N\Pi(0)}. \quad (2.16)$$

For a cutoff of 1 GeV the maximum λ_r is 13.

As a final comment we would like to point out that the effective action method we employed can also be used to compute higher order corrections in $1/N$. In that case it is necessary to use a formalism that enables us to derive real and causal equations for the expectation values (otherwise one cannot even pose the initial value problem). This formalism is known as the “closed time path” method developed by Schwinger, Mahanthappa, and Keldysh [29]. The analysis of the $1/N$ corrections obtained using this approach (that carry the effects of the collisions that may produce relaxation towards equilibrium) will be analyzed elsewhere [21] (see also Refs. [7, 14, 20] for recent applications of this formalism in the DCC and other related contexts).

III. COOLING BY EXPANSION AND BAKED ALASKA SCENARIO

A. The basic idea

To analyze the possibility of forming DCC’s we should take into account the specific features characterizing the situation after a highly energetic heavy-ion collision. Experimentally, a flat plateau in the distribution of produced particles per unit rapidity is observed in the central rapidity region (these results are obtained in p - \bar{p} and other collisions). This suggests the existence of an approximate Lorentz boost invariance. Thus, the simplest picture of a collision, due to Landau [30], is one in which the excited nuclei are highly contracted pancakes receding away from the collision point at approximately the speed of light. The boost invariance implies that the evolution of the “hot plasma” that is left in between the nuclei looks the same when viewed from different inertial frames. Of course, this is an approximate picture that is not valid for large values of the spatial rapidity and for transverse distances of the order of the nucleus size. The existence of this approximate symmetry can be used to make a very simple hydrodynamical model [16] that, in some cases, may describe the evolution of the plasma. It is worth reviewing very briefly these ideas. First, one should recognize that the natural coordinates to make a boost invariant model are the proper time τ and the spatial rapidity η defined as

$$\tau \equiv (t^2 - x^2)^{1/2}, \quad \eta \equiv \frac{1}{2} \ln \left(\frac{t - x}{t + x} \right). \quad (3.1)$$

The observed symmetry will be respected by the model

if one imposes initial values on a $\tau = \text{const}$ hypersurface (and not at constant laboratory time t).

If we had local thermodynamic equilibrium during the expansion so that the relaxation rate is faster than the expansion rate and assumed that we could approximate the field theory dynamics with a hydrodynamic flow, one would find by solving the hydrodynamical equations with this type of initial conditions (homogeneity along the constant τ surface) that the energy density drops as $\tau^{-\alpha}$. Here $\alpha = 1 + c_0^2$ where c_0 is the speed of sound, which depends on the equation of state of the fluid $p = c_0^2 \epsilon$. In the ultrarelativistic case $c_0^2 = 1/3$ and the temperature falls as $\tau^{-1/3}$. Of course, with an effective field theory of the O(4) σ model one replaces these hydrodynamical ansätze with the evolution Eqs. (2.2)–(2.4) instead.

According to this simple model of a collision, the plasma evolves in a highly inhomogeneous way when viewed from the laboratory frame. In fact, analyzing a constant t surface we realize that the field configurations strongly depend on the spatial coordinate x . Near the light cone $|x| = t$ the system is “hot” (corresponding to small values of the proper time τ). On the contrary, for small values of x (that correspond to larger values of τ) the system is “colder.” This type of configuration, hot on the outside and cold on the inside, is schematically known as baked alaska.

In this paper we want to study the formation of DCC’s using some of the ideas presented above. We *will not* assume a quasiequilibrium situation or use a phenomenological hydrodynamical (or kinetic) model to describe the evolution of our system. On the contrary, we will study the nonequilibrium evolution in its full glory and solve Eqs. (2.2)–(2.4), which include thermal and quantum effects. Using the coordinates (3.1) and fixing boost invariant initial conditions at an initial proper time τ_0 (whose numerical value we discuss below) we introduce the expansion and “cooling” of the plasma in a natural way. Therefore, we do not need to introduce any ad hoc cooling mechanism by hand. The cooling, if any, will appear as a result of the evolution, which is fully out of equilibrium. In this way, we can really test the validity of the “quenching” approximation that has been almost universally used when analyzing the evolution of DCC’s.

B. The equations

Our treatment is very similar to the one required to study quantum field theory in a curved spacetime [31]. Thus, in the coordinates (3.1) Minkowski’s arc element is

$$ds^2 = d\tau^2 - \tau^2 d\eta^2 - dx_\perp^2, \quad (3.2)$$

which has the same form of a Kasner universe (an anisotropically expanding universe, which in this case is nevertheless flat, see [31]). To study the evolution of the mean fields and correlations in this coordinates our first task is to rewrite Eqs. (2.2)–(2.4) using the new variables. Assuming that the mean values Φ and χ are functions of τ only (homogeneity in the constant τ hypersurface) we have

$$\tau^{-1}\partial_\tau \tau\partial_\tau \Phi_i(\tau) + \chi(\tau) \Phi_i(\tau) = H\delta_{i1}, \quad (3.3)$$

$$\chi(\tau) = \lambda(-v^2 + \Phi_i^2(\tau) + N\langle\phi^2(x, \tau)\rangle), \quad (3.4)$$

where the quantum field $\phi(x, \tau)$ satisfies the Klein-Gordon equation

$$(\tau^{-1}\partial_\tau \tau\partial_\tau - \tau^{-2}\partial_\eta^2 - \partial_\perp^2 + \chi(x))\phi(x, \tau) = 0. \quad (3.5)$$

The quantum field $\phi(x, \tau)$ defined here is an auxiliary field which allows us to calculate the Wightman function $G_0(x, y)$ by taking the expectation value:

$$G_0(x, y; \tau) \equiv \langle\phi(x, \tau) \phi(y, \tau)\rangle. \quad (3.6)$$

When the expectation value of the pion field is zero, then this field corresponds to one component of the pion field.

As usual, we expand this field in an orthonormal basis:

$$\phi(\eta, x_\perp, \tau) \equiv \frac{1}{\tau^{1/2}} \int [d^3k] (\exp(ik \cdot \mathbf{x}) f_k(\tau) a_k + \text{H.c.}) \quad (3.7)$$

where $k \cdot \mathbf{x} \equiv k_\eta \eta + \mathbf{k}_\perp \cdot \mathbf{x}_\perp$, $[d^3k] \equiv dk_\eta d^2k_\perp / (2\pi)^3$ and the mode functions $f_k(\tau)$ evolve according to (an overdot here denotes the derivative with respect to the proper time τ)

$$\ddot{f}_k + \left(\frac{k_\eta^2}{\tau^2} + \mathbf{k}_\perp^2 + \chi(\tau) + \frac{1}{4\tau^2} \right) f_k = 0. \quad (3.8)$$

For the creation and annihilation operators to obey the usual canonical commutation relations $[a_k, a_k^\dagger] = 1$, then the mode functions must satisfy the Wronskian condition

$$f \dot{f}^* - f^* \dot{f} = i.$$

The expectation value $\langle\phi^2(x, \tau)\rangle$ can be expressed in terms of the mode functions f_k and of the distribution functions

$$n_k \equiv \langle a_k^\dagger a_k \rangle, \quad g_k \equiv \langle a_k a_k \rangle, \quad (3.9)$$

which entirely characterize the initial state of the quantum field. For simplicity, we will assume that the initial state is described by a density matrix which is diagonal in the number basis (like a thermal state). In such case, the only nonvanishing distribution is n_k . Thus, replacing the above expressions in (3.4) we have

$$\chi(\tau) = \lambda \left(-v^2 + \Phi_i^2(\tau) + \frac{1}{\tau} N \int [d^3k] |f_k(\tau)|^2 \times (1 + 2 n_k) \right). \quad (3.10)$$

We assume the initial density matrix is one of local thermal equilibrium in the comoving frame. In the comoving frame, the expectation value of the energy-momentum tensor is diagonal and of the form

$$\text{diag}(\epsilon, p, p, p). \quad (3.11)$$

We then have at $\tau = \tau_0$ (the surface of constant energy density and temperature T_0) that

$$n_k = \frac{1}{e^{\beta_0 E_k^0} - 1}, \quad (3.12)$$

where $\beta_0 = 1/T_0$, $E_k^0 = \sqrt{k^2(\tau_0) + \chi(\tau_0)}$, and $k^2 = k_\eta^2/\tau^2 + k_\perp^2$.

We can also use a time-dependent set of creation and annihilation operators to describe the quantum field ϕ . If we use the first order adiabatic mode function,

$$f_k^0 = \frac{e^{-iy_k(\tau)}}{\sqrt{2\omega_k}}, \quad dy_k/d\tau = \omega_k, \quad (3.13)$$

where $\omega_k(\tau) \equiv \left[\frac{k_\eta^2 + 1/4}{\tau^2} + \mathbf{k}_\perp^2 + \chi(\tau) \right]^{1/2}$. Then if we expand the field in terms of these mode functions we have

$$\phi(\eta, x_\perp, \tau) \equiv \frac{1}{\tau^{1/2}} \int [d^3k] [\exp(ik \cdot \mathbf{x}) f_k^0(\tau) a_k(\tau) + \text{H.c.}] \quad (3.14)$$

We can now define the first order adiabatic interpolating number and pair distribution functions via

$$n_k(\tau) \equiv \langle a_k^\dagger(\tau) a_k(\tau) \rangle, \quad g_k(\tau) \equiv \langle a_k(\tau) a_k(\tau) \rangle. \quad (3.15)$$

The time-dependent creation and annihilation operators satisfy

$$\dot{a} f^0 + a^\dagger \dot{f}^{*0} = 0, \quad (3.16)$$

in order that the usual canonical commutation relations hold. Then we can rewrite the Green's function G^0 in the two bases

$$G^0(x, y; \tau) = \frac{1}{\tau} \int [d^3k] e^{ik(x-y)} \frac{1}{2\omega_k(\tau)} (1 + 2 n_k(\tau) + 2\text{Re}[g_k(\tau) e^{-i2y_k(\tau)}]) \quad (3.17)$$

or

$$G^0(x, y; \tau) = \frac{1}{\tau} \int [d^3k] e^{ik(x-y)} [1 + 2 n_k(0)] |f_k(\tau)|^2. \quad (3.18)$$

The τ -dependent variables n and g now have the physical interpretation as the interpolating phase space number and pair density for the case when $\langle\pi_i\rangle = 0$. These operators agree with the time-independent number and pair densities defined by (3.9) at τ_0 , and in the out regime become the physically measurable number and pair densities. For $\langle\pi_i\rangle \neq 0$ the actual pion two-point function is more complicated than G^0 , and this is only one piece of that Green's function. Of course in the vacuum isospin conservation requires $\langle\pi_i\rangle = 0$, so that once we are approaching the final true vacuum state during the cooling process, one can use these interpolating number and pair operators to describe the physics of the problem. Also note that during the instability phase, ω_k^2 can become negative, and for those modes the adiabatic basis does not exist because the Wronskian condition for the mode functions is no longer satisfied.

