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Covariant spin tensors in meson spectroscopy

V. Filippini, A. Fontana, and A. Rotondi
Dipartimento di Fisica Nucleare e Teorica, Via Bassi 7 27100 Pavia, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, via Bassi 6 27100 Pavia, Italy
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It is shown that the decay amplitude of a particle with spin to one spinless meson and a resonance with spin can be expressed in a general and compact form using the covariant tensor (also named Rarita-Schwinger) formalism. The identity of this formalism with the covariant helicity formalism recently proposed by Chung is shown. Many angular distributions are derived, showing that in some cases there are large differences with the distributions calculated with noncovariant (Zemach or helicity) amplitudes. These differences are shown in detail for some Dalitz plots relative to the annihilation $p\bar{p} \to \pi\pi\pi$ at low energy. Although the worked examples refer to binary decays with spins ≤ 2 only, the covariant tensor formalism is presented in a general form to permit its extension to more complicated cases.

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I. INTRODUCTION

This paper deals with the phenomenological spin-parity determination of resonances and intends to show in detail, as recently emphasized by Chung [1], that the use of covariant spin formalisms is needed to obtain reliable results. The discussion will be limited to the so-called spin-dependent part of the decay amplitude, assuming that the energy-dependent part can be properly parametrized with relativistic Breit-Wigner functions or similar forms.

The spin dependence of the decay amplitude of resonances is usually written in terms of tensor or helicity formalisms. The former was pioneered by Zemach [2,3] and is very often used in the spin-parity determination of resonances produced by the $p\bar{p}$ annihilation, while the latter one was developed by Jacob and Wick [4], and is the most employed one.

Both formalisms are often used in phenomenological analyses in a noncovariant form.

In the noncovariant Zemach formalism the spin part of the amplitude is usually defined in terms of three-dimensional spin tensors written in the rest frame of each decaying state, and no boost to a general reference frame is made. Since the final equations contain tensors defined in different frames, they are noncovariant. Examples of the application of this technique can be found in [5] and, more recently, in the analysis of low energy $p\bar{p}$ annihilation data [6–8].

In the helicity formalism the spin rotation functions $D^J_{mm'}$ are used for the angular dependence. They are multiplied by a helicity-coupling amplitude which is written empirically in terms of a Breit-Wigner function. To obtain the same results as in the noncovariant Zemach formalism, it is necessary to introduce into the helicity-coupling amplitude the moduli of the Zemach tensors relative to each resonance rest frame involved in the decay. These terms, that are usually named centrifugal barrier factors, make also the helicity-coupling amplitude noncovariant.

The extended use of noncovariant formalisms is rather surprising, because the square modulus of the decay amplitude, which gives the decay probability of a certain configuration, should be independent of any particular frame, that is, a Lorentz scalar. The reasons for this puzzling situation are not very clear to us. In the original paper of Zemach [2] the formalism was developed at the beginning in the noncovariant form, then the recipe for making it fully covariant was correctly given (see, for example, the Sec. V of that work, in particular page B1216). However, in a subsequent paper [3], Zemach made the statement *"relativity is not an essential complication"* (see Sec. II.4 of that work), which probably has been misunderstood by many analysts. One possible reason for this misunderstanding is that in many phenomenological analyses good fits are obtained with the noncovariant formalism and in many (but not in all) cases there is not a big difference between the covariant and noncovariant angular distributions.

The fully covariant tensor formalism is often named the Rarita-Schwinger formalism, because usually one recalls a brief paper of these authors [9] in which the importance of the spin-tensor orthogonalization to the four-velocity of the decaying system was stressed. However, in our opinion it should be historically more correct to speak about covariant and noncovariant Zemach formalisms. In the following we will refer to the Rarita-Schwinger or covariant Zemach tensor formalism as the same thing.

The noncovariant approaches have been recently criticized by Chung [1], who proposed to write the helicity-coupling amplitude as a Lorentz scalar. The resulting helicity amplitude presents an E/m dependence (where E and m are the energy and the mass of the resonance in the parent rest frame) which is absent in the noncovariant formalism.

Although the helicity amplitude is intrinsically noncovariant, because the spin $D^J_{mm'}$ functions are expressed in each resonance rest frame, in the following we will refer for brevity to Chung's work [1] as the covariant helicity formalism.

We think that some important points have still to be clarified: (i) the connection between the covariant helicity formalism and the tensor one, (ii) the quantitative evaluation of the differences between covariant and noncovariant angular distributions, and (iii) the practical effects of these differences (if any) in a phenomenological analysis.

This paper intends to answer these questions in a precise way.

In Sec. II and III we try to present in a concise and compact form the covariant tensor (Rarita-Schwinger) formalism and in Sec. IV we apply it to the binary decay of resonances. Many spin angular distributions and their comparison with the noncovariant ones are shown there. In Sec. V we demonstrate that, for binary decays, the above results are identical to those obtained with the covariant helicity formalism of Chung [1]. In Sec. VI some striking differences between covariant and noncovariant Dalitz plots are shown. The examples are taken from the low energy $p\bar{p}$ annihilation, studied extensively for many years at the Low Energy Antiproton Ring (LEAR) of CERN.

Although the distribution calculations are made for binary decays with spins ≤ 2, and the examples are drawn from the LEAR physics only, the formalism is presented in Secs. II and III in a general form, to permit easily its extension to more complicated cases.

Finally, in Sec. VII we conclude that the use of covariant spin formalisms is essential in many practical cases to obtain reliable results.

II. COVARIANT SPIN TENSORS

We consider the decay of a spin J state with fixed P parity into two steps:

$$\begin{aligned} J^P &\longrightarrow j^p + c \\ j^p &\longrightarrow a + b\,, \end{aligned} \tag{1}$$

where (j^p) is a resonance of spin j, parity p and mass m_R, a, b, and c are the measured spin-0 mesons (pions or kaons) with mass m_a, m_b, m_c. The particle four-momenta are also labeled with a, b, and c. In the following we will use for this decay the notation $J \rightarrow j + l$, where l is the orbital angular momentum quantum number between the produced (j^p) resonance and the recoil particle c. Obviously, for reactions of this type only integer spins are involved. We assume also the absence of polarization effects.

For the reaction of type (1) a connection exists between the spin dynamics and the only final-state observables, represented by the momenta of the particles a, b, and c. This connection is given by the symmetric traceless Cartesian tensors formed by the particle four-momenta. A rank j tensor of this type has $2j + 1$ independent components, and represents an element of an irreducible subspace. In other words, it is isomorphous to a rank-j spinor. This correspondence is valid in the decaying particle rest frame and it has been developed in detail by Zemach [2,3], so that in particle physics Cartesian tensors are often named Zemach tensors.

However, this description, being valid only in the resonance rest frame, is noncovariant. It is possible to restore covariance by noting that, since the spin represents how the particle at rest behaves under spatial rotations, a spin tensor of rank 1 must have no time component in the resonance rest frame. This condition, put in a covariant form, is named the Rarita-Schwinger condition [9]. For spin-1 this condition reads

$$Su = S_\mu u^\mu = 0\,, \tag{2}$$

where $u = (a + b)/m$ is the four-velocity ($u^2 = 1$) of the resonance of mass $m^2 = (a + b)^2$.

The vector S_μ, being orthogonal to the timelike vector u_μ, has to be spacelike:

$$S^2 < 0\,. \tag{3}$$

The most elementary object with these properties, that is, a spin-1 covariant tensor, is called a pure spin tensor and is given by

$$S_\mu = q_\mu - (qu)u_\mu\,, \tag{4}$$

where $q_\mu = a_\mu - b_\mu$ is the break-up four-momentum. It is easily shown that (2) and (3) are satisfied. In particular, the negative norm is assured by the equation

$$S^2 = q^2 - (qu)^2 = - \mid \boldsymbol{q}_R \mid^2\,, \tag{5}$$

where $\boldsymbol{q}_R = (\boldsymbol{a}-\boldsymbol{b})_R$ is the the break-up three-momentum in the resonance rest frame.

The spin-2 tensor is a symmetric, traceless rank-2 tensor obeying condition (2):

$$T_{\mu\nu} = S_\mu S_\nu - \tfrac{1}{3} S^2 (g_{\mu\nu} - u_\mu u_\nu)\,, \tag{6}$$

where $g_{\mu\nu}$ is the metric tensor. Note also that $T^2 > 0$, and that zero trace means $T^\mu_\mu = 0$.

The formulas displayed up to now refer to the decay of a spin-1 or spin-2 resonance having four-velocity u. However, the general rule is to orthogonalize to the four-velocity of the system where the spin or the angular momentum are defined. Hence, if there is a relative orbital angular momentum l between the resonance and the spectator particle c of reaction (1), the $l = 1$ and $l = 2$ spin tensors are given by (4) and (6) where all the quantities refer now to the J^P resonance rest frame: $q = c - (a + b)$ is the break-up four-momentum and u is the four-velocity, which is $u = (1; \mathbf{0})$ in the case of the c.m. frame.

One appealing characteristic of the tensor formalism appears here: intrinsic spins and orbital angular momenta are described by the same object, the pure spin tensor.