The two sets of creation and annihilation operators are connected by a Bogoliubov transformation:

$$a_k(\tau) = \alpha(k, \tau)a_k + \beta(k, \tau)a_{-k}^\dagger. \quad (3.19)$$

α and β can be determined from the exact time evolving mode functions via

$$\alpha(k, \tau) = i \left(f_k^{0*} \frac{\partial f_k}{\partial \tau} - \frac{\partial f_k^{0*}}{\partial \tau} f_k \right), \quad (3.20)$$

$$\beta(k, \tau) = i \left(f_k^0 \frac{\partial f_k}{\partial \tau} - \frac{\partial f_k^0}{\partial \tau} f_k \right). \quad (3.21)$$

In terms of the initial distribution of particles $n_k(\tau_0) \equiv n_k$ and the Bogoliubov coefficients α and β we have

$$n_k(\tau) = n_k + |\beta(k, \tau)|^2 (1 + 2n_k), \quad (3.22)$$

$$g_k(\tau) = \alpha(k, \tau)\beta(k, \tau)(1 + 2n_k). \quad (3.23)$$

C. Initial conditions and renormalization

We desire to solve Eqs. (3.3), (3.8), and (3.10) as an initial value problem. To do this we need to give Cauchy data (the function and its derivatives) for the mean values $\Phi_i(\tau)$, $\dot{\Phi}_i(\tau)$, and for the mode functions $f_k(\tau)$. This, together with the distribution function n_k and g_k at τ_0 , fully defines the problem. Notice that the initial value of $\chi(\tau)$ is determined self-consistently by solving the gap equation (3.10) at the initial time τ_0 .

The quantum state is fully determined by the distribution n_k , g_k and by the initial data $f_k(\tau_0)$, $\dot{f}_k(\tau_0)$. These data fix the vacuum state upon which the Fock space is built. If we choose the initial data $f_k(\tau_0)$ and $\dot{f}_k(\tau_0)$ so that the vacuum state coincides with the ordinary Minkowski vacuum for high momentum, then we have the “zero order adiabatic” vacuum described by

$$f_k(\tau_0) = \frac{1}{2\omega_k(\tau_0)}, \quad (3.24)$$

$$\dot{f}_k(\tau_0) = \left(i\omega_k(\tau_0) - \frac{\dot{\omega}_k(\tau_0)}{2\omega_k(\tau_0)} \right) f_k(\tau_0),$$

where $\omega_k(\tau) \equiv [k_\eta^2/\tau^2 + \mathbf{k}_\perp^2 + \chi(\tau)]^{1/2}$. We will fix these initial conditions for the normal modes, which are normalized according to the Wronskian condition $f^* \dot{f} - \dot{f}^* f = i$. The distribution n_k will be taken as thermal, characterized by a temperature T :

$$n_k = [\exp(\omega_k(\tau_0)/k_B T) - 1]^{-1} \quad (3.25)$$

and $g_k = 0$.

The existence of ultraviolet divergences is not a very delicate issue here since we are dealing with a theory that has a natural cutoff. However, it is worth mentioning how the removal of divergences can be implemented. For the initial conditions (3.24), the divergences in $\langle \phi^2(x, \tau) \rangle$ can be shown to be the same ones obtained in the lowest order adiabatic approximation: i.e.,

$$\langle \phi^2(x, \tau) \rangle_{\text{div}} = \frac{1}{\tau} \int [d^3k] \frac{1}{2\omega_k(\tau)}. \quad (3.26)$$

Introducing a cutoff at the physical momenta k_η/τ and k_\perp , we can easily see that the above integral has both quadratic and logarithmic divergences. To renormalize the theory we absorb the quadratic divergences in the

bare mass λv^2 and the logarithmic ones in the coupling constant λ . After a few simple manipulations [that involve adding and subtracting the appropriate terms in (3.10)] we can write the equation for χ as

$$\chi(\tau) = \chi(\tau_0) + \frac{\lambda_r}{Z} (\Phi_i^2(\tau) - \Phi_i^2(\tau_0)) + \frac{N\lambda_r}{\tau Z} \int [d^3k] \times \left\{ |f_k(\tau)|^2 (1 + 2n_k) - \frac{1}{2\tilde{\omega}_k(\tau)} \right\} - \frac{N\lambda_r}{\tau_0 Z} \int [d^3k] \frac{1}{\omega_k(\tau_0)} n_k, \quad (3.27)$$

where $\tilde{\omega}_k(\tau) \equiv [k_\eta^2/\tau^2 + \mathbf{k}_\perp^2 + \chi(\tau)]^{1/2}$ and the renormalized coupling λ_r and Z are defined as

$$\lambda = \lambda_r/Z, \quad Z = 1 - \lambda_r \delta\lambda, \quad \delta\lambda \equiv \frac{N}{4\tau} \int [d^3k] \frac{1}{\tilde{\omega}_k^3(\tau)}. \quad (3.28)$$

The initial value $\chi(\tau_0)$ comes from solving the gap equation (3.10) at the initial time. As we discussed earlier it is useful to rewrite Eq. (3.28) in order to discuss triviality. Namely we have the inverse equation

$$\lambda_r = \frac{\lambda}{1 + \delta\lambda/\lambda}. \quad (3.29)$$

The result is that at a fixed value of Λ there is a maximum renormalized coupling which decreases with the logarithm of the cutoff. As we take the cutoff away, the renormalized coupling constant goes to zero, signifying the triviality of the theory as we remove the cutoff as first discussed by Baker and Kincaid [27] and Bender *et al.* [28].

To make our presentation more clear it may be useful to put together the set of equations we will solve. They are

$$\ddot{\Phi}_i(\tau) + \tau^{-1} \dot{\Phi}_i(\tau) + \chi(\tau) \Phi_i(\tau) = H\delta_{i1}, \quad (3.30)$$

$$\chi(\tau) = \chi(\tau_0) + \frac{\lambda_r}{Z} (\Phi_i^2(\tau) - \Phi_i^2(\tau_0)) + \frac{N\lambda_r}{\tau Z} \int [d^3k] \times \left\{ |f_k(\tau)|^2 (1 + 2n_k) - \frac{1}{2\tilde{\omega}_k(\tau)} \right\} - \frac{N\lambda_r}{\tau_0 Z} \int [d^3k] \frac{1}{\omega_k(\tau_0)} n_k, \quad (3.31)$$

$$\ddot{f}_k + \left(\frac{k_\eta^2}{\tau^2} + \mathbf{k}_\perp^2 + \chi(\tau) + \frac{1}{4\tau^2} \right) f_k = 0. \quad (3.32)$$

It is worth noticing that in these equations the “bare parameters” λ and v^2 do not appear. Thus, the equations are entirely written in terms of renormalized quantities λ_r and χ_0 . The value of χ_0 is obtained from the gap equation (with the same cutoff in all the integrals). Once this is done, the value of the cutoff can be changed at will in (3.31) (provided we do not cross the Landau pole).

For future use, it is convenient to compute the initial value of χ , which is determined by Eq. (3.31). It simply reads

$$\begin{aligned}
\dot{\chi}(\tau_0) & \left(1 + \frac{N\lambda_r}{2\tau_0} \int [d^3k] \frac{n(k)}{\omega_k^3(\tau_0)}\right) \\
& = 2\lambda_r \Phi_i(\tau_0) \dot{\Phi}_i(\tau_0) \\
& \quad - \frac{N\lambda_r}{\tau_0^2} \int [d^3k] \frac{n(k)}{\omega_k(\tau_0)} \left(1 - \frac{k_\eta^2}{\tau_0^2 \omega_k^2}\right). \quad (3.33)
\end{aligned}$$

From here we can see that $\dot{\chi}(\tau_0)$ has two contributions. The first term in the rhs of (3.33) carries the contribution of the mean values: as expected, $\dot{\chi}(\tau_0)$ is sensitive to the existence of initial velocities. The second term, which is always negative, arises from the finite temperature part of the initial state. It clearly shows how the expansion tends to reduce the initial value of χ .

It is instructive at this point to look at the simpler equations that arise in flat space for a spatially homogeneous expansion. In that case we have

$$\ddot{\Phi}_i(t) + \chi(t) \Phi_i(t) = H\delta_{i1}, \quad (3.34)$$

$$\chi(t) = -\lambda v^2 + \lambda \Phi_i^2(t) + N\lambda \int [d^3k] |f_k(t)|^2 (1 + 2n_k), \quad (3.35)$$

$$\ddot{f}_k + [k^2 + \chi(t)] f_k = 0. \quad (3.36)$$

We also have the vacuum relations

$$m_\pi^2 = -\lambda v^2 + \lambda f_\pi^2 + N\lambda \int [d^3k] \frac{1}{2\sqrt{k^2 + m_\pi^2}}, \quad (3.37)$$

which yields

$$\begin{aligned}
\chi(t) & = m_\pi^2 + \lambda \Phi_i^2(t) - \lambda f_\pi^2 \\
& \quad + N\lambda \int [d^3k] \left[|f_k(t)|^2 (1 + 2n_k) - \frac{1}{2\sqrt{k^2 + m_\pi^2}} \right]. \quad (3.38)
\end{aligned}$$

These equations govern the growth of instabilities. We see that there are unstable modes f_k whenever $k^2 + \chi$ is negative so that only the long wavelength modes can go unstable. If we ignore the quantum fluctuation contribution, χ can go negative only when

$$m_\pi^2 + \lambda \Phi_i^2(t) - \lambda f_\pi^2 < 0, \quad (3.39)$$

so that the most unstable case has $\Phi = 0$. One also has that the bare coupling must satisfy

$$\lambda > \frac{m_\pi^2}{f_\pi^2} \quad (3.40)$$

for there to be any instabilities. (Growth of instabilities has also been discussed in Refs. [3, 10, 13–15].) An important point, usually neglected is that once instabilities grow, then they cause exponential growth of the modes in the mode sum which contributes a positive quantity to the equation for χ . Thus the stronger the coupling, the quicker χ returns to a positive value, and the instability tends to shut itself off rather quickly. The stronger the self-coupling, the more quickly this occurs. In determin-

ing the parameters of our effective field theory we found that we are in the strong coupling regime ($\lambda_r \gg 1$). Thus we have the situation where any instabilities are quickly suppressed. This is not true at weak coupling such as found in the early Universe problems. In our approximation, we found that because of the triviality constraint the renormalized coupling constant is constrained to be less than or equal to about 10 if we use $\Lambda = 1$ GeV. We merely comment here that if we were able to consider higher values of λ_r , then the relaxation of instabilities would be even faster and this would only dampen the possibility of producing disoriented chiral condensates.