These concepts can be generalized [10] by considering a particle of integer spin s as a tensor of rank s that satisfies the wave equation

$$(p^2 + m^2)\Phi_{\mu_1\mu_2\cdots\mu_s} = 0 \tag{7}$$

and the conditions of symmetry, orthogonality to the four-velocity and null trace:

$$\Phi_{\cdots\mu_i\cdots\mu_j\cdots} = \Phi_{\cdots\mu_j\cdots\mu_i\cdots}, \tag{8}$$
$$p^{\mu_i}\Phi_{\cdots\mu_i\cdots} = 0, \tag{9}$$
$$g^{\mu_i\mu_j}\Phi_{\cdots\mu_i\cdots\mu_j\cdots} = 0. \tag{10}$$

If one now considers a rank-$2s$ tensor Θ that, in addition to (8)–(10), satisfies also the property

$$\Theta^{\rho_1\cdots\rho_s}_{\mu_1\cdots\mu_s}\Theta^{\nu_1\cdots\nu_s}_{\rho_1\cdots\rho_s} = \Theta^{\nu_1\cdots\nu_s}_{\mu_1\cdots\mu_s} \tag{11}$$

one sees that, for any solution $\Psi_{\mu_1\cdots}$ of (7) not necessarily satisfying (8)–(10), the tensor

$$\Phi_{\mu_1\cdots} = \Theta^{\nu_1\cdots}_{\mu_1\cdots}\Psi_{\nu_1\cdots} \tag{12}$$

satisfies all the conditions (7)–(10); i.e., it is that part of $\Psi_{\mu_1\cdots}$ that describes a particle of spin s. Hence, one sees that Θ behaves like a spin projector. The general expression for Θ valid for any spin can be found in [10]. Here we give only the cases $s = 1, 2, 3$:

$$\Theta^{\nu}_{\mu} = \tilde g^{\nu}_{\mu}, \tag{13}$$
$$\Theta^{\nu_1\nu_2}_{\mu_1\mu_2} = \tfrac{1}{2}(\tilde g^{\nu_1}_{\mu_1}\tilde g^{\nu_2}_{\mu_2} + \tilde g^{\nu_2}_{\mu_1}\tilde g^{\nu_1}_{\mu_2}) - \tfrac{1}{3}\tilde g_{\mu_1\mu_2}\tilde g^{\nu_1\nu_2}, \tag{14}$$
$$\begin{aligned}\Theta^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3} = {}& \tfrac{1}{3}(\tilde g^{\nu_1}_{\mu_1}\tilde g^{\nu_2}_{\mu_2}\tilde g^{\nu_3}_{\mu_3} + \tilde g^{\nu_2}_{\mu_1}\tilde g^{\nu_3}_{\mu_2}\tilde g^{\nu_1}_{\mu_3} + \tilde g^{\nu_3}_{\mu_1}\tilde g^{\nu_1}_{\mu_2}\tilde g^{\nu_2}_{\mu_3}) \\ & -\tfrac{1}{5}(\tilde g_{\mu_1\mu_2}\tilde g^{\nu_1}_{\mu_3}\tilde g^{\nu_2\nu_3} + \tilde g_{\mu_2\mu_3}\tilde g^{\nu_2}_{\mu_1}\tilde g^{\nu_3\nu_1} + \tilde g_{\mu_3\mu_1}\tilde g^{\nu_3}_{\mu_2}\tilde g^{\nu_1\nu_2}),\end{aligned} \tag{15}$$

where we have defined the tensor

$$\tilde g_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \tag{16}$$

that has the useful properties

$$u_\mu \tilde g^{\mu\nu} = 0^\nu\,, \quad \tilde g^{\mu}_{\mu} = \tilde g_{\mu\nu}\tilde g^{\mu\nu} = 3\,, \quad \tilde g_{\mu\rho}\tilde g^{\rho}_{\nu} = \tilde g_{\mu\nu}\,. \tag{17}$$

If S_μ is a generic *pure* spin vector, orthogonalized to its own four-velocity (eventually different from that of $\tilde g$), one has

$$\tilde g_{\mu\nu}\, S^\nu = S_\mu - (Su)\, u_\mu \equiv \tilde S_\mu\,, \tag{18}$$

where we have adopted the convention to indicate as tilded tensors the pure spin tensors orthogonalized to the same four velocity of $\tilde g$.

If now S and P are two generic pure spin vectors orthogonalized to their own four-velocities (eventually different from that of $\tilde g$), from properties (17) and (18) one has

$$\begin{aligned}\tilde g_{\mu\nu}\,\tilde S^{\nu} &= \tilde S_\mu = \tilde g_{\mu\nu}\, S^\nu = g_{\mu\nu}\,\tilde S^\nu, \\ \tilde g_{\mu\nu} S^\mu P^\nu &= \tilde S_\nu P^\nu = \tilde g_{\mu\nu}\tilde S^{\mu} P^\nu \\ &= \tilde S^{\mu}\tilde P_\mu = (\tilde g^{\mu\nu} S_\nu)\,(\tilde g_{\mu\rho}P^\rho)\,.\end{aligned} \tag{19}$$

We see that the contraction of two generic pure spin tensors with $\tilde g$ gives the scalar product of the parts orthogonal to the $\tilde g$ four-velocity. This is a particular case of the general property (11), which is valid for any projector.

Finally, we note that with the aid of the projectors (13) and (14), (4) and (6) can be expressed in a more compact form:

$$S_\mu = \Theta^{\nu}_{\mu} q_\nu,$$
$$T_{\mu\nu} = \Theta^{\rho\sigma}_{\mu\nu}\, q_\rho q_\sigma.$$

III. TENSOR AMPLITUDES

The spin dependence of the decay amplitude can be obtained by using the above defined tensors and the spin-1 Lorentz invariant wave functions. For a particle of momentum p, these functions can be defined by their spin components along z, and are given by [11]

$$e^\mu(\mp 1) = \frac{\pm 1}{m\sqrt{2}} \begin{pmatrix} p_x \mp i p_y \\ m + p_x(p_x \mp i p_y)/(E+m) \\ \mp i m + p_y(p_x \mp i p_y)/(E+m) \\ p_z(p_x \mp i p_y)/(E+m) \end{pmatrix}, \tag{20}$$

$$e^\mu(0) = \frac{1}{m} \begin{pmatrix} p_z \\ p_z p_x/(E+m) \\ p_z p_y/(E+m) \\ m + p_z^2/(E+m) \end{pmatrix}. \tag{21}$$

This spinor is isomorphous to the spin vector of (4). Indeed, it is orthogonal to the four-velocity and spacelike [see the analogous (3) for tensors],

$$e^*_\mu(m)\, e^\mu(m') = -\delta_{mm'}\,, \tag{22}$$

and it gives, when summed over the polarizations, the projector operator $\tilde g_{\mu\nu}$ for spin-1 states [1,11]:

$$\sum_m e^*_\mu(m) e_\nu(m) = -\tilde g_{\mu\nu}\,, \tag{23}$$

where $u = p/m$. The minus sign is due to the Condon and Shortley choice for the arbitrary phases in (20) and (21), and has no physical significance.

If we indicate with S_μ a pure spin-1 tensor orthogonal to its own four-velocity, the decay probability $1 \to 1 + 0$ or $1 \to 0 + 1$ is given by

$$\begin{aligned} W &= \sum_m \left|\langle e^{(J)}_\mu(m) \mid S^{\mu}\rangle\right|^2 \\ &= \sum_m e^*_\mu(m)\, e_\nu(m)\, S^\mu S^\nu \\ &= -(\tilde g^{\mu\nu} S_\nu)\,(\tilde g_{\mu\rho} S^\rho)\,, \end{aligned} \tag{24}$$

where (19) and (23) have been used. This equation defines a tensor amplitude

$$A_\mu^{(1)} \equiv \tilde{g}_{\mu\nu}\, S^\nu \tag{25}$$

for the decay of a spin-1 resonance to two particles having spin zero and one, with no relative angular momentum, or to two spinless particles with relative angular momentum equal to one. In this last case the four-velocities of $\tilde{g}$ and S are the same.

Using the tensor amplitude $A_\mu^{(1)}$ the decay probability (24) is

$$W = -A_\mu^{(1)} A^{(1)\mu}\ , \tag{26}$$

where the minus sign is a common feature of the decay probabilities computed from tensor amplitudes with an odd number of indices.

To have a notation independent of the tensor rank, in the following we will write the decay probabilities as the absolute value of tensor scalar products.

At this point it is very important to recall that, when the spin tensor S_ν of (25) represents an intrinsic spin, it is orthogonalized to the four velocity of the produced resonance [j^p in the notation of (1)], whereas A_μ is orthogonalized to the four-velocity of the initial state [J^P in the notation of (1)]. This second orthogonalization, which corresponds to a Lorentz boost from the produced resonance to the initial state, is usually ignored in the so-called Zemach formalism, and this fact is precisely the source of the differences between the covariant approach and the noncovariant one. We will return to this point in more detail in the following.

The general integer spin functions are obtained from the spin-1 wave functions using the Clebsch-Gordan series. The three rank-2 functions $\mid J\, M\rangle$ with $J = 0, 1, 2$ are then given by

$$e_{\mu\nu}^{(J)}(M) = \sum_{M=m_1+m_2} \langle 1 m_1 1 m_2 \mid J\, M\rangle\, e_\mu(m_1) e_\nu(m_2)\ . \tag{27}$$

The sum over the polarizations defines three new projectors [1]:

$$P_{\mu\nu\rho\sigma}^{(J)} = \sum_m e_{\mu\nu}^{*(J)}(m) e_{\rho\sigma}^{(J)}(m)\ .$$

For the cases $0 \to 1+1$, $1 \to 1+1$, $2 \to 1+1$ one obtains, respectively,

$$\begin{aligned} P_{\mu\nu\rho\sigma}^{(0)} &= \tfrac{1}{3}\tilde{g}_{\mu\nu}\, \tilde{g}_{\rho\sigma}, \\ P_{\mu\nu\rho\sigma}^{(1)} &= \tfrac{1}{2}\left(\tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma} - \tilde{g}_{\mu\sigma}\tilde{g}_{\nu\rho}\right) \\ &= -\tfrac{1}{2}(\varepsilon_{\alpha\mu\nu\beta}\, u^\beta)\, (\varepsilon_{\gamma\rho\sigma\delta}\, u^\delta)\, g^{\alpha\gamma}, \\ P_{\mu\nu\rho\sigma}^{(2)} &= \tfrac{1}{2}\left(\tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma} + \tilde{g}_{\mu\sigma}\tilde{g}_{\nu\rho}\right) - \tfrac{1}{3}\tilde{g}_{\mu\nu}\tilde{g}_{\rho\sigma}\ , \end{aligned} \tag{28}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor. Note that the higher spin projector $P_{\mu\nu\rho\sigma}^{(2)}$ has been already introduced in (14). As in the spin-1 case, if S_μ and P_ν are two pure spin-1 tensors orthogonalized to their own four-velocities, the decay probability $J \to 1+1$ with $J = 0, 1, 2$ is given by

$$\begin{aligned} W &= \sum_m \left|\langle e_{\mu\nu}^{(J)}(m) \mid S^\mu P^\nu\rangle\right|^2 \\ &= \sum_m \left| e_{\mu\nu}^{*(J)}(m)\, S^\mu\, P^\nu\right|^2 \\ &= P_{\mu\nu\rho\sigma}^{(J)}\, S^\mu P^\nu S^\rho P^\sigma\ . \end{aligned} \tag{29}$$

Equations (28) and (29) define the spin-2 tensor algebra to be used when there are no polarization effects and there is no interference between different spin-J states. The decay $0 \to 1+1$ defines a scalar amplitude $A^{(2)} = \tilde{g}_{\mu\nu}\, S^\mu\, P^\nu$:

$$\begin{aligned} W &= P_{\mu\nu\rho\sigma}^{(0)} S^\mu P^\nu S^\rho P^\sigma \\ &= \tfrac{1}{3}(\tilde{g}_{\mu\nu} S^\mu P^\nu)\, (\tilde{g}_{\rho\sigma} S^\rho P^\sigma) \\ &\equiv \tfrac{1}{3} A^{(2)} A^{(2)}\ , \end{aligned} \tag{30}$$

the decay $1 \to 1+1$ defines the vector amplitude $A_\mu^{(2)} = \varepsilon_{\mu\alpha\beta\gamma}\, S^\alpha P^\beta u^\gamma$:

$$\begin{aligned} W &= P_{\mu\nu\rho\sigma}^{(1)} S^\mu P^\nu S^\rho P^\sigma \\ &= -\tfrac{1}{2} g^{\alpha\gamma}\, (\varepsilon_{\alpha\mu\nu\beta}\, S^\mu P^\nu u^\beta)\, (\varepsilon_{\gamma\rho\sigma\delta}\, S^\rho P^\sigma u^\delta) \\ &\equiv \tfrac{1}{2}\left| A_\alpha^{(2)} A^{(2)\alpha}\right|\ . \end{aligned} \tag{31}$$

Finally, the decay $2 \to 1+1$ defines the symmetric traceless rank-2 tensor amplitude $A_{\mu\nu}^{(2)} = \frac{1}{2}(\tilde{S}_\mu \tilde{P}_\nu + \tilde{S}_\nu \tilde{P}_\mu) - \frac{1}{3}(\tilde{S}\tilde{P})\, \tilde{g}_{\mu\nu}$:

$$\begin{aligned} W &= P_{\mu\nu\rho\sigma}^{(2)} S^\mu P^\nu S^\rho P^\sigma \\ &= \left[\tfrac{1}{2}(\tilde{S}_\rho \tilde{P}_\sigma + \tilde{S}_\sigma \tilde{P}_\rho) - \tfrac{1}{3}(\tilde{S}\tilde{P})\, \tilde{g}_{\rho\sigma}\right]\, S^\rho P^\sigma \\ &\equiv A_{\rho\sigma}^{(2)}\, S^\rho P^\sigma = A_{\rho\sigma}^{(2)}\, A^{(2)\rho\sigma}\ , \end{aligned} \tag{32}$$

where the last equality can be demonstrated by repeated use of (19) or more directly from (11).

Equations (4), (6), (30)–(32) contain all the elements necessary for the correct use of the tensor formalism. Firstly, it is necessary to form, by means of the break-up four-momenta and (4) and (6), the *spin tensors* which represent spin-1 and spin-2 functions on the Cartesian basis. Then, the decay probabilities of (30)–(32), apart from unimportant constant factors, can be interpreted as the square moduli of the *tensor amplitudes* $A_{\mu\ldots}^{(J)}$. If we define the tensor

$$\tilde{\varepsilon}_{\mu\nu\rho} \equiv \varepsilon_{\mu\nu\rho\sigma}\, u^\sigma \tag{33}$$

and recall the projectors (13) and (14) we can express all the tensor amplitudes (25), (30)–(32) in the compact forms

$$\begin{aligned} A_\mu^{(1)} &= \Theta_\mu^\nu S_\nu, \\ A^{(2)} &= \tilde{g}_{\mu\nu}\, S^\mu P^\nu, \\ A_\mu^{(2)} &= \tilde{\varepsilon}_{\mu\nu\sigma}\, S^\nu P^\sigma, \\ A_{\mu\nu}^{(2)} &= \Theta_{\mu\nu}^{\rho\sigma}\, S_\rho P_\sigma. \end{aligned} \tag{34}$$

To generalize the discussion to higher spins we should have to start from the corresponding spinors, to write the projectors $P^{(J)}_{\mu\nu\cdots}$ as in (28) and to find the tensor amplitudes. This is a very cumbersome procedure. We can avoid it by using the tensor rules which emerge from (34): starting from a space of spin J, the tensor amplitudes relative to the invariant subspaces of lower spin are obtained by contraction with the tensor $\tilde{g}_{\mu\nu}$ for the subspace of spin $J-2$, by contraction of two indices ν, ρ with the tensor $\tilde{\varepsilon}_{\mu\nu\rho}$ for the subspace of spin $J-1$ and by multiplication with the symmetric traceless projector of rank $2J$ [as in (13)–(15) for spin $s = 1, 2, 3$] to obtain the irreducible subspace of the highest spin J. In addition, all the free tensor indices must be orthogonalized to the four-velocity of the system considered by association to one of the projectors $\tilde{\varepsilon}$, $\tilde{g}$, or Θ.

Since in what follows we deal with spins ≤ 2 we apply this technique here to extract some tensor amplitudes of spin ≤ 2 from tensors belonging to spaces of higher spin.

If $\Phi_{\mu_1\mu_2\mu_3}$ is a solution of (7) representing a generic spin-3 tensor, the tensor amplitude of spin 3 is obtained by multiplication of this tensor with the spin-3 projector (15). Instead, the tensor amplitude representing an element of the irreducible spin-2 subspace is given by

$$A^{(3)}_{\mu_1\mu_2} = \Phi^{\nu_1\nu_2\nu_3}\Big[\frac{1}{6}\sum_{(i,j,k)}\Big(\tilde{\varepsilon}_{\mu_1\nu_i\nu_j}\tilde{g}_{\nu_k\mu_2} + \tilde{\varepsilon}_{\mu_2\nu_i\nu_j}\tilde{g}_{\nu_k\mu_1}\Big) - \frac{1}{3}\tilde{\varepsilon}_{\nu_1\nu_2\nu_3}\,\tilde{g}_{\mu_1\mu_2}\Big], \tag{35}$$

where from now on we use the notation $(i, j, k, \ldots)$ to indicate the sum over the even (cyclic) permutations of $(i, j, k, \ldots)$, with $i, j, k, \ldots = 1, 2, 3, 4$ and $i \neq j \neq k, \ldots$. In this way one takes properly into account the antisymmetry of $\tilde{\varepsilon}$ and the symmetry of $\tilde{g}$.

The tensor amplitude (35) is symmetric, traceless, and orthogonal to the four-velocity of the system where the spin 2 under consideration is defined.

The spin-1 amplitude is given by

$$A^{(3)}_{\mu} = \tfrac{1}{3}\,\Phi^{\nu_1\nu_2\nu_3}\big(\tilde{g}_{\mu\nu_1}\tilde{g}_{\nu_2\nu_3} + \tilde{g}_{\mu\nu_2}\tilde{g}_{\nu_3\nu_1} + \tilde{g}_{\mu\nu_3}\tilde{g}_{\nu_1\nu_2}\big), \tag{36}$$

whereas the spin-zero scalar is given by

$$A^{(3)} = \frac{1}{3}\,\Phi^{\nu_1\nu_2\nu_3}\sum_{(i,j,k)}\big(\tilde{\varepsilon}_{\mu\nu_i\nu_j}\,\tilde{g}^{\mu}_{\nu_k}\big) = \Phi^{\nu_1\nu_2\nu_3}\tilde{\varepsilon}_{\nu_1\nu_2\nu_3}, \tag{37}$$

where the notation is the same as in (35).

Without giving the $\Theta^{\nu_1\nu_2\nu_3\nu_4}_{\mu_1\mu_2\mu_3\mu_4}$ projection operator corresponding to $s = 4$ we can write the solution of rank $0, 1, 2$ that can be extracted from the $s = 4$ solution of (7). The tensor that corresponds to the solutions with $s = 2$ is given by

$$A^{(4)}_{\mu_1\mu_2} = \frac{1}{12}\Phi^{\nu_1\nu_2\nu_3\nu_4}\sum_{(i,j,k,l)}\Big[\frac{1}{2}(\tilde{g}_{\mu_1\nu_i}\tilde{g}_{\mu_2\nu_j}\tilde{g}_{\nu_k\nu_l} + \tilde{g}_{\mu_2\nu_i}\tilde{g}_{\mu_1\nu_j}\tilde{g}_{\nu_k\nu_l}) - \frac{1}{3}\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\nu_i\nu_j}\tilde{g}_{\nu_k\nu_l}\Big]. \tag{38}$$

The vector

$$A^{(4)}_{\mu} = \frac{1}{6}\Phi^{\nu_1\nu_2\nu_3\nu_4}\sum_{(i,j,k<l)}(\tilde{\varepsilon}_{\mu\nu_i\nu_j}\tilde{g}_{\nu_k\nu_l}), \tag{39}$$

where the sum extends over the even permutations of the four indices i, j, k, l with $k < l$, corresponds to the solutions with $s = 1$. Finally, the scalar

$$A^{(4)} = \frac{1}{6}\,\Phi^{\nu_1\nu_2\nu_3\nu_4}\sum_{i,j,k<l}\big(\tilde{g}_{\nu_i\nu_j}\tilde{g}_{\nu_k\nu_l}\big) \tag{40}$$

corresponds to the solutions with $s = 0$.