IV. INITIAL CONDITIONS, CORRELATIONS, AND DOMAINS

A. Reasonable initial conditions

How can we study the possibility of forming large domains in which the pion field is disoriented? As we described above, the picture we have in mind is the following: After the collision, a “baked-alaska”-type configuration is formed. When viewed in the natural “boost invariant” coordinate system this configuration is described as a state characterized by homogeneous mean values $\Phi_i(\tau)$ and by quantum and thermal fluctuations. The quantum state after the collision is clearly unknown, and we will be forced to assume reasonable initial values for the parameters consistent with being near local thermal equilibrium in a disordered phase at a time τ_0 following the collision. Ideally we would like to be able to study the growth of inhomogeneous instabilities in real space and time. However to simplify our calculation we are assuming that the system evolves such that all expectation values just depend on the proper time. This rules out a detailed study of domain growth, and we have to obtain information about the growth of domains indirectly from the quantum correlation functions. These are parametrized by the proper time evolving interpolating phase space number and pair densities. We will study the evolution of different reasonable initial states constrained to have expectation values which depend only on the proper time τ , focusing on the possibility of the development of instabilities and on the potential growth of long range correlations. The question is how to choose the initial conditions defining a “peculiar,” but not entirely unrealistic initial state? Without any rigorous justification, we will assume an initial state that is a *disoriented (or displaced) thermal state*. Let us describe it in more detail.

Let us forget for the moment about the expansion and consider a much simpler situation: thermal equilibrium at temperature T . This is characterized by a value of $\chi = \chi_T$ and by mean fields $\pi = \pi_T = 0$, $\sigma = \sigma_T = H/\chi_T$. The value of χ_T , which is obtained by solving the gap equation given in the Appendix, is positive and fixes the “effective mass” of the quasiparticles (at very high temperatures we have $\chi_T \propto \lambda_r T^2$ and $\sigma_T \propto H/T^2$, i.e., the explicitly broken symmetry is restored at high temperatures).

In choosing an initial state to model the situation after the collision, it is reasonable to assume the conditions

of local thermal equilibrium above the critical temperature so that we start in the disordered phase where the chiral symmetry is unbroken. In particular, we can simply take the initial values $\sigma(\tau_0) = \sigma_T$, $\pi(\tau_0) = \pi_T$, and $\chi(\tau_0) = \chi_T$ and turn on the expansion at τ_0 (the results of this simulation will be explained in detail later). One should also consider the possibility of exciting the initial state with some extra kinetic energy. For this we should consider an initial distribution of velocities $\dot{\sigma}(\tau_0)$ that kick the initial mean values making it change in time (it is worth noticing that the expansion destroys the initial equilibrium and the fields start to change even without any kick). Other initial conditions we will consider are those in which the initial state is a “disoriented” equilibrium where $\pi^2(\tau_0) + \sigma^2(\tau_0) = \sigma_T^2 + \pi_T^2$ but the state is pointing in the wrong direction (say, along π^0). In fact, the formation of such disoriented state is not at all unlikely: simple estimates show that the energy required to “tilt” the initial state orienting it along the π^0 direction (instead of the σ direction) is only 10 MeV/fm³ [5]. Another possibility is to change the initial value of the magnitude of the O(4) vector making it different from its high temperature equilibrium one. For all the above cases, the initial value of χ will always be determined by the solution of the initial gap equation (the same for its initial derivative) evaluated for an equilibrium configuration at a temperature T_0 .

However, we should point out that the initial value of χ will always be restricted to be positive. As we said, $\chi(\tau)$ is the effective mass of the quasiparticles. Therefore, taking a negative initial value for χ implies that we are “turning the effective potential upside down.” In our view, this cannot be the consequence of forming a “peculiar” initial state but must rather be the consequence of the cooling mechanism, which is entirely produced by the expansion. In fact, starting with a negative χ_0 is what is done when studying this problem by using the quench approximation: one starts with a hot initial state and lets it evolve in the low temperature effective potential. The quench approximation was studied in detail in [13]. It should be clear by now that this is drastically different from our approach. We will study the self-consistent evolution of χ starting from a “hot” initial value and follow its evolution. What we find is that for some of our initial conditions which lead to quenching we get similar results to the above-mentioned authors who imposed this behavior by hand.

B. Instabilities and correlations

During the nonequilibrium evolution it is possible that, for certain time intervals, the value of $\chi(\tau)$ becomes negative (we will discuss some examples below). When this happens there is an instability in the system. As is clear from (3.32), if $\chi \leq 0$ there are long wavelength modes that become unstable: their amplitudes start growing exponentially (the factor $1/4\tau^2$ very soon becomes negligible). The existence of such unstable modes is the crucial ingredient needed for the development of structure through the mechanism of spinodal decomposition. Let us briefly describe now how we will study this issue here.

The intuitive picture is clear: the system cools as it expands evolving (in a fully nonequilibrium way) towards a stable low temperature state. In our case, such state (the vacuum) is characterized by the values $\pi = 0$, $\sigma = f_\pi$, and $\chi = m_\pi^2$. The existence of an instability means that the homogeneous configuration is not energetically preferred and that any small inhomogeneity seed will tend to grow. As the O(4) symmetry is explicitly broken, the growth of structure is only transient since in the long run the stable state is again the homogeneous vacuum. The question one should try to answer is if during the nonequilibrium stage the instability is strong enough to form large domains in which the field is correlated and disoriented. If such domains do form, the correlations in the emitted pions can be detected. We should then examine the evolution of the correlation length (in the rapidity space).

A natural question that arises is how can one study the existence of long range correlations within our scheme, which is basically a mean field theory analysis. The textbook answer to this is that mean field theory can still be used to provide useful information about the typical scale of the correlations despite of the fact that the mean values are taken as homogeneous [32]. In fact, this analysis enables us to compute the two-point correlation functions which determine the behavior of the system when perturbed away from its homogeneous configuration. Thus, the correlation length obtained from the two-point functions will characterize the growth of structure, at least in the linear regime. This point of view was used in [14, 19]). Our feeling, however, is that it is not so easy to interpret directly the two-point correlation function. In the first order adiabatic basis we have

$$G^0(x, y; \tau) = \frac{1}{\tau} \int [d^3k] e^{ik(x-y)} \frac{1}{2\omega_k(\tau)} (1 + 2n_k(\tau) + 2\text{Re}[g_k(\tau)e^{-i2y_k(\tau)}]). \quad (4.1)$$

We see that there are phases in the fourier transform $G(k, \tau)$ between the number density $n_k(\tau)$ and the pair density $g_k(\tau)$ which can make the interpretation difficult. These interpolating operators only make physical sense when $\langle \pi \rangle = 0$ and for the stable modes. So at best we can look at the two quantities n_k and g_k and study their momentum dependence. If we have a particular model for these quantities which have proper time-dependent masses as well as temperatures then can one extract correlation lengths (or mass scales) in a model-dependent fashion. For example if the system expanded in local thermal equilibrium in a comoving frame we would have

$$n_k(\tau) = \frac{1}{e^{\beta(\tau)E_k(\tau)} - 1}, \quad (4.2)$$

where $\beta(\tau) = 1/T(\tau)$ and $E_k(\tau) = \sqrt{k^2 + \chi(\tau)}$.

The effective temperature could be calculated by first determining the hydrodynamical quantities $\epsilon(\tau)$ and $p(\tau)$ from the diagonal entries of the expectation value of the energy momentum tensor in the comoving frame as discussed in [33]. Then one determines the temperature (and entropy) from the relations $\epsilon + p = Ts$; $d\epsilon = Tds$.

We notice that this parametrization fails exactly when

the pion mass goes negative which is the case we want to study. Also from our previous study of the production of pairs from strong electric fields in a related $1/N$ expansion [33], we know that the effective temperature in lowest order is not monotonically decreasing but is oscillating about a decreasing function with the plasma oscillation frequency. Thus until we add scattering (which occurs at next order in the $1/N$ expansion), the temperature parameter does not have this monotonicity property. At late times one expects after a period of entropy production that $s\tau = \text{const}$ and $T(\tau) = T_0(\frac{\tau_0}{\tau})^{1/3}$ as in the thermal equilibrium case. This is obviously one possible parametrization of the data in terms of two τ -dependent inverse length scales, the temperature and the mass of the pion. As we shall discover, this parametrization of the number density does not agree with our nonequilibrium evolution so that there are at least three proper time evolving length scales in this problem—the inverse mass of the pion, the inverse of the effective temperature, and possible length scales describing domain growth. In what follows we will present our results for these two interpolating densities to see their general proper time development without making specific models in terms of various length scales except for the quasithermal model which does not reproduce the results.

The computation of the pion two-point function in general is straightforward. The two-point function is obtained by inverting the inverse propagator matrix as discussed in the Appendix. In general we get for the inverse of the pion two-point function:

$$G_{\pi\pi}^{-1}(x, y)_{ij} = -\delta_{ij}[\square + \chi(x)]\delta(x - y) - \pi_i(x)G_{0\chi\chi}(x - y)\pi_j(y). \quad (4.3)$$

We see that only when $\pi_i(x) = 0$ can one obtain the Green's function directly in terms of the modes used to calculate the quantum field ϕ . Also only in that case do we get a direct interpretation of the Fourier transform of the Green's function in terms of the (proper) time-dependent interpolating phase space number and pair densities described above. As discussed earlier it is simple to extract n and g from the solution f of the mode equation.

Although the equal proper time correlation function depends only on the time-dependent number and pair densities, the full nonequal time correlation function has more phase information and is given by (when $\langle \pi \rangle = 0$)

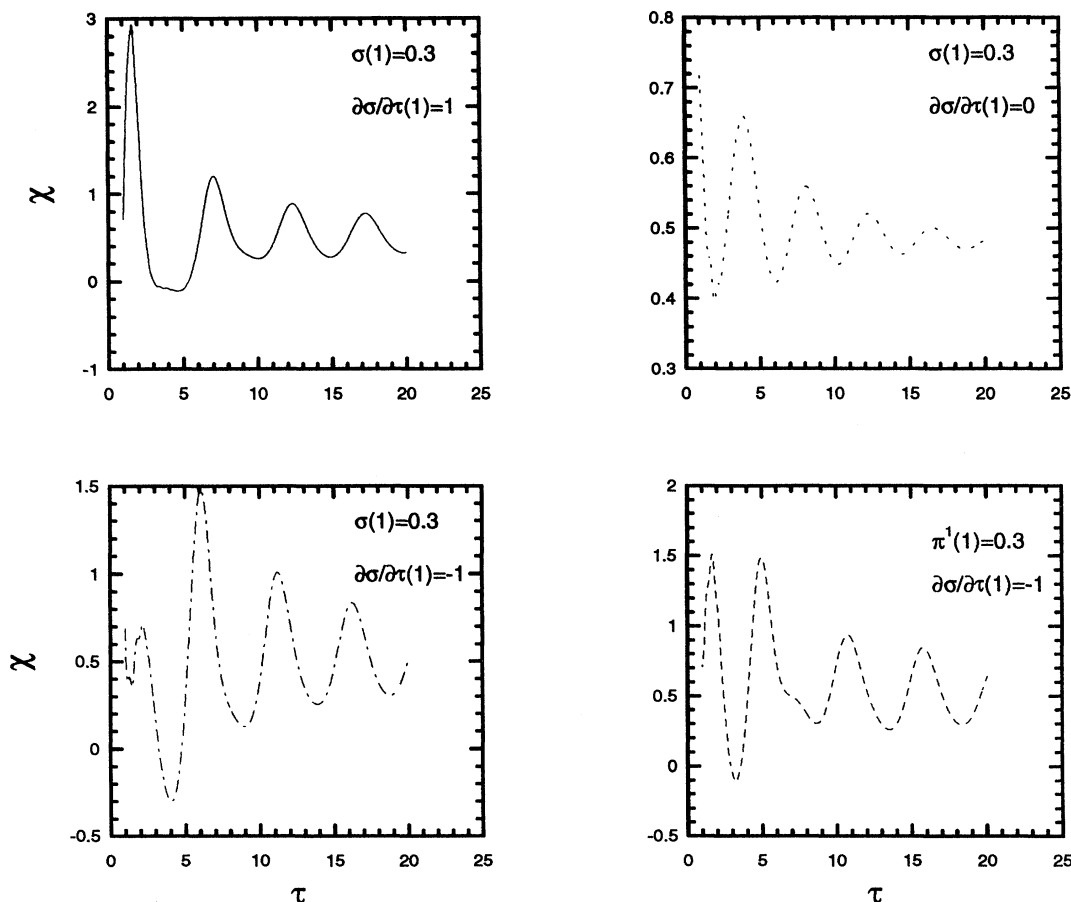


FIG. 1. Proper time evolution of the χ field for four different initial conditions with $f_\pi = 92.5$ MeV.

$$\begin{aligned}
G^0(x_\perp - x'_\perp; \eta - \eta'; \tau, \tau') &= \frac{1}{\sqrt{\tau}} \frac{1}{\sqrt{\tau'}} \int [d^3k] e^{-ik(x-x')} \\
&\times \{(1 + 2n_k) \text{Re}[f_k^*(\tau) f_k(\tau')] \\
&+ 2\text{Re}[g_k f_k(\tau) f_k(\tau')]\}, \quad (4.4)
\end{aligned}$$

where here n, g refer to their value at τ_0 .

The only thing left to discuss is the choice of initial proper time τ . One can estimate this quantity in a hydrodynamical model with Landau's choice of initial conditions (namely energy density of a Lorentz contracted pancake being given at a time zero). If one assumes that the collision produces a quark-gluon plasma in equilibrium, one can run the hydrodynamical code for initial energy densities appropriate to the collision of Lorentz contracted disks and determine the proper time when the critical temperature is around 200 MeV or slightly above the chiral phase transition. We want to start in the quark-gluon plasma phase above the chiral phase transition which places constraints on the initial energy density present at τ_0 . We then assume that slightly above this transition it is reasonable to model the chiral transition with the effective Lagrangian of the σ model. Not know-

ing exactly what this proper time is, we will here consider reasonable initial proper times ($1 \text{ fm}/c < \tau_0 < 4 \text{ fm}/c$) and study the effect of the initial proper time τ on the production of instabilities. Taking larger proper times as the starting point for our calculation reduces even further the possibility of instability growth so that our results present an upper limit on the growth of domain sizes in this model.

V. RESULTS

We have performed numerical simulations on the connection machine CM-5 using a grid that has 10000τ modes. (We start at $\tau_0 = 1$.) The grid is 100 modes wide in the transverse direction and $100 \frac{\tau}{\tau_0}$ modes in the η direction. That is we choose

$$dk_\perp = \frac{\Lambda}{100}, \quad \frac{dk_\eta}{1 \text{ fm}} = \frac{\Lambda}{100}.$$

The first issue we will examine here is the existence of instabilities. In previous works the presence of unstable modes was assumed by imposing the quench approximation. In our numerical studies we find that, because of

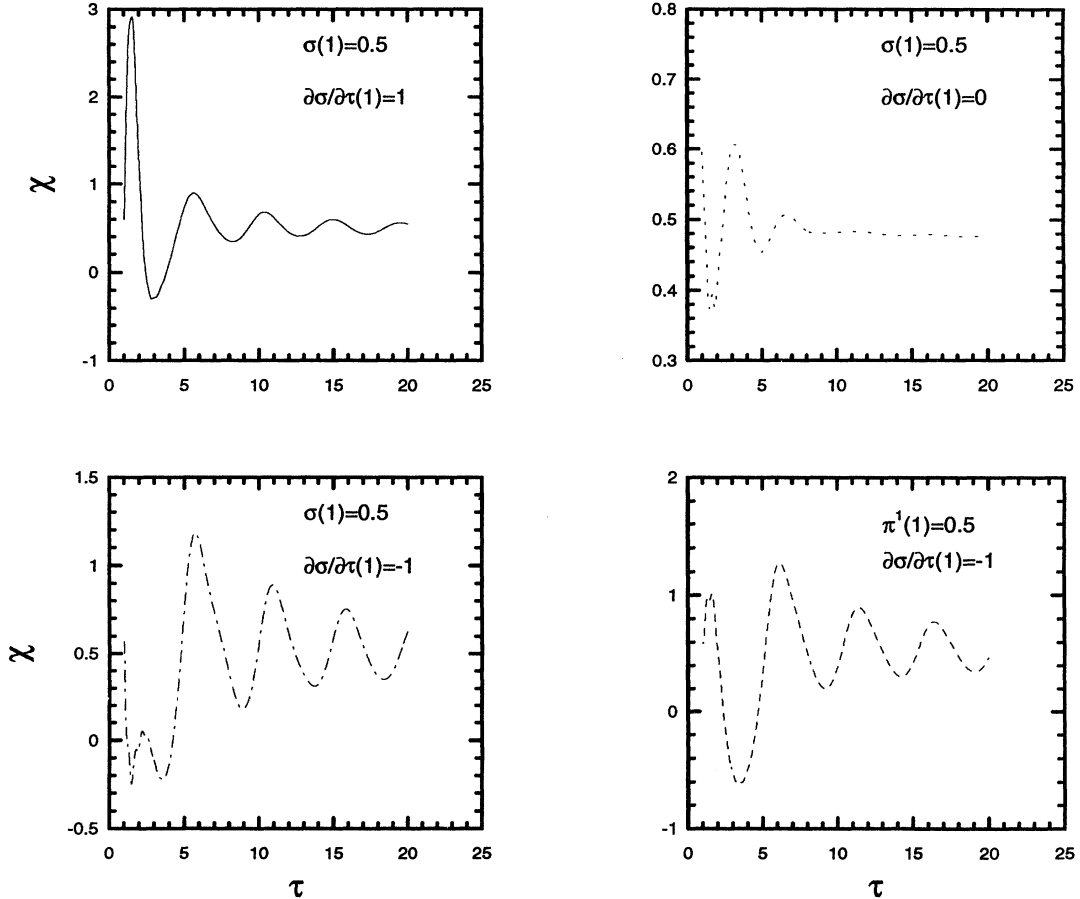


FIG. 2. Same as Fig. 1, but for $f_\pi = 125 \text{ MeV}$.

the strong coupling, that a wide class of initial conditions consistent with reasonable fluctuations found in a thermal distribution no instabilities develop. In our simulations the one thing we did not change was the value of the composite field χ which was fixed to be the solution of the gap equation in the initial thermal state. We also maintained the constraint that the initial values of the expectation values of the field, namely $\pi(\tau_0)$ and $\sigma(\tau_0)$ satisfied the relationship

$$\pi^2(\tau_0) + \sigma^2(\tau_0) = \sigma_T^2,$$

where σ_T is the equilibrium value of Φ at the initial temperature T . This last constraint is reasonable since it does not cause much energy to make a rotation in the direction of π_i . Thus we chose different initial σ and π_i consistent with the above constraints and then probed the effect of different initial values for $\dot{\sigma}$ and $\dot{\pi}_i$. Since the results for $\dot{\pi}_i \neq 0$ were similar to those when $\dot{\sigma} \neq 0$ we mainly present here the results for the latter case. We have also surveyed other possibilities that violate the above constraints such as choosing initial $\sigma = 0$. In that case we found no instabilities even with $\dot{\sigma} \neq 0$.

First, let us consider the case when the initial value $\sigma(\tau_0)$ is the thermal equilibrium one corresponding to a temperature of 200 MeV. We varied the value of the initial proper time derivative of the σ field expectation value and found that there is a narrow range of initial values that lead to the growth of instabilities. Namely

$$0.25 < |\dot{\sigma}| < 1.3.$$

Surprisingly when $|\dot{\sigma}| > 1.3$ instabilities no longer occur.

Figures 1 and 2 summarize the results of the numerical simulation for the evolution of the system (3.30)–(3.32). We display the auxiliary field χ in units of fm^{-2} , the classical fields Φ in units of fm^{-1} , and the proper time in units of fm^{-1} ($1 \text{ fm}^{-1} = 197 \text{ MeV}$). In Figs. 1 and 2 the proper time evolutions of the auxiliary field χ field are presented for two different values of f_π , where the initial conditions were fixed at $\tau = 1$. The initial conditions in all the cases are with the equilibrium value of the χ field at the corresponding initial temperature, where as the various initial conditions of the classical fields Φ are chosen to satisfy $\pi^2(\tau_0) + \sigma^2(\tau_0) = \sigma_T^2$ where σ_T^2 is the equilibrium value of Φ at the initial temperature T . In