The tensor amplitudes so defined are expressed on the Cartesian basis. In the case of in-flight interactions, it is necessary to project on the spherical basis in order to sum incoherently over different polarizations. This projection is easily realized by means of scalar products of the type $A(m) = \langle e_\mu(m) \mid A^{(J)\mu}\rangle$, $A(m) = \langle e_{\mu\nu}(m) \mid A^{(J)\mu\nu}\rangle$ where the spinors have been defined in (20), (21) and (27).

The relativistic spin tensor formalism is probably the simplest one to describe reactions of type (1): the amplitudes are fully covariant, no Lorentz boosts are required, and the final distributions are obtained with standard tensor calculus. The algebra is not more complicated than that of other formalisms if one works always in terms of pure spin tensors with the metric of (13)–(16), and (33). Moreover, the computer calculations are faster than those employing the helicity formalism (at least a factor 4, in our experience).

IV. THE DECAY $J \to j + L$

Here we apply the formalism developed up to now to the case of reaction (1), where a spin-J initial state decays to a spin-j resonance and a spinless particle, in a final state of relative orbital angular momentum l. We consider only the cases where spins and orbital angular momenta ≤ 2 are involved. We have a twofold purpose: to find the angular distributions and to compare them with the helicity corresponding ones. In this context we have to solve at the beginning the problem relative to the moduli of the spin tensors, which give the so-called centrifugal barrier factors [for spin vectors they are given by

(5)]. These factors are absent when one uses spinors (20) and (21), that are normalized according to (22), whereas they remain into the relative orbital angular momentum terms, that are usually described as tensors [1,11].

The centrifugal barrier factors are merely constant when very sharp resonances are produced, whereas they are slowly varying functions of the momenta in the real cases. Hence, they do not affect very much the angular distributions and can be inserted into the Breit-Wigner functions of the decay amplitude. To simplify our calculations, here we make the choice to normalize all the spin tensors. In this way the tensors assume length -1 for odd order and $+1$ for even order, as required [11]. The kinematical factors due to the tensor moduli are removed, the tensors represent only pure spin effects and the corresponding decay amplitude can be directly compared with that coming from the helicity formalism. Anyway, we emphasize that this choice has no deep physical significance, being chosen here for computational purposes only.

With the notation of (1), the four-velocities of the J^P (the c.m.) and j^p resonances are given by $u_\mu = (a+b+c)_\mu/\sqrt{s}$ and $w_\mu = (a+b)_\mu/m_R$, respectively. The square of the total energy is $s=(a+b+c)^2$.

From (4) one obtains the following normalized spin-1 tensor for the resonance j^p:

$$\mathcal{S}_\mu = N_1\left[(a-b)_\mu - (a+b)_\mu\frac{(m_a^2-m_b^2)}{m_R^2}\right], \qquad (41)$$

where

$$N_1 = \frac{m_R}{[m_R^2-(m_a+m_b)^2]^{\frac12}\,[m_R^2-(m_a-m_b)^2]^{\frac12}}$$

is the normalization factor.

By inserting (41) into (6) one obtains the normalized spin-2 tensor:

$$\mathcal{T}_{\mu\nu} = \sqrt{\frac{3}{2}}\left[\mathcal{S}_\mu\mathcal{S}_\nu + \frac{1}{3}(g_{\mu\nu}-w_\mu w_\nu)\right]. \qquad (42)$$

Following the same procedure one obtains also the normalized tensors describing the states of relative orbital angular momenta $l=1$ and $l=2$ (p and d waves):

$$\mathcal{L}_\mu = N_2\left([c-(a+b)]_\mu - [c+(a+b)]_\mu\frac{(m_c^2-m_R^2)}{s}\right), \qquad (43)$$

$$\mathcal{M}_{\mu\nu} = \sqrt{\frac{3}{2}}\left(\mathcal{L}_\mu\mathcal{L}_\nu + \frac{1}{3}\tilde{g}_{\mu\nu}\right), \qquad (44)$$

where

$$N_2 = \frac{\sqrt{s}}{[s-(m_R+m_c)^2]^{\frac12}\,[s-(m_R-m_c)^2]^{\frac12}}$$

is the normalization factor, and

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu.$$

We note that in (41) and (42) and (43) and (44) two different four-velocities appear: the tensors $\mathcal{S}$ and $\mathcal{T}$ are orthogonal to the resonance four velocity w, whereas the tensors $\mathcal{L}$ and $\mathcal{M}$ are orthogonal to the c.m. four-velocity u.

We can now compute the amplitudes and the angular distributions for all the possible combinations of spins and angular momenta, under the assumption that the initial state is unpolarized. We recall that the notation is $J \to j + l$.

A. $J \to j + l$ with $j = 0$

Apart from the trivial case $0 \to 0+0$, which gives a flat angular distribution, from (34) and (41)–(44) we have the tensor amplitudes

$$A_\mu(1\to 0+1) = \Theta_\mu^\nu(1\mathcal{L}_\nu) = \mathcal{L}_\mu, \qquad (45)$$

$$A_{\mu\nu}(2\to 0+2) = \Theta_{\mu\nu}^{\rho\sigma}(1\mathcal{M}_{\rho\sigma}) = \mathcal{M}_{\mu\nu}, \qquad (46)$$

whose square modulus gives the flat angular distributions

$$W(1\to 0+1) \propto |\,\mathcal{L}_\mu\mathcal{L}^\mu\,| = 1,$$
$$W(2\to 0+2) \propto |\,\mathcal{M}_{\mu\nu}\mathcal{M}^{\mu\nu}\,| = 1,$$

since both $\mathcal{L}_\mu$ and $\mathcal{M}_{\mu\nu}$ are normalized.

As expected, we see that, when the spin of the resonance is zero, the angular distributions are flat.

B. $J \to j + l$ with $j = 1$

In this case, when the final-state particles are in s and p waves, one has to use again (35) and (41)–(44) to obtain the tensor amplitudes

$$A_\mu(1\to 1+0) = \tilde{g}_{\mu\nu}(\mathcal{S}^\nu 1) = \tilde{g}_{\mu\nu}\mathcal{S}^\nu, \qquad (47)$$

$$A(0\to 1+1) = \tilde{g}^{\mu\nu}(\mathcal{S}_\mu\mathcal{L}_\nu), \qquad (48)$$

$$A_\mu(1\to 1+1) = \tilde{\varepsilon}_{\mu\nu\rho}(\mathcal{S}^\nu\mathcal{L}^\rho), \qquad (49)$$

$$A_{\mu\nu}(2\to 1+1) = \Theta_{\mu\nu}^{\rho\sigma}(\mathcal{S}_\rho\mathcal{L}_\sigma). \qquad (50)$$

The square moduli of these amplitudes give the corresponding angular distributions. After a straightforward but lengthy calculation we obtain[1]

$$W(1\to 1+0) \propto |A_\mu A^\mu| = 1 + (\mathcal{S}u)^2, \qquad (51)$$

$$W(0\to 1+1) \propto |A|^2 = (\mathcal{SL})^2, \qquad (52)$$

$$W(1\to 1+1) \propto |A_\mu A^\mu| = 1 + (\mathcal{S}u)^2 - (\mathcal{SL})^2, \qquad (53)$$

$$W(2\to 1+1) \propto |A_{\mu\nu}A^{\mu\nu}| \propto 1 + (\mathcal{S}u)^2 + \tfrac{1}{3}(\mathcal{SL})^2. \qquad (54)$$

In the case of the d wave, there are three possible values of $J = 1,2,3$, but we will compute the amplitude only for the states with $J = 1,2$.

[1]We neglect any unimportant constant factor multiplying the amplitudes.

In the case $1 \to 1+2$ we have to combine $\mathcal{S}_\mu$ and $\mathcal{M}_{\mu\nu}$ in order to obtain a vector. This can be done with the help of (36), where

$$\Phi^{\nu_1\nu_2\nu_3} = \mathcal{S}^{\nu_1}\mathcal{M}^{\nu_2\nu_3}. \qquad (55)$$

Being $\mathcal{M}^{\nu_2\nu_3}$ traceless and orthogonal to u, the term in (36) with $\tilde{g}_{\nu_2\nu_3}$ is equal to 0, while the other two are identical. Moreover, $\mathcal{M}_{\mu\nu}$ contains the same four-velocity of $\tilde{g}$, so that $\tilde{g}_{\mu\nu}\mathcal{M}^{\nu\rho} = \mathcal{M}^\rho_\mu$. Therefore, we obtain the amplitude

$$A_\mu(1 \to 1+2) \propto \tilde{g}_{\mu\rho}\,\mathcal{S}^\nu\mathcal{M}^\rho_\nu, \qquad (56)$$

corresponding to the angular distribution

$$W(1 \to 1+2) \propto |A_\mu A^\mu| \propto 1 + (\mathcal{S}u)^2 + 3(\mathcal{S}\mathcal{L})^2. \qquad (57)$$

For the case $2 \to 1+2$ we have to combine $\mathcal{S}_\mu$ and $\mathcal{M}_{\mu\nu}$ in order to obtain a tensor. This can be done with the help of (35), applied to the tensor (55). Being $\mathcal{M}^{\nu_2\nu_3}$ symmetric, the terms in (35) with $\tilde{\varepsilon}_{\nu_1\nu_2\nu_3}$ are equal to 0, while all the others are identical. Hence, we obtain the amplitude

$$A_{\mu_1\mu_2}(2 \to 1+2) \propto \mathcal{S}^\nu\mathcal{M}^{\rho\sigma}\left(\tilde{\varepsilon}_{\mu_1\nu\rho}\tilde{g}_{\mu_2\sigma} + \tilde{\varepsilon}_{\mu_2\nu\rho}\tilde{g}_{\mu_1\sigma}\right) \qquad (58)$$

and the angular distribution

$$W(2 \to 1+2) \propto |A_{\mu_1\mu_2}A^{\mu_1\mu_2}| \propto 1 + (\mathcal{S}u)^2 - (\mathcal{S}\mathcal{L})^2. \qquad (59)$$