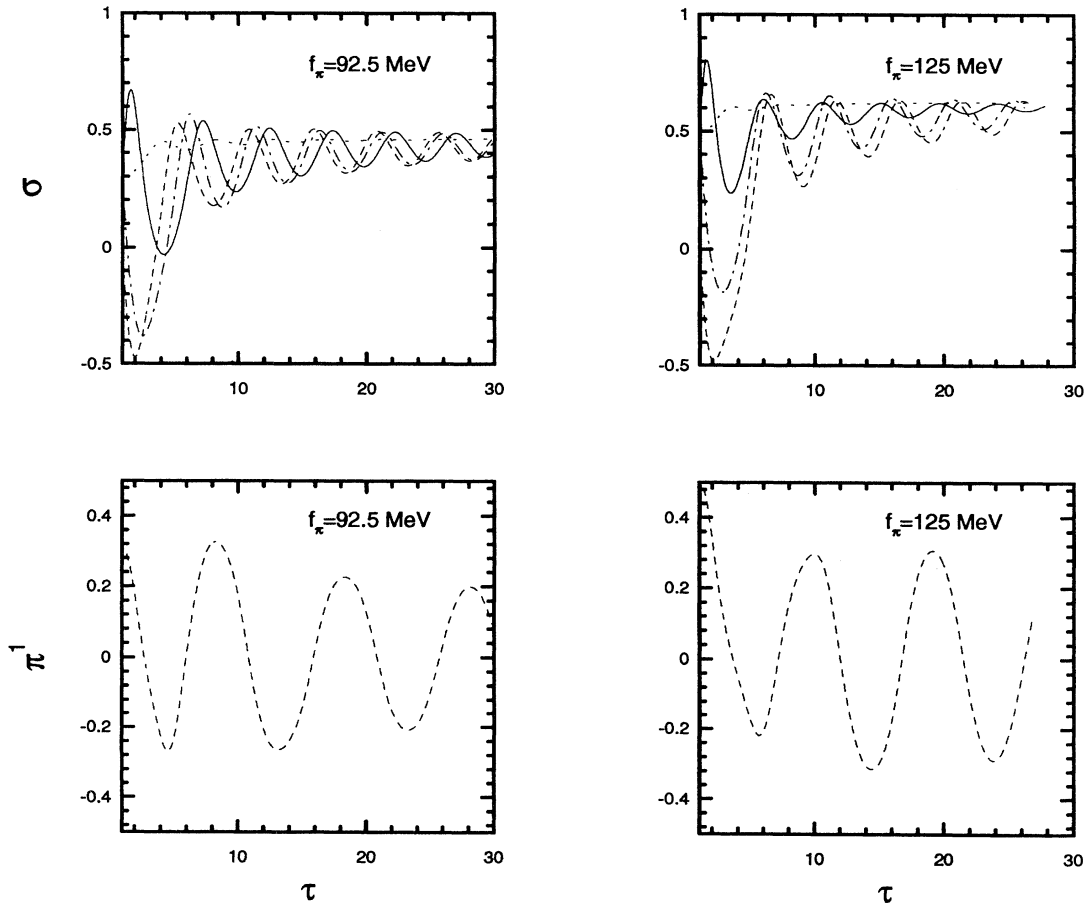


FIG. 3. Proper time evolution of the σ and π fields for the following initial conditions: Solid line is for $\sigma(1) = \sigma_T$, $\pi^i(1) = 0$, and $\dot{\sigma}(1) = 1$. Dotted line is for $\sigma(1) = \sigma_T$, $\pi^i(1) = 0$, and $\dot{\sigma}(1) = 0$. Dashed-dotted line is for $\sigma(1) = \sigma_T$, $\pi^i(1) = 0$, and $\dot{\sigma}(1) = -1$. Dashed line is for $\sigma(1) = 0$, $\pi^i(1) = \sigma_T$, and $\dot{\sigma}(1) = -1$. At $T = 200 \text{ MeV}$, $\sigma_T = 0.3 \text{ fm}^{-1}$ for $f_\pi = 92.5 \text{ MeV}$ and $\sigma_T = 0.5 \text{ fm}^{-1}$ for $f_\pi = 125 \text{ MeV}$.

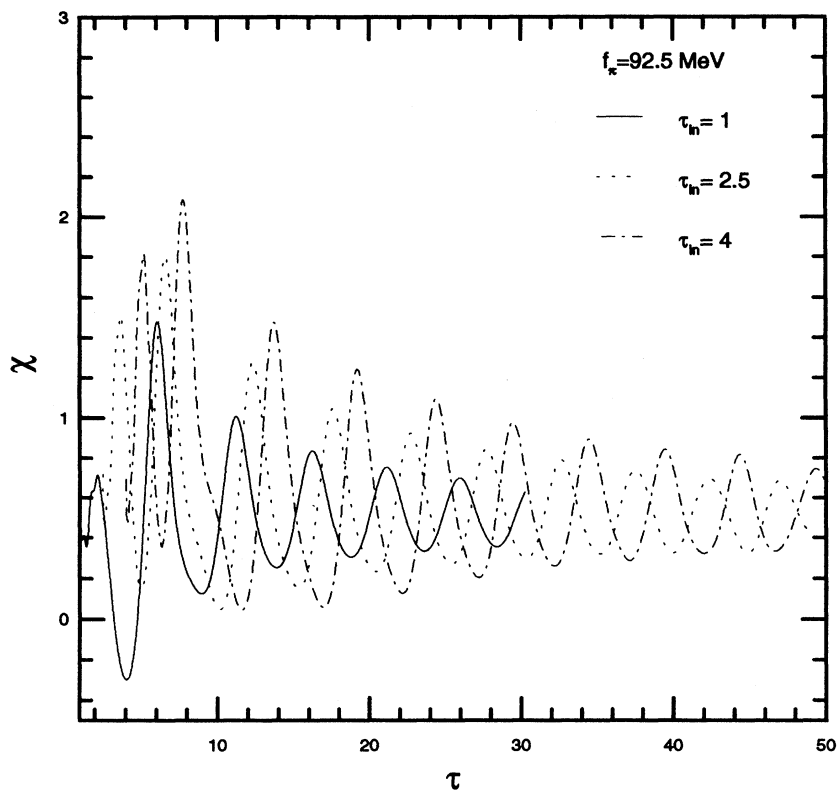


FIG. 4. Proper time evolution of the χ field for three different initial proper times $\tau_0 = 1, 2.5, 4$ with $f_\pi = 92.5$ MeV, $\sigma(\tau_{in}) = \sigma_T$, $\pi^i(\tau_{in}) = 0$, and $\dot{\sigma}(\tau_{in}) = -1$.

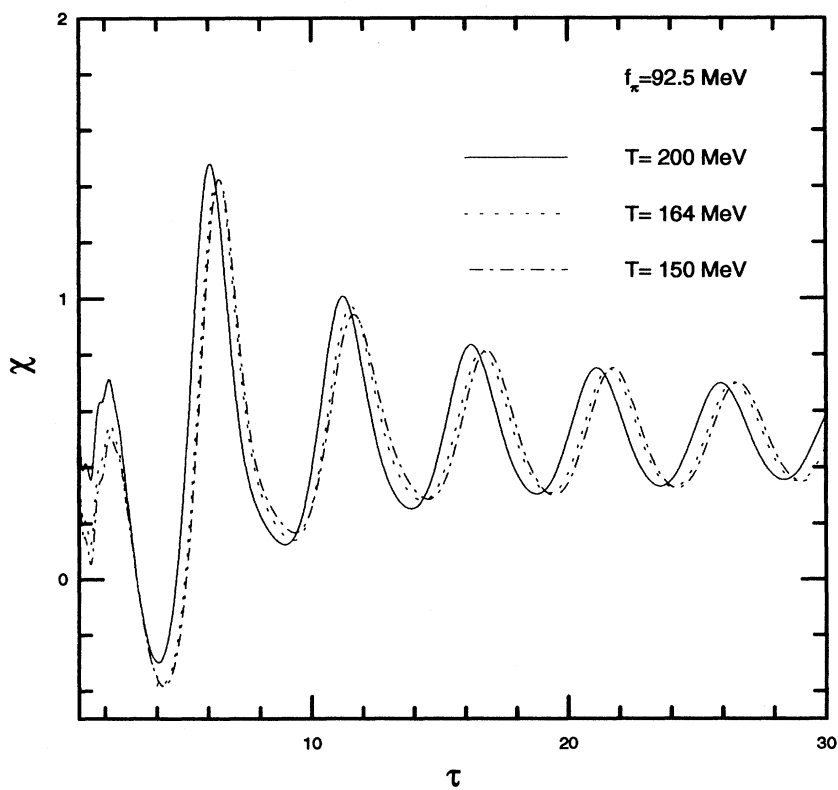


FIG. 5. Proper time evolution of the χ field for three different initial thermal distributions with $T = 200, 164, 150$ MeV for the initial conditions $\sigma(\tau_{in}) = \sigma_T$, $\pi^i(\tau_{in}) = 0$, and $\dot{\sigma}(\tau_{in}) = -1$.

the case of $f_\pi = 92.5$ MeV we see that the instability lasts for less than 3 fm. As discussed earlier, the regime of exponential growth of the unstable modes occurs whenever $\chi < 0$. We notice that when we rotate the expectation value from the σ to the π^1 direction initially then the instability regime has the same typical "survival" time. We also notice that if we choose the time derivative to be zero and start from an equilibrium configuration, then the expansion alone is insufficient to generate instabilities. We find that in order to generate instabilities we require fluctuations in the classical kinetic terms such as $\dot{\sigma}$ or $\dot{\pi}_i$, and as discussed earlier these initial conditions must be in a very narrow range to produce instabilities. Thus the rapid quench conditions assumed by other authors comprises only a small region of the phase space of initial fluctuations expected in an initial thermal distribution. As expected, at late proper times, the auxiliary field approaches its equilibrium value of m_π^2 . For the case when $f_\pi = 125$ MeV, a value favored by fitting the low energy scattering, we see a very similar behavior, showing that our main results concerning the small range of initial conditions that lead to instabilities are not affected by a 30% change in the value of one of our parameters.

A crude estimate for the size of a disoriented chiral condensate is to multiply a typical survival time length for the instabilities by the speed of light. If this estimate is valid then the size of these regions are of the order of a few fm. Of course, one needs to study the growth of inhomogeneous instabilities to make any definite statements about this size. We were hoping that the correlation functions we have calculated could be inter-

preted easily in terms of a length parameter associated with the size of these regions. However as we will see below such a naive hope is not to be satisfied. In Fig. 3 we plot the proper time evolution of the classical fields σ and π for the same two choices for f_π as in Figs. 1 and 2. We see that the σ field asymptotes to its vacuum value f_π and the π field gradually converges to its equilibrium value of zero. In Fig. 4 we show the effect of starting the initial value problem at later times, namely $\tau_0 = 2.5, 4$ fm and compare them to the case previously studied with $\tau_0 = 1$ fm which had a modest region of instability growth. We see that as we increase τ_0 we decrease the possibility of instability growth, and by these late times it is not possible to produce instabilities even with kinetic energy fluctuations. In Fig. 5 we study the effect of the initial temperature on our time evolution problem. For $f_\pi = 92.5$ the critical temperature is 160 MeV in the absence of explicit symmetry breaking. We see that in the vicinity of the critical temperature the effects of varying the initial temperature is minor. In Fig. 6 we study the proper time evolution of the effective number density for various initial conditions. In thermal equilibrium one expects for an isentropic expansion in boost invariant coordinates that $s\tau = \text{const}$. Since the number density is proportional to the entropy density one expects that once particle production stops that $n(\tau)\tau \rightarrow \text{const}$. We see this trend in this figure, showing that we are reaching the out regime as the system expands. The breaks in some of these curves at early values of τ are a result of the fact that the interpolating number density cannot be defined when χ is negative.