C. $J \to j + l$ with $j = 2$

The case $2 \to 2+0$ (s wave) is easily solved by the last of (34) applied to the spin tensor (42):

$$A_{\mu_1\mu_2}(2 \to 2+0) = \Theta^{\nu_1\nu_2}_{\mu_1\mu_2}\left(\mathcal{T}_{\nu_1\nu_2}1\right). \qquad (60)$$

The distribution is then given by

$$W(2 \to 2+0) \propto |A_{\mu_1\mu_2}A^{\mu_1\mu_2}| \propto 1 + \left[\tfrac{1}{3} + (\mathcal{S}u)^2 - \tfrac{1}{3}(uw)^2\right]^2 + (\mathcal{S}u)^2 - \tfrac{1}{3}\left[1 - (uw)^2\right]. \qquad (61)$$

In the case of the p wave, there are three possible spins for the initial state: $J = 1,2,3$. Here we compute the amplitude only for the states with $J = 1,2$.

For the $1 \to 2+1$ case we have to combine $\mathcal{L}_\mu$ and $\mathcal{T}_{\mu\nu}$ to obtain a vector. We have to use (36), where

$$\Phi^{\nu_1\nu_2\nu_3} = \mathcal{L}^{\nu_1}\mathcal{T}^{\nu_2\nu_3}. \qquad (62)$$

Being $\mathcal{T}^{\nu_2\nu_3}$ traceless and symmetric, the term in (36) with $\tilde{g}_{\nu_2\nu_3}$ is equal to 0, while the other two are identical. Hence, we obtain the amplitude

$$A_\mu(1 \to 2+1) \propto \tilde{g}_{\mu\rho}\,\mathcal{L}^\nu\mathcal{T}^\rho_\nu. \qquad (63)$$

The corresponding distribution is

$$W(1 \to 2+1) \propto |A_\mu A^\mu| \propto 1 + (\mathcal{L}w)^2 + 3(\mathcal{S}\mathcal{L})^2 + \left[3(\mathcal{S}u)(\mathcal{S}\mathcal{L}) - (uw)(\mathcal{L}w)\right]^2. \qquad (64)$$

For the case $2 \to 2+1$ we have to combine $\mathcal{L}_\mu$ and $\mathcal{T}_{\mu\nu}$ in order to obtain a tensor. This can be done with the help of (35) applied to the tensor (62). Also in this case, one sees that, since $\mathcal{T}^{\nu_2\nu_3}$ is symmetric, the terms in (35) with $\tilde{\varepsilon}_{\nu_1\nu_2\nu_3}$ and $\tilde{g}_{\nu_2\nu_3}$ are equal to 0, while all the others are identical. Hence, the amplitude and the corresponding distribution are

$$A_{\mu_1\mu_2}(2 \to 2+1) \propto \mathcal{L}^\nu\mathcal{T}^{\rho\sigma}\left(\tilde{\varepsilon}_{\mu_1\nu\rho}\tilde{g}_{\mu_2\sigma} + \tilde{\varepsilon}_{\mu_2\nu\rho}\tilde{g}_{\mu_1\sigma}\right), \qquad (65)$$

$$W(2 \to 2+1) \propto |A_{\mu_1\mu_2}A^{\mu_1\mu_2}| \propto \left[1 + (\mathcal{S}u)^2 - (\mathcal{S}\mathcal{L})^2\right] \times \left[1 + (\mathcal{S}u)^2\right]. \qquad (66)$$

In the case of the d wave, the spin values of the initial state are $J = 0,1,2,3,4$. Here we list the amplitudes for $J = 0,1,2$ only, which are obtained from (38)–(40) applied to the spin tensors (41) and (44), in the same way as explained above:

$$A(0 \to 2+2) \propto \tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma}\,\mathcal{T}^{\mu\nu}\mathcal{M}^{\rho\sigma}, \qquad (67)$$

$$A_\mu(1 \to 2+2) \propto \tilde{\varepsilon}_{\mu\nu\rho}\mathcal{T}^{\sigma\nu}\mathcal{M}^\rho_\sigma, \qquad (68)$$

$$A_{\mu_1\mu_2}(2 \to 2+2) \propto \left[\tfrac{1}{2}\left(\tilde{g}_{\mu_1\nu}\tilde{g}_{\mu_2\rho} + \tilde{g}_{\mu_2\nu}\tilde{g}_{\mu_1\rho}\right) - \tfrac{1}{3}\tilde{g}_{\mu_1\mu_2}\,\tilde{g}_{\nu\rho}\right]\mathcal{T}^{\nu\sigma}\mathcal{M}^\rho_\sigma. \qquad (69)$$

The corresponding distributions are obtained by the square modulus of these amplitudes.

D. Explicit form of angular distributions

Here we calculate explicitly some of the distributions previously obtained. To this end, the more convenient reference frame is that of the produced resonance j^P, where the relevant kinematical quantities assume the form [see (1), (4)–(6), (41)–(44) and the notation there used]

$$\begin{aligned} a &= (\sqrt{m_a^2 + q^2/4}; \boldsymbol{q}/2), \\ b &= (\sqrt{m_b^2 + q^2/4}; -\boldsymbol{q}/2), \\ c &= (\sqrt{m_c^2 + p^2}; \boldsymbol{p}), \\ a + b &= (m_R; \boldsymbol{0}), \\ w &= (1; \boldsymbol{0}), \\ a + b + c &= (\sqrt{s + p^2}; \boldsymbol{p}), \\ u &= (\sqrt{1 + p^2/s}; \boldsymbol{p}/\sqrt{s}). \end{aligned} \tag{70}$$

If we now introduce the angle θ between the spectator and the break-up three-momenta and the ratio z between the modulus of the spectator three-momentum and the total energy,

$$\cos\theta = \boldsymbol{qp}/(|\,\boldsymbol{q}\,||\,\boldsymbol{p}\,|), \qquad z = |\,\boldsymbol{p}\,|\,/\sqrt{s}\,, \tag{71}$$

we can use (41)–(44) and compute the following invariants in the resonance rest frame:

$$\mathcal{S}u = -z\cos\theta, \tag{72}$$

$$\mathcal{S}\mathcal{L} = -\sqrt{1+z^2}\cos\theta, \tag{73}$$

$$\mathcal{L}w = z, \tag{74}$$

$$uw = \sqrt{1+z^2}. \tag{75}$$

Now it is easy, using these quantities, to rewrite the spin distributions (51)–(54), (57), (59), (61), (64), and (66) in the form

$$W(0 \to 1+1) \propto (1+z^2)\cos^2\theta, \tag{76}$$

$$W(1 \to 1+0) \propto 1 + z^2\cos^2\theta, \tag{77}$$

$$W(1 \to 1+1) \propto 1 - \cos^2\theta, \tag{78}$$

$$W(1 \to 1+2) \propto 1 + \left(3 + 4z^2\right)\cos^2\theta, \tag{79}$$

$$W(1 \to 2+1) \propto (1+z^2)\Big[1 + 3\cos^2\theta + 9z^2\left(\cos^2\theta - \tfrac{1}{3}\right)^2\Big], \tag{80}$$

$$W(2 \to 1+1) \propto 3 + (1+4z^2)\cos^2\theta, \tag{81}$$

$$W(2 \to 1+2) \propto 1 - \cos^2\theta, \tag{82}$$

$$W(2 \to 2+1) \propto 1 + \frac{z^2}{9} + \left(\frac{z^2}{3} - 1\right)\cos^2\theta - z^2(\cos^2\theta - \tfrac{1}{3})^2, \tag{83}$$

$$W(2 \to 2+0) \propto 1 + \frac{z^2}{3} + z^2\cos^2\theta + z^4\left(\cos^2\theta - \tfrac{1}{3}\right)^2, \tag{84}$$

where the familiar expressions of the Legendre polynomials appear.

It is interesting now to compare these distributions with those coming from the noncovariant approach. We can consider the simplest (perhaps the most striking) case (77): the noncovariant result is, namely, the flat distribution, because the unnormalized Zemach amplitude is in this case $\boldsymbol{q}$, the break-up three-momentum in the resonance rest frame. The flat distribution is obtained from (77) by putting $z = 0$. It is easy to recognize that this is a general rule: all the covariant distributions tend to the noncovariant (Zemach) ones in the case of the production of a heavy resonance, when the recoil particle is nearly at rest, that is when $z \to 0$. In this case the Lorentz boost between the initial and the resonance rest frames tends to be negligible and this effect is measured by the magnitude of z, which in this context assumes a crucial role. Since this parameter is the ratio between a noncovariant quantity and the total energy, which is covariant [see (71)], it is by no means obliged to be a small quantity. By using relativistic kinematics it is possible to show that for the decay $J \to (ab)c$ one has

$$z^2 = \gamma^2 - 1\,, \tag{85}$$

where $\gamma = E/m$ for the system (ab) in the J rest frame, and z refers to the spectator c in the rest frame of (ab).

The values of z for the reaction $p\bar{p} \to (\pi_1\pi_2)\pi_3$ at rest are shown in Fig. 1 as a function of the $(\pi_1\pi_2)$ invariant mass. In the case of the production of the $\rho(770)$ and $f_2(1270)$ resonances one has $z^2 = 1.016$ and $z^2 = 0.152$, respectively, at the nominal mass values.