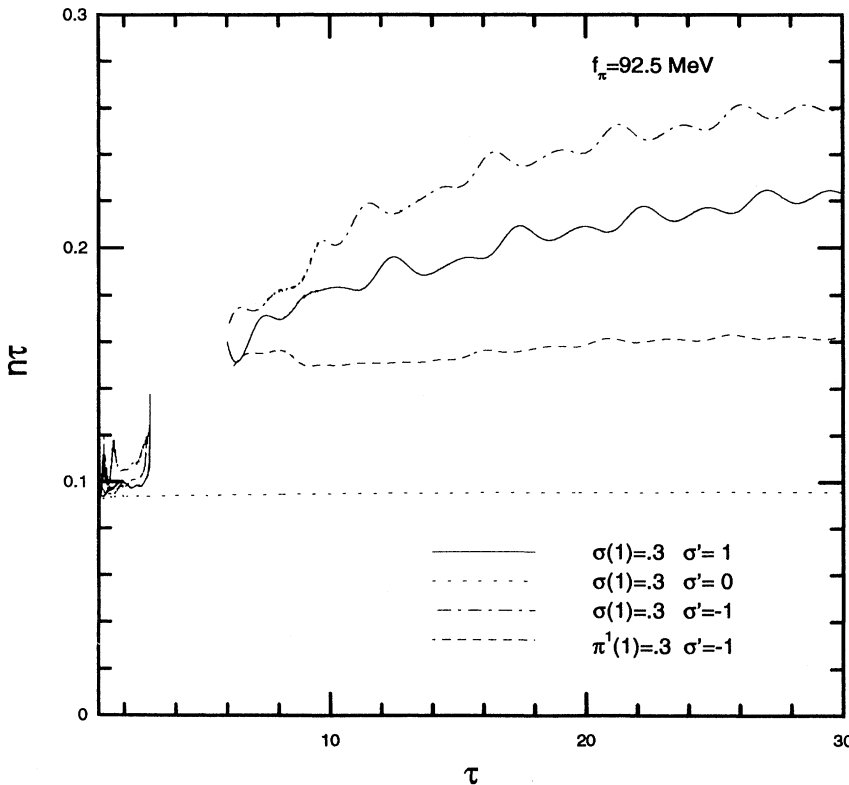


FIG. 6. Proper time evolution of the produced particle density $n\tau \equiv dN/d\eta dx_\perp$ for the same evolution shown in Fig. 1. The particle density is not plotted for those τ for which $x(\tau) < 0$.

Next we are interested in knowing how our results differ from the case where the system evolves in local thermal equilibrium which is described by two correlation lengths, the inverse of the effective pion mass associated with χ , and the inverse of the proper time evolving effective temperature $T(\tau) = T_0(\frac{\tau_0}{\tau})^{1/3}$ discussed earlier. We see from Fig. 7 that in the case that $\sigma(1) = \sigma_T$, $\pi^i(1) = 0$, and $\dot{\sigma}(1) = -1$, where maximum instability exists, complex structures are formed as contrasted to the local thermal equilibrium evolution. The interpolating phase space distribution $n(k_\eta, \mathbf{k}_\perp, \tau)$ obtained numerically, clearly exhibits a larger correlation length in the transverse direction than the equilibrium one and has correlation in rapidity of the order of 1–2 units of rapidity. We notice that in both directions there is structure which does not lend itself to a simple interpretation. On the other hand the local thermal equilibrium evolution is quite regular apart from the normalization of the distributions that are changing with time due to oscillation in the quantity $\chi(\tau)$ which is damped to its equilibrium value once the system expands sufficiently. Other authors have suggested that it is possible to extract the sized of the domains of DCC's from the coordinate space corre-

lation function $G(x, y, \tau)$ by considering the width of the distribution as a measure of the size of the domain. However, it is clear from our detailed numerical results that there are several length scales affecting the momentum space distribution apart from the effective pion mass and temperature. The spatial Green's function depends on not only n_k and g_k , but also the phases y_k , and would be even harder to interpret than n and g separately. In Fig. 8 we look at the time evolution of the interpolating number operator for the case when we did not have any fluctuations in the kinetic energy. Here we are still far from equilibrium, and we see that unlike the equilibrium case the correlation length in rapidity space is not decreasing with proper time. In this case where there are no instabilities we do not see complicated structures. The transverse distribution is similar to the equilibrium case. The pair density function $g(k_\eta, \mathbf{k}_\perp, \tau)$ is even more elusive to parametrize than the single particle distribution function. In the case where there are no instabilities [$\dot{\sigma}(1) = 0$] we see in Fig. 9 that although the transverse distribution is relatively simple, the distribution in the η direction (whose fourier transform gives the rapidity distribution) has many length scales. When we have in-

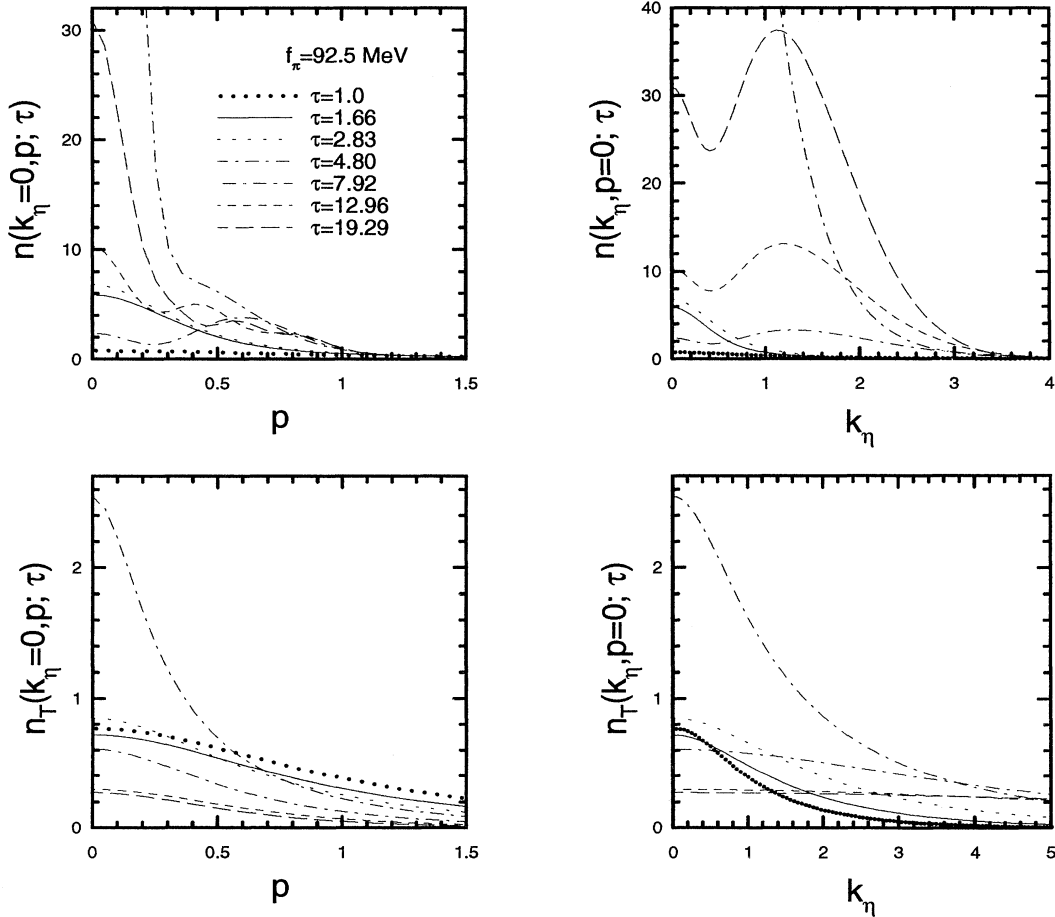


FIG. 7. Slices of $k_\eta = 0$ and $p \equiv |\mathbf{k}_\perp| = 0$ of the proper time evolution of the interpolating phase space particle number density $n(k_\eta, \mathbf{k}_\perp, \tau)$ for $\sigma(1) = \sigma_T$, $\pi^i(1) = 0$, and $\dot{\sigma}(1) = -1$ compared with the corresponding local thermal equilibrium densities $n_T(k_\eta, \mathbf{k}_\perp, \tau)$.

stabilities [$\dot{\sigma}(1) = -1$] then both distributions are complicated and possess several length scales as seen in the lower part of Fig. 9.

VI. CONCLUSIONS

In this paper we have performed numerical simulations in the regime of the chiral phase transition in the linear σ model for a wide variety of initial conditions starting above the critical temperature for an expanding plasma of pions and sigmas. Assuming that this model gives a reasonable description in this temperature range, we found initial conditions where instabilities grew in the scenario required for the formation of disoriented chiral condensates. However, low energy phenomenology also requires a moderately large renormalized coupling constant. This has the effect of rapidly damping the instabilities, and we found no evidence in this model for large domains of disoriented chiral condensates. We did, however, see rather large departures in the phase space number density from one which would result from an evolution in local thermal equilibrium. These departures show a narrowing of the momentum space distribution which could be interpreted as a larger spatial correlation length. However, we found no simple method of extracting cor-

relation lengths from our results. In our simulations we assumed a reasonable mechanism for cooling, namely the expansion of the plasma following its production in a collision. However we did ignore scattering, which occurs at next order in the $1/N$ expansion. We also did not allow for general inhomogeneous fluctuations which would allow us actually to study the growth of disoriented domains. This would require a study of the growth of inhomogeneous perturbations about our homogeneous background and would involve a much more difficult numerical computation. We hope to carry out such a computation in the future. In the future we also hope to include scattering effects which will introduce another time scale, namely the equilibration time scale into the problem, as well as study the nonlinear σ model to see if the results differ significantly from those found here.

ACKNOWLEDGMENTS

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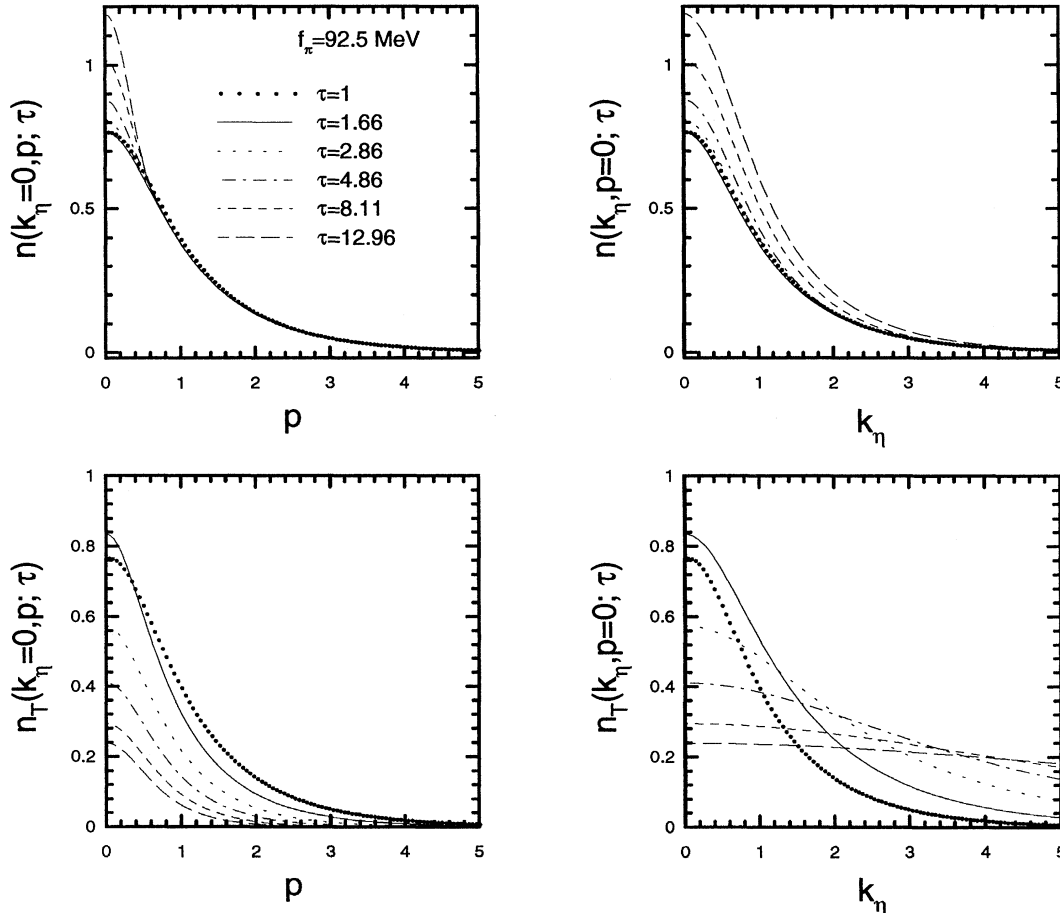


FIG. 8. As in Fig. 7, but for $\sigma(1) = \sigma_T$, $\pi^i(1) = 0$, and $\dot{\sigma}(1) = 0$.

us to do this calculation. We also benefitted from several discussions with D. Boyanovsky and Ian Kogan. This research was performed in part using the resources located at the Advanced Computing Laboratory of Los Alamos National Laboratory, Los Alamos, NM 87545.