Since the $p\bar{p} \to \pi\pi\pi$ annihilation at low energy is at present extensively studied at LEAR [13,14,8] we report in Table I, as an example, the covariant and noncovariant spin distributions (angular weights), calculated at the nominal $\rho(770)$ and $f_2(1270)$ masses, for all the channels of importance. The results are purely indicative, because the resonance widths are neglected.

From this table one can note that for initial states $0^{-+}, 1^{--}$, corresponding to the S wave protonium orbitals, there is no difference between the covariant and noncovariant spin distributions. Therefore, the results coming from the data analysis of experiments on $p\bar{p} \to \pi\pi\pi$ from ${}^1S_0, {}^3S_1$ protonium states [12,6] are rather insensitive to the spin formalism used in the analysis.

On the contrary, when the annihilation into three pions from protonium P orbitals is concerned, there is a big difference between the covariant and noncovariant spin distributions. The difference is particularly evident for the decay $\rho\pi$ from ${}^1P_1, {}^3P_1$ protonum states. This important point has been completely neglected in the recent Dalitz plot analysis of this reaction, where the noncovariant formalism has been used [7].

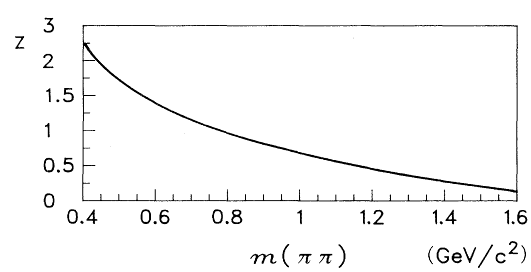

FIG. 1. The parameter z of (71) for the reaction $p\bar{p} \to \pi_1\pi_2\pi_3$ at rest as a function of the $(\pi_1\pi_2)$ invariant mass.

TABLE I. Covariant and noncovariant spin distributions for the low energy $p\bar{p} \to \pi\pi\pi$ annihilation. The first two columns give the $p\bar{p}$ initial state quantum numbers in the J^{PC} and spectroscopic notations, respectively. In the third column the decay final state is reported with the notation $(R\pi)_l$, where R is the produced resonance and l the wave corresponding to its relative angular momentum with the recoil pion. The production of the $\rho(770)$ and $f_2(1270)$ resonances is considered. However, one should note that the $\rho(770)$ production is forbidden in the $p\bar{p} \to \pi^0\pi^0\pi^0$ channel. The fourth column reports the decay type with the notation used in the text. Finally, the last two columns give the angular distributions resulting from (76)–(84) by calculating z at the resonance nominal mass values (covariant weights) and by putting $z = 0$ (noncovariant weights).

Initial state J^{PC}	Initial state $^{2S+1}L_J$	Final state $(R\pi)_l$	Decay type $J \to j + l$	Covariant weights	Noncovariant weights
(0^{-+})	$^1S_0$	$(\rho\pi)_p$	$0 \to 1+1$	$\cos^2\theta$	$\cos^2\theta$
(1^{--})	$^3S_1$	$(\rho\pi)_p$	$1 \to 1+1$	$\sin^2\theta$	$\sin^2\theta$
$(1^{+-}, 1^{++})$	$^1P_1, ^3P_1$	$(\rho\pi)_s$	$1 \to 1+0$	$1 + 1.02\cos^2\theta$	1
$(1^{+-}, 1^{++})$	$^1P_1, ^3P_1$	$(\rho\pi)_d$	$1 \to 1+2$	$1 + 7.06\cos^2\theta$	$1 + 3\cos^2\theta$
(2^{++})	$^3P_2$	$(\rho\pi)_d$	$2 \to 1+2$	$\sin^2\theta$	$\sin^2\theta$
$(2^{-+}, 2^{--})$	$^1D_2, ^3D_2$	$(\rho\pi)_p$	$2 \to 1+1$	$1 + 1.69\cos^2\theta$	$1 + (1/3)\cos^2\theta$
(0^{-+})	$^1S_0$	$(f_2\pi)_d$	$0 \to 2+2$	$(\cos^2\theta - 1/3)^2$	$(\cos^2\theta - 1/3)^2$
(1^{++})	$^3P_1$	$(f_2\pi)_p$	$1 \to 2+1$	$1 + 3\cos^2\theta$ $+1.37(\cos^2\theta - 1/3)^2$	$1 + 3\cos^2\theta$
(2^{++})	$^3P_2$	$(f_2\pi)_p$	$2 \to 2+1$	$\sin^2\theta(1 + 0.15\cos^2\theta)$	$\sin^2\theta$
(2^{-+})	$^1D_2$	$(f_2\pi)_s$	$2 \to 2+0$	$1 + 0.14\cos^2\theta$ $+0.02(\cos^2\theta - 1/3)^2$	1

V. COMPARISON WITH THE HELICITY AMPLITUDE

The helicity amplitude for the two-body decay of a spin-J resonance to two particles of spin s_1, s_2 is usually written in the form [1,4,11]

$$A_{\lambda\nu M} \propto \langle \vartheta\varphi\lambda\nu \mid JM\lambda\nu\rangle\langle JM\lambda\nu \mid A \mid JM\rangle \propto D^{*J}_{M\delta}(\varphi,\vartheta,0)F^J_{\lambda\nu}\,, \qquad (86)$$

where M is the z component of the spin J in a coordinate system fixed with the resonance rest frame, λ, ν are the helicities of the two final-state particles, $\delta = \lambda - \nu$, D is the usual Wigner spin rotation matrix, and the angles (ϑ, φ) define the direction of the decay products in the resonance rest frame. The helicity-coupling amplitude $F^J_{\lambda\nu} = \langle JM\lambda\nu \mid A \mid JM\rangle$ is a rotationally invariant and is related to the ls-coupling amplitudes $G_{ls} = \langle JM\, l\, s \mid A \mid JM\rangle$ via the relation

$$F^J_{\lambda\nu} = \sum_{ls}\left(\frac{2l+1}{2J+1}\right)^{1/2}\langle ls0\delta \mid J\delta\rangle\langle s_1 s_2\, \lambda - \nu \mid s\delta\rangle G_{ls}\,. \qquad (87)$$

The helicity amplitude $A_{\lambda\nu M}$ is a matrix with $(2s_1+1)(2s_2+1)$ rows and $(2J+1)$ columns. In the case of sequential decays as in the reaction (1), it is necessary to calculate, with (86), two different helicity amplitudes in the two rest frames of (j^p) and (J^P). Then, the matrix product of these two matrices gives the total decay amplitude, that in our case becomes a complex vector of $(2J+1)$ components. In [15] it is explained in detail how to perform such practical calculations.

The helicity formalism is more general than the tensor one, because it is valid for any mass and spin of the particles involved in the decay. Obviously, in our case of spin-zero massive particles measured in the final state, the two approaches should coincide. The terms for the comparison are the amplitudes G_{ls} or $F^J_{\mu\nu}$ of (87), that contain the energy dependence of the interaction. This dependence, usually unknown, is parametrized in terms of complex constants, Breit-Wigner (or phase shifts) functions and some further energy-momentum-dependent factor. Since at this stage we are interested in the angular dependence predicted by the two formalisms, we do not consider for the moment Breit-Wigner or equivalent terms.

Firstly, we note that if one puts $G_{ls} = 1$ the helicity gives (apart from inessential multiplicative constants) exactly the same results as the noncovariant Zemach tensors, when these are properly normalized. Usually the comparison is made with unnormalized tensors, and, for the case of reaction (1), the well-known empirical rule is used [1]:

$$G_{ls} \equiv G_{j0} \propto \mid \boldsymbol{q} \mid^j \quad \text{for the decay} \quad (j^p) \longrightarrow a + b, \qquad (88)$$

$$G_{ls} \equiv G_{l0} \propto \mid \boldsymbol{p} \mid^l \quad \text{for the decay} \quad (J^P) \longrightarrow j^p + c\,, \qquad (89)$$

where $\boldsymbol{q}$ and $\boldsymbol{p}$ are the break-up momenta calculated in the (j^p) and (J^P) rest frames and l is the relative orbital angular momentum between (j^p) and c in the (J^P) rest frame. The two factors of (88) and (89) are exactly the moduli of the two Zemach tensors defined in the two different systems.

The helicity-coupling amplitude, corrected with the recipe of (88) and (89), is manifestly noncovariant. The covariance is restored by imposing that the amplitudes $F^J_{\lambda\mu}$ or G_{ls} of (87) be Lorentz scalars. The most general scalar has to be constructed, in each helicity frame, by the polarization four-vectors associated with the particle spins and by the covariant tensors associated to the orbital angular momenta [1]. The procedure is based on the following steps. One starts with the covariant polarization four vectors of (20) and (21), relative to a spin-1 particle of four-momentum p and mass m, that in the helicity frame assume the form [1,11]

$$\begin{aligned} e^\mu(\pm) &= \mp\tfrac{1}{\sqrt{2}}\ (0;\quad 1,\ \pm i,\quad 0), \\ e^\mu(0) &= \left(\frac{p_z}{m};\ 0,\quad 0,\ \frac{p_0}{m}\right). \end{aligned} \tag{90}$$

Then, the formalism is completed with the usual rules for the addition of angular momenta to obtain spinors of higher order and with (4) and (6), written in the helicity frame, to obtain the orbital angular momenta involved in the decay process. With the tensors and spinors defined in this way, one can write the initial- and final-state tensors of the decay reaction. Finally, the scalar amplitude is obtained by contraction of the spin tensor T of the decay final state with the conjugate tensor σ of the initial state. Since by construction these tensors have the same rank, the result is a scalar. If we indicate with a square bracket a generic tensor contraction, the amplitude can be expressed in the form

$$\begin{aligned} F^J_{\lambda\mu} &= \langle JM\lambda\nu \mid A \mid JM\rangle \\ &= \sum_{ls} C_{ls}\ [\sigma^*(\lambda-\mu)T(\lambda,\mu,l,s)]\ , \end{aligned} \tag{91}$$

where the sum is on the set of the l, s variables, and C are complex constants which take into account the unknown decay dynamics contained in A. Explicit forms of (91) will be given later on for the cases discussed in the examples.

As the momenta involved are all parallel with the z axis, (91) gives the momentum dependence of the helicity-coupling amplitudes, but no angular dependence, which is already contained in the D function of (86). However, in a multistep reaction as in (1), the Lorentz boosts to each resonance rest frame, necessary in the helicity formalism, can give rise to differences in the angular distributions in the general frame (the c.m.). These differences depend on the form of the amplitude $F^J_{\mu\nu}$.

We are interested here in the special case of an initial J^P particle that decays into a j^p resonance and a spinless spectator in an l wave of relative angular momentum. The resonance has a subsequent decay into two spinless particles.

We then need two helicity amplitudes. The first one is a $(2j+1)\times(2J+1)$ matrix which describes the J^P decay:

$$\begin{aligned} A_{\lambda M} &\propto \langle\vartheta\varphi\lambda 0 \mid JM\lambda 0\rangle\langle JM\lambda 0 \mid A \mid JM\rangle \\ &\propto D^{*J}_{M\lambda}(\varphi,\vartheta,0)F^J_{\lambda 0}\ , \end{aligned} \tag{92}$$

where λ runs from $-j$ to $+j$, and M runs from $-J$ to $+J$.

The second amplitude is a column matrix with $(2j+1)$ elements, which describes the j^p decay:

$$\begin{aligned} A'_\lambda &\propto \langle\vartheta'\varphi' 00 \mid j\lambda 00\rangle\langle j\lambda 00 \mid A' \mid j\lambda\rangle \\ &\propto D^{*j}_{\lambda 0}(\varphi',\vartheta',0)\ . \end{aligned} \tag{93}$$

The total amplitude is now the product of the two matrices:

$$A_M \propto \sum_\lambda F^J_{\lambda 0}D^{*J}_{M\lambda}(\varphi,\vartheta,0)D^{*j}_{\lambda 0}(\varphi',\vartheta',0)\ . \tag{94}$$

If the spin density matrix ρ^J of the initial state is known, then the final probability can be obtained by the product

$$\begin{aligned} W &\propto \sum_{MM'} \rho^J_{MM'}A^J_M A^{*J}_{M'} \\ &\propto \sum_{MM'} \rho^J_{MM'} \sum_{\lambda\lambda'} F^J_{\lambda 0}F^{*J}_{\lambda' 0}\,[D^{*J}_{M\lambda}(\varphi,\vartheta,0)D^J_{M'\lambda'}(\varphi,\vartheta,0)D^{*j}_{\lambda 0}(\varphi',\vartheta',0)D^j_{\lambda' 0}(\varphi',\vartheta',0)]\ . \end{aligned} \tag{95}$$

The angular distribution in ϑ' is obtained by integrating over φ', φ, and ϑ. Integrating over φ' results in a $\delta_{\lambda\lambda'}$ factor, while integrating over φ and ϑ results in another $\delta_{MM'}$ factor. One is then left with a factor $\sum_M \rho^J_{MM} = \mathrm{Tr}[\rho^J] = 1$, and with the angular distribution in ϑ' given by

$$W \propto \sum_\lambda \mid F^J_{\lambda 0}\mid^2\ \left[d^{(j)}_{\lambda 0}(\vartheta')\right]^2\ . \tag{96}$$

For $j=1$ (96) can be written using the explicit form of the $d^{(1)}$ functions,

$$W \propto f^J_1 + (f^J_0 - f^J_1)\cos^2\vartheta'\ , \tag{97}$$

and the same can be done for $j=2$,

$$\begin{aligned} W \propto\ & \tfrac{4}{9}(f^J_1 + 2f^J_2) + \tfrac{4}{3}(f^J_1 - f^J_2)\cos^2\vartheta' \\ & + (3f^J_0 - 4f^J_1 + f^J_2)\left(\cos^2\vartheta' - \tfrac{1}{3}\right)^2, \end{aligned} \tag{98}$$

where we have used the shorthand notation

$$f_0^J = | F_{00}^J |^2 , \quad (99)$$

$$f_1^J = \tfrac{1}{2}\Big(| F_{+10}^J |^2 + | F_{-10}^J |^2 \Big) = | F_{+10}^J |^2 , \quad (100)$$

$$f_2^J = \tfrac{1}{2}\Big(| F_{+20}^J |^2 + | F_{-20}^J |^2 \Big) = | F_{+20}^J |^2 . \quad (101)$$

We performed complete numerical and analytical calculations for many of the $\bar{p}p$ annihilation channels at present under study at LEAR, and we found that the results obtained with covariant tensors and with covariant helicity-coupling amplitudes always coincide.[2] Here we report some explicit calculations to explain in more detail this aspect.

A. $p\bar{p}(1^+) \to \rho\pi$

The first case we discuss is the decay $\bar{p}p \to \rho\pi^0$, $\rho \to \pi^+\pi^-$ from the ${}^1P_1$ protonium state. Following our notation we denote as a, b the momenta of the pions coming from the ρ decay, as c the momentum of the recoiling neutral pion. The four-velocities of the ρ and of the $\bar{p}p$ system are denoted as u and w, respectively. The decay is of the type $1^+ \to 1^- + 0^-$ so that the orbital angular momentum l has to be even by parity conservation.

Firstly, we consider the case $l = 0$, which corresponds to (77) in the tensor formalism. Here the scalar product of (91) is the contraction between the spinor of the initial helicity ${}^1P_1$ state

$$\begin{aligned} \sigma^\mu(\pm) &= \mp\tfrac{1}{\sqrt{2}}\ (0;\ 1,\ \pm i,\ 0), \\ \sigma^\mu(0) &= \phantom{\mp\tfrac{1}{\sqrt{2}}}\ (0;\ 0,\ 0,\ 1), \end{aligned} \quad (102)$$

and the ρ spinor

$$\begin{aligned} \Phi^\mu(\pm) &= \mp\tfrac{1}{\sqrt{2}}\ (0;\ 1,\ \pm i,\ 0), \\ \Phi^\mu(0) &= \left(\frac{p}{m_\rho};\ 0,\ 0,\ \frac{p_0}{m_\rho}\right), \end{aligned} \quad (103)$$

where $p = (a + b)$ is the ρ four-momentum in the $\bar{p}p$ helicity frame. Hence, (91) gives (see also Chung [1], Eq. (59))

$$\begin{aligned} F^1_{\pm 10} &= C_{01}\,[\sigma^*(\pm)\Phi(\pm)] = C_{01}, \\ F^1_{00} &= [\sigma^*(0)\Phi(0)] = C_{01}\left(\frac{p_0}{m_\rho}\right) = \gamma C_{01}. \end{aligned} \quad (104)$$

These terms have now to be inserted in (97) to obtain

$$W(1 \to 1 + 0) \propto 1 + (\gamma^2 - 1)\cos^2\vartheta'. \quad (105)$$

Since ϑ', has the same meaning as θ in (71), from (85) the identity between this result and (77) is assured. The noncovariant result (uniform distribution) is obtained by putting $\gamma = 1$. This corresponds to set all the three amplitudes of (104) equal to the same complex constant C_{01}.

We note that (105) relies on covariance, Lorentz spinors, and spin rotation matrices only, whereas (77) is obtained with pure tensor calculus.

Now we treat the case with $l = 2$. In this case the tensor describing the d wave in the $\rho\pi$ helicity frame is given by (44) with the ingredients given by

$$\begin{aligned} u_\mu &= (1; 0, 0, 0), \\ l_\mu &= c_\mu - (a + b)_\mu = (l_0; 0, 0, l_z), \\ \mathcal{L}_\mu &= \frac{1}{|l|}\,[l_\mu - (l \cdot u)u_\mu] = (0; 0, 0, 1). \end{aligned} \quad (106)$$

The amplitude $T(1 \to 1 + 2)$ is given by (56) and (103) and (106):

$$\begin{aligned} T_\mu(1 \to 1 + 2) &= \tilde{g}_{\mu\rho}\Phi^\nu \mathcal{M}^\rho_\nu \\ &= g_{\mu\rho}\Phi^\nu \mathcal{M}^\rho_\nu = \Phi^\nu \mathcal{M}_{\mu\nu}. \end{aligned} \quad (107)$$

Following (91) the covariant helicity-coupling amplitude is then given by

$$\begin{aligned} F^J_{\lambda 0} &= C_{21}\,\sigma^{*\mu}(\lambda)\,\Phi^\nu(\lambda)\,M_{\mu\nu} \\ &= C_{21}\,\{[\mathcal{L}\phi(\lambda)]\,[\mathcal{L}\sigma^*(\lambda)] \\ &\quad + \tfrac{1}{3}[\Phi(\lambda)\sigma^*(\lambda)]\} . \end{aligned} \quad (108)$$

With the same notation of (104) the helicity components are

$$F^1_{+10} = F^1_{-10} = -C_{21}\,, \qquad F^1_{00} = 2C_{21}\left(\frac{p_0}{m_\rho}\right) , \quad (109)$$

This equation is analogous to (59) of Chung [1], apart from the factor l^2, which in our case is missing because we use normalized spin tensors.

By inserting (109) into (97) we obtain

$$W(1 \to 1 + 2) \propto 1 + (4\gamma^2 - 1)\cos^2\vartheta'$$

which, due to (85), is identical to (79).

B. $p\bar{p}(1^+) \to f_2(1270)\pi$

Here we refer to the decay $p\bar{p} \to f_2\pi$, $f_2 \to \pi\pi$ from ${}^1P_1, {}^3P_1$ protonium states. The lowest order allowed angular momentum is $l = 1$. In this case the tensor amplitude $A(1 \to 2 + 1)$ involving a spin-2 particle produced in p wave is given by (27), (63), (103), and (106):

TABLE II. Covariant helicity-coupling amplitudes for the decay $J \to j + l$ with $j = 1$, as a function of γ from (85).