APPENDIX: σ MODEL IN THE LARGE N APPROXIMATION

The $O(4)$ σ model is described by the Lagrangian

$$L = \left\{ \frac{1}{2} \partial \Phi \cdot \partial \Phi - \frac{1}{4} \lambda (\Phi \cdot \Phi - v^2)^2 + H \sigma \right\}, \quad (\text{A1})$$

where the mesons are organized in an $O(4)$ vector $\Phi = (\boldsymbol{\pi}, \sigma)$. The counting of the large N expansion is made explicit [20] by introducing a composite field χ defined by $\chi = \lambda(\Phi \cdot \Phi - v^2)$. It is easy to show by appropriate rescalings that the large N expansion is obtained by integrating out the Φ field and then performing a steepest descent calculation on the remaining χ path integral. That is the large N expansion is a loop expansion in the composite field χ propagator [22]. Thus we add to the

above Lagrangian the constraint equation

$$\frac{[\chi - \lambda(\Phi \cdot \Phi - v^2)]^2}{4\lambda}, \quad (\text{A2})$$

which exactly cancels the quadratic term. We thus obtain the alternate Lagrangian

$$L_2 = -\frac{1}{2} \phi_i (\square + \chi) \phi_i + \frac{\chi^2}{4\lambda} + \frac{1}{2} \chi v^2 + H \sigma. \quad (\text{A3})$$

The generating functional of the Green's functions is given by the path integral

$$Z[J, S] = \int D\chi D\Phi \exp \left\{ i \int d^4x [L_2 + J \cdot \Phi + S\chi] \right\} \\ \equiv \exp[iW(J, \chi)]. \quad (\text{A4})$$

We can now perform the Gaussian path integration over the Φ field. Evaluating the remaining χ integral at the stationary phase point of the resulting effective action, and then Legendre transforming,

$$\Gamma[\Phi, \chi] = W[J, S] - \int d^4x [J(x) \cdot \Phi(x) + S(x)\chi(x)], \quad (\text{A5})$$

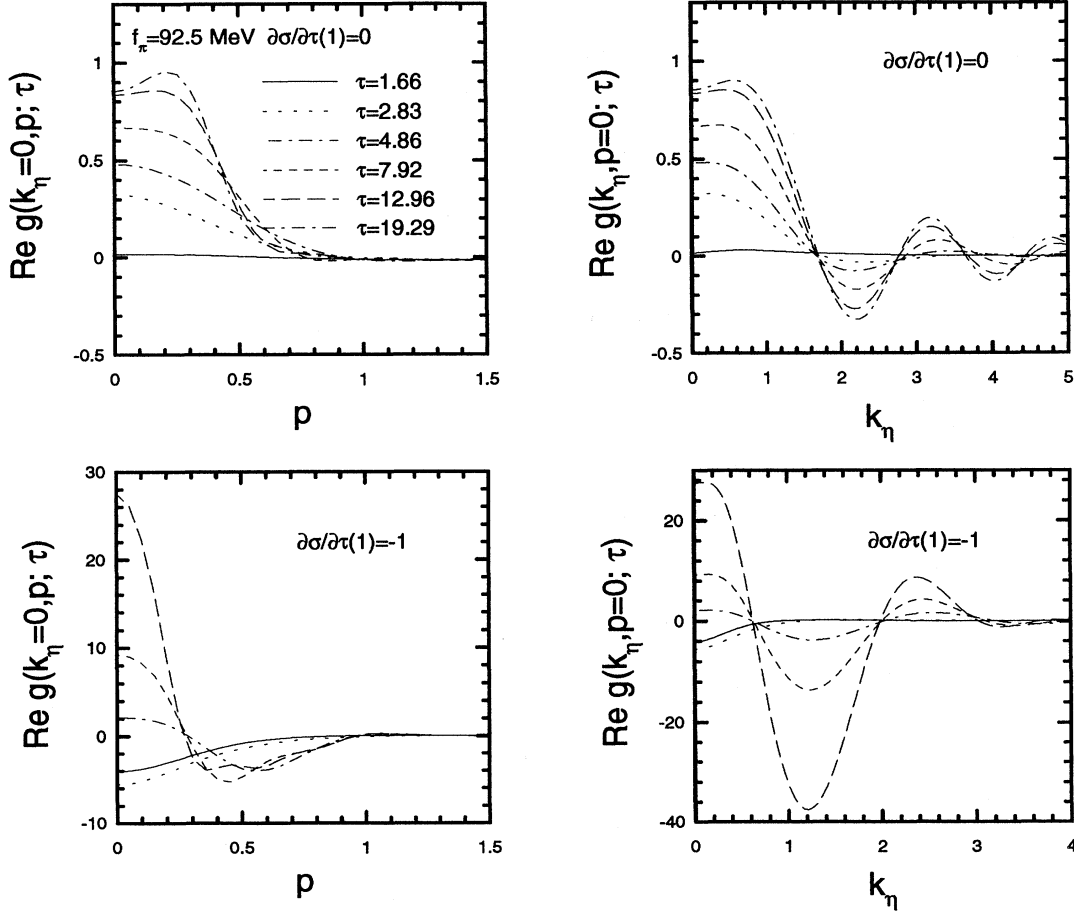


FIG. 9. Slices of $k_\eta = 0$ and $p \equiv |\mathbf{k}_\perp| = 0$ of the proper time evolution of the real part of the phase space pair density $g(k_\eta, \mathbf{k}_\perp, \tau)$ for $\sigma(1) = \sigma_T$, $\pi^i(1) = 0$, and $\dot{\sigma}(1) = 0, -1$.

we obtain the lowest order (in large N) result

$$\Gamma[\Phi, \chi] = \int d^4x \left[L_2(\Phi, \chi, H) + \frac{i}{2} N \text{tr} \ln G_0^{-1} \right], \quad (\text{A6})$$

where

$$G_0^{-1}(x, y) = i[\square + \chi(x)] \delta^4(x - y). \quad (\text{A7})$$

For the O(4) sigma model $N = 4$. From this effective action we immediately get the equations of motion for the fields Φ and the equation of constraint (gap equation) for the composite field χ :

$$[\square + \chi(x)]\pi_i = 0, \quad [\square + \chi(x)]\sigma = H, \quad (\text{A8})$$

$$\chi = -\lambda v^2 + \lambda(\sigma^2 + \pi \cdot \pi) + \lambda N G_0(x, x). \quad (\text{A9})$$

In the static case when we are considering symmetry breaking, we obtain the lowest energy state when

$$\chi\sigma = H, \quad \chi = -\lambda v^2 + \lambda\sigma^2 + \lambda N G_0. \quad (\text{A10})$$

By considering the fields (Φ, χ) to be a five-dimensional vector Ψ , and the matrix inverse propagator to be

$$\hat{G}^{-1} = \frac{\delta^2 \Gamma}{\delta \Psi \delta \Psi}, \quad (\text{A11})$$

we obtain by inverting this matrix the relevant π, σ , and χ propagators. Performing this inversion we obtain for the vacuum Feynman χ inverse propagator

$$\hat{G}_{\chi\chi}^{-1}(p^2) = G_0^{-1}{}_{\chi\chi}(p^2) + \frac{\sigma^2}{\chi - p^2}, \quad (\text{A12})$$

where

$$\hat{G}_0^{-1}{}_{\chi\chi}(p^2) = \frac{1}{2\lambda} + \frac{N}{2}\Pi(p^2) \quad (\text{A13})$$

is the inverse propagator in the absence of symmetry breaking, and the polarization $\Pi = iG_0^2$ is given by

$$\Pi(p^2) = -i \int [d^4q] (\chi - q^2)^{-1} (\chi - (p+q)^2)^{-1}. \quad (\text{A14})$$

In our simulations we have a three-dimensional cutoff. If we integrate over q^0 and perform the remaining three-dimensional integration with a cutoff Λ we obtain explicitly

$$\begin{aligned} \Pi(p^2) = \frac{1}{8\pi^2} \left[\ln \left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} \right) + \frac{1}{2} \sqrt{1 - \frac{4m^2}{p^2}} \right. \\ \left. \times \ln \left(\frac{\sqrt{(1 - \frac{4m^2}{p^2})(1+x^2)} - 1}{1 + \sqrt{(1 - \frac{4m^2}{p^2})(1+x^2)}} \right) \right], \quad (\text{A15}) \end{aligned}$$

where $x = m/\Lambda$ and m is the pion mass: $\chi = m^2$.

For the pion inverse propagator we obtain, in the vacuum sector,

$$\hat{G}_{\pi\pi}^{-1}(p^2)_{ij} = \delta_{ij}(p^2 - \chi) = \delta_{ij}(p^2 - m^2). \quad (\text{A16})$$

Thus we see that $\chi = m^2$ is the pion mass squared. In our initial value problem one can have $\pi_i \neq 0$. For that case

(or when there are external isospin sources present), the pion inverse propagator becomes, for constant external sources,

$$\hat{G}_{\pi\pi}^{-1}(p^2)_{ij} = \delta_{ij}(p^2 - \chi) - \pi_i \hat{G}_{0\chi\chi}(p^2) \pi_j. \quad (\text{A17})$$

For the σ inverse propagator we obtain

$$\hat{G}_{\sigma\sigma}^{-1}(p^2) = p^2 - m^2 - \sigma^2 \hat{G}_{0\chi\chi}(p^2). \quad (\text{A18})$$

Thus the σ mass m_σ^2 is determined by the relationship

$$m_\sigma^2 = m^2 + \sigma^2 \text{Re}[\hat{G}_{0\chi\chi}(m_\sigma^2)]. \quad (\text{A19})$$

The axial vector current in this model is given by

$$A_\mu^i(x) = [\pi^i \partial_\mu \sigma - \sigma \partial_\mu \pi^i]. \quad (\text{A20})$$

The PCAC (partial conservation of axial vector current) condition is given by

$$\partial_\mu A_\mu^i(x) = H \pi^i(x). \quad (\text{A21})$$

Therefore one has

$$H = f_\pi m^2. \quad (\text{A22})$$

Since we also have that the vacuum is defined by $\chi\sigma = m^2\sigma = H$, we immediately find that $\sigma = f_\pi$. We can rewrite the gap equation in terms of physical quantities as

$$m^2 = -\lambda v^2 + \lambda f_\pi^2 + \lambda N G_0(x, x), \quad (\text{A23})$$

where

$$G_0(x, x) = \int_0^\Lambda [d^3k] \frac{1}{2\sqrt{k^2 + m^2}}. \quad (\text{A24})$$

This relationship allows us to determine the bare mass $-\lambda v^2$ in terms of the renormalized mass m , the pion decay constant f_π , and the cutoff Λ .

To determine λ_r we must look at either the σ mass which is not well determined by experiment, or the low energy scattering data such as the s -wave scattering $I = 0$ scattering amplitude. Although at tree level the σ mass is an arbitrary parameter, that is not true for the quantum theory. The reason for this is that we only want to consider this model as an effective field theory with a cutoff, so that the bare coupling constant is always positive. (Renormalized ordinary perturbation theory requires that the bare coupling constant is negative as we take away the cutoff.) Once there is a cutoff, then the property of the exact O(N) σ model, as determined by lattice or strong coupling expansions, is that the renormalized coupling is a monotonically increasing function of the bare coupling reaching a finite maximum as the bare coupling goes to infinity. In four dimensions, this maximum value decreases logarithmically with the cutoff. This exact feature of the O(N) sigma model is preserved in the large N approximation which is a virtue of our approximation. However, for reasonable cutoff (say 1 GeV) this puts an upper bound on the σ mass of around 3 m as we shall see below. The low energy scattering data presents another problem. This data is quite

difficult to fit in perturbation theory because perturbative results violate partial wave unitarity. Basdevant and Lee [25] were able to fit early s -wave data by first calculating to one loop and then enforcing unitarity by doing a Padé approximant. Our $1/N$ approximation is related to the Padé approximant since it sums the bubbles of the perturbative result, but it does not automatically obey partial wave unitarity. Since that work there has been some further experimental and theoretical work on the π - π phase shifts [26]. It is important for us to see what values of λ_r give reasonable s -wave phase scattering amplitudes in our approximation. We realize that this model cannot totally describe low energy pion physics, but we will do the best we can with its shortcomings. We are mostly interested in the region just above threshold since those are the energy regimes relevant for the phase transition. Let us therefore turn our attention to determining the low energy scattering amplitude in the large N limit. The pion-pion scattering invariant T matrix is given by

$$T = \delta_{ij}\delta_{kl}A(s) + \delta_{ik}\delta_{jl}A(t) + \delta_{il}\delta_{jk}A(u), \quad (\text{A25})$$

where the isospin indices i, j, k, l are coupled to the four momenta p_1, p_2, p_3, p_4 such that $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$. In leading order in large N the amplitudes $A(s)$, $A(t)$, and $A(u)$ are exactly the χ propagator. Namely

$$A(s) = -\hat{G}_{\chi\chi}(p^2 = s). \quad (\text{A26})$$

The large N expansion preserves the current algebra so that the usual low energy theorem is exact. That is, for small s, m^2 one easily shows that

$$A(s) \rightarrow \frac{(s - m^2)}{f_\pi^2}.$$

This amplitude is independent of the coupling constant λ . We thus need to go above threshold to determine the coupling constant. The $I = 0$ scattering amplitude is

$$A^0 = 3A(s) + A(t) + A(u). \quad (\text{A27})$$

The s -wave scattering amplitude is obtained by integrating the $I = 0$ scattering amplitude over angles:

$$f_{I=0} = e^{i\delta(s)} \sin \delta(s) = \frac{1}{32\pi} \sqrt{1 - \frac{4m^2}{s}} \int_{-1}^1 dz A^0, \quad (\text{A28})$$

where $z = \cos \theta$ and θ is the scattering angle in the s channel center of mass system.

It is useful to describe the theory in terms of the renormalized coupling constant λ_r , which depends on λ as well as the cutoff Λ . The running renormalized coupling constant is determined by the renormalization group invariant composite field propagator $G_{\chi\chi}$. We choose to define the renormalized coupling constant λ_r as the running coupling constant at $q^2 = 0$, for the unbroken mode of the theory. That is,

$$\lambda_r = \frac{\lambda}{1 + N\lambda\Pi(q^2 = 0)}. \quad (\text{A29})$$

Rewriting the propagator in terms of λ_r , one immediately gets a finite expression for the running coupling constant:

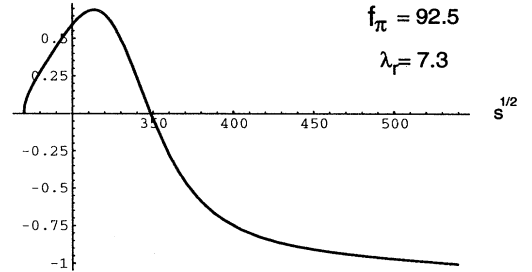
$$2\lambda_r(q^2) = \hat{G}_{\chi\chi}(q^2) = \frac{2\lambda_r}{1 + \lambda_r N\Pi_r(q^2) - \frac{2\lambda_r f_\pi^2}{q^2 - m^2}}, \quad (\text{A30})$$

with

$$\Pi_r(q^2) = \Pi(q^2) - \Pi(q^2 = 0). \quad (\text{A31})$$

In Fig. 10 we plot the real part of the s -wave scattering amplitude for $\lambda_r = 7.3$, $f_\pi = 125$ MeV, and $\lambda_r = 7.3$, $f_\pi = 92.5$ MeV. By comparing these curves to the unitarized perturbation theory results of Basdevant and Lee [25] we find reasonable agreement in the regime $2 < \sqrt{s/m} < 2.6$ for $f_\pi = 125$ MeV. This value of f_π is obviously the preferred one if we want a closer agreement of our results with those of Basdevant and Lee. (However since in our large N expansion we have only a modest value of N , namely $N = 4$ we might expect 25% corrections at next order in $1/N$). We notice the existence of the σ resonance at a mass of around 3 m. These values of f_π and m_σ are in reasonable agreement with the fits of Basdevant and Lee using their Padé

$$1/16\pi (1-4m^2/s)^{1/2} \int dz A^0$$



$$\text{Re}\{1/16\pi (1-4m^2/s)^{1/2} \int dz A^0\}$$

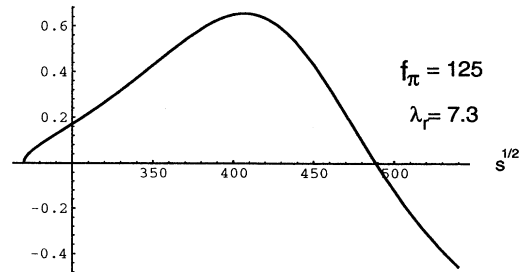


FIG. 10. The real part of the partial wave amplitude $2f_{I=0}$ for $I = 0$ computed from the linear σ model in the lowest order of the $1/N$ expansion for $f_\pi = 92.5$ MeV and for $f_\pi = 125$ MeV. The value of λ_r used was 7.3.

analysis of ordinary perturbation theory. We see that in our approximation there is a slight breakdown in s -wave unitarity near the peak. (s -wave unitarity requires that the real part of this amplitude to be less than $1/2$.) This breakdown in s -wave unitarity occurs in the leading order in large- N approximation for renormalized couplings which are larger than three.

One can approximately calculate the σ mass from the zero of the real part of the inverse propagator. In terms of the renormalized coupling defined above we have

$$m_\sigma^2 = m^2 + \text{Re} \left[\frac{2\lambda_r f_\pi^2}{1 + \lambda_r N \Pi_r(m_\sigma^2)} \right]. \quad (\text{A32})$$

$$W(\Phi \cdot \Phi, \chi) = \frac{N}{64\pi^2} \chi^2 \ln \left[\frac{e^{1/2} \Lambda^2}{\chi} \right] + \frac{\lambda}{4} \left[\Phi \cdot \Phi - v^2 - \frac{N}{16\pi^2} \chi \ln \left(\frac{e\Lambda^2}{\chi} \right) \right]^2. \quad (\text{A33})$$

Minimizing this functional with respect to χ gives the gap equation

$$\chi = \lambda(\Phi \cdot \Phi - f_\pi^2) - \frac{N\lambda\chi}{16\pi^2} \ln \left(\frac{e\Lambda^2}{\chi} \right), \quad (\text{A34})$$

which then gives χ as a functional of $\Phi \cdot \Phi$. This then implicitly determines the energy functional as a functional of only $\Phi \cdot \Phi$. Bardeen and Moshe point out that if one uses the end point solution $\chi = 0$ below the minimum of the potential, then one gets a real “effective potential.”

At finite temperature the phase structure of the cutoff theory is determined instead by the free energy density functional

$$\begin{aligned} W(\Phi \cdot \Phi, \chi) = & \frac{N}{64\pi^2} \chi^2 \ln \left[\frac{e^{1/2} \Lambda^2}{\chi} \right] \\ & - \frac{N}{48} T^4 \int_0^{\chi/T^2} dy y \left[\frac{dF(y)}{dy} \right] \\ & + \frac{\lambda}{4} \left[\Phi \cdot \Phi - v^2 + \frac{N}{16\pi^2} \Lambda^2 - \frac{N}{16\pi^2} \chi \right. \\ & \left. \times \ln \left(\frac{e\Lambda^2}{\chi} \right) + \frac{N}{24} T^2 F \left(\frac{\chi}{T^2} \right) \right]^2, \quad (\text{A35}) \end{aligned}$$

This equation, which leads to a similar value for the mass of the σ as found from the peak in the scattering, shows that there is an upper bound on the σ mass which depends on the chosen cutoff, since λ_r decreases monotonically with the cutoff.

The phase structure of the σ model in this approximation has been studied extensively by Bardeen and Moshe [23]. For zero temperature the expectation value $\langle T_{00} \rangle$ in our initial density matrix is the correct energy density functional to study. At finite temperature it is the free energy density expectation value that is needed to determine the phase structure. Explicitly at zero temperature the energy density functional is:

where

$$F(x) = \frac{6}{\pi^2} \int_0^\infty \frac{dy y^2}{\sqrt{y^2 + x}} [\exp(\sqrt{y^2 + x}) - 1]^{-1}. \quad (\text{A36})$$

The gap equation is now

$$\chi = \lambda(\Phi \cdot \Phi - f_\pi^2) - \frac{N\lambda\chi}{16\pi^2} \ln \left(\frac{e\Lambda^2}{\chi} \right) + \frac{\lambda N}{12} T^2 F \left(\frac{\chi}{T^2} \right). \quad (\text{A37})$$

Using this free energy one finds that the critical temperature is determined from:

$$\frac{NT_c^2}{12} = f_\pi^2. \quad (\text{A38})$$

These relations are used to start our calculation above T_c .

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