$J \to j + l$	F^J_{00}	F^J_{+10}
$0 \to 1{+}1$	γ	0
$1 \to 1{+}0$	γ	1
$1 \to 1{+}1$	0	1
$1 \to 1{+}2$	$\frac{2}{3}\gamma$	$-\frac{1}{3}$
$2 \to 1{+}1$	$\sqrt{\frac{2}{3}}\gamma$	$\frac{1}{\sqrt{2}}$
$2 \to 1{+}2$	0	1

[2] Apart from Eq. (128) of Chung [1] that should read $F_0^{(0)} = [(q_0/m)^2 + 1/2]gr^2$.

$$T_\mu(m) = \tilde{g}_{\mu\rho}\,\omega^{\rho\nu}\mathcal{L}_\nu = \sum_{m=m_1+m_2} \langle 1m_1 1m_2 \mid 2m\rangle \tilde{g}_{\mu\rho}\Phi^\rho(m_1)\Phi^\nu(m_2)\,\mathcal{L}_\nu. \quad (110)$$

When contracted with the initial state spinor (102), which is orthogonal to the same four-velocity of $\tilde{g}$, this amplitude transforms into the covariant helicity-coupling amplitude (91):

$$F^1_{\lambda 0} = C_{12} \sum_{\lambda=m_1+m_2} \langle 1m_1 1m_2 \mid 2\lambda\rangle [\sigma^*(\lambda)\Phi(m_1)][\Phi(m_2)\mathcal{L}]. \quad (111)$$

The covariant helicity components are, in the same notation of (104)

$$F^1_{\pm 20} = 0, \qquad F^1_{\pm 10} = \frac{1}{\sqrt{2}} C_{12}\gamma, \qquad F^1_{00} = \sqrt{\frac{2}{3}} C_{12}\gamma^2, \quad (112)$$

which are analogous to Eqs. (116) of Chung [1], apart from the factor l, that in our case is missing due to spin tensor normalization.

Using (98) we obtain

$$W(1 \to 2+1) \propto \gamma^2[1 + 3\cos^2\vartheta' + 9(\gamma^2 - 1)(\cos^2\vartheta' - \tfrac{1}{3})^2]. \quad (113)$$

This result, taking into account (85), is identical to (80).

If one uses the factors $F^J_{\lambda 0}$ reported in Tables II and III, and (85), (97)–(101), all the distributions (76)–(84) are easily obtained. In this way the already-mentioned identity of the tensor formalism with the covariant helicity formalism can be explicitly shown for all the cases considered.

VI. DALITZ PLOTS

Here we make some comparison between Dalitz plots calculated with the covariant and noncovariant spin formalisms. As in the previous sections, the examples are taken from the low energy $p\bar{p}$ annihilation into three pions, with the production of a resonance: $p\bar{p} \to R\pi_3 \to \pi_1\pi_2\pi_3$.

The Dalitz plots are obtained in the standard way, by weighting Monte Carlo phase space events with the term

$$|F_{\rm BW}(m_R)\,A_{\mu\ldots}(J \to j + l)|^2\,, \quad (114)$$

where $A(J \to j + l)$ is the tensor amplitude (covariant or noncovariant) describing the spin dynamics, and $F_{\rm BW}(m_R)$ is the Breit-Wigner (BW) function describing the two pion $\pi_1\pi_2$ resonant state. For this function we choose the form [17]

$$F_{\rm BW}(m) = \frac{m_0\,\Gamma_0}{m_0^2 - m^2 - im_0\Gamma(m)},$$
$$\Gamma(m) = \Gamma_0\left(\frac{Q}{Q_0}\right)^{2j+1}\frac{m_0}{m}\,, \quad (115)$$

where m and m_0 are the effective and the nominal resonance masses, Γ_0 is the nominal width, Q and Q_0 are the moduli of the break-up three-momenta of the two decay products, in the resonance rest frame, corresponding to the masses m and m_0, $(Q/Q_0)^{2j+1}$ is the centrifugal barrier factor, j is the spin of the resonance. The BW function is not multiplied by any additional centrifugal barrier factor, because here we return to use the unnormalized tensors defined in Secs. II and III. Functions more sophisticated than (115) could be employed, but they are not important in this context.

The first case we present is the annihilation at rest $p\bar{p}(1^{+-}) \to \rho\pi$ from the ${}^1P_1$ protonium state with ρ and π in a relative s wave. The mass and width of the ρ resonance are fixed at 770 and 149 MeV, respectively. We recall that the covariant spin distribution for relative angular momenta $l = 0$ is given by (77). This equation gives also the noncovariant result (the flat distribution) under the condition $z = 0$.

The result is shown in Fig. 2. We see that the covariant distribution differs noticeably from the noncovariant one.

The annihilation into three pions has been studied experimentally at LEAR by forming S and P wave protonium states with an antiproton beam stopped in a gaseous hydrogen target. The data of the reaction $p\bar{p} \to \pi^+\pi^-\pi^0$ in protonium P waves have been reported by the ASTERIX collaboration [7]. The waves ${}^3P_1(1^{++})$ and ${}^3P_2(2^{++})$, with the production of the neutral ρ only are also present in the reaction in flight $p\bar{n} \to \pi^+\pi^+\pi^-$ recently studied at LEAR by the OBELIX Collaboration [13,14]. New data in P wave will also be available from the low pressure target technique developed recently by OBELIX [16].

All these experiments have used in their published analyses the noncovariant spin formalism. In this context it is interesting to note that the covariant result presents a bump, due merely to the reflection of the ρ spin, exactly in the region where the existence of the new resonance 2^{++} $AX/f_2(1520)$ has been recently claimed by ASTERIX [7] and confirmed by Crystal Barrel [8] and OBELIX [14].

TABLE III. Covariant helicity-coupling amplitudes for the decay $J \to j + l$ with $j = 2$, as a function of γ from (85).

$J \to j + l$	F^J_{00}	F^J_{+10}	F^J_{+20}
$0 \to 2{+}2$	$\gamma^2 + \frac{1}{2}$	0	0
$1 \to 2{+}1$	$\sqrt{\frac{2}{3}}\gamma^2$	$\frac{1}{\sqrt{2}}\gamma$	0
$1 \to 2{+}2$	0	γ	0
$2 \to 2{+}0$	$\frac{2}{3}\gamma^2 + \frac{1}{3}$	γ	1
$2 \to 2{+}1$	0	$\frac{1}{2}\gamma$	1
$2 \to 2{+}2$	$\frac{4}{9}\gamma^2 - \frac{1}{9}$	$\frac{1}{6}\gamma$	$-\frac{1}{3}$

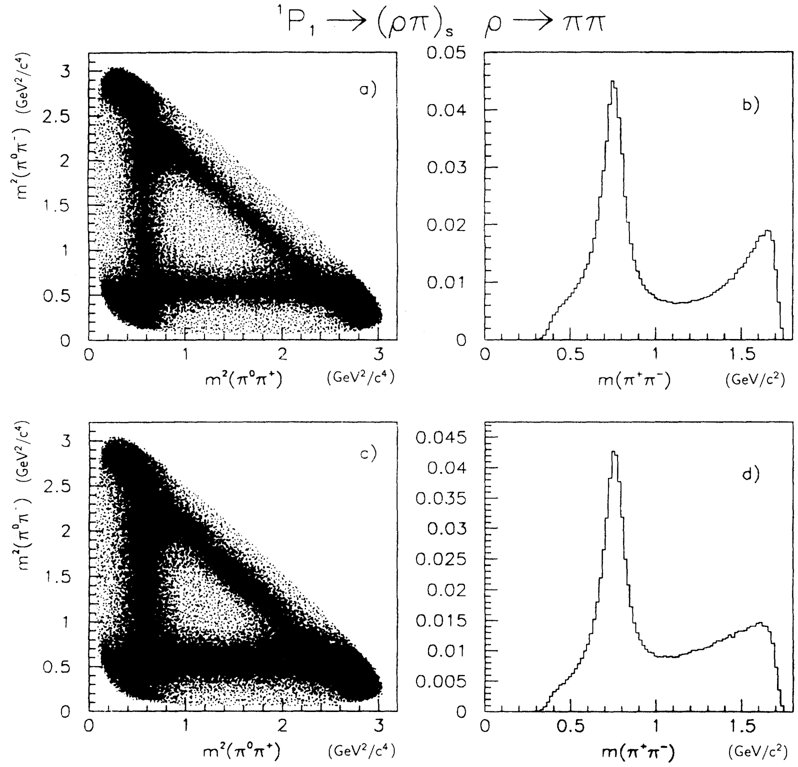

FIG. 2. Dalitz plots for the decay $p\bar{p} \to \pi^+\pi^-\pi^0$, from the initial $^1P_1$ protonium state, with the production of the $\rho(770)$ resonance. (a) Covariant Dalitz plot; (b) its projection on the $\pi^+\pi^-$ invariant mass; (c) noncovariant Dalitz plot; (d) its projection on the $\pi^+\pi^-$ invariant mass. For the definition of covariant and noncovariant (Zemach) Dalitz plots see the text.

The correct $\cos\theta$ dependence is obtained in the noncovariant Zemach formalism by summing two $l = 0$ and $l = 2$ relative orbital angular momenta between ρ and π [7]. This corresponds, in the covariant formalism, to the sum of the amplitudes (47) and (56), which give the angular distributions (77) and (79). We note, however, that in the noncovariant approach the dependence on z of (71) is lost and that the complex coefficients for the coupling of the two angular momenta assume completely different values.

It is not our intention here to go further into the discussion and to draw any premature conclusion about this new meson AX/f_2, because the signal is evident also in the channel $p\bar{p} \to \pi^0\pi^0\pi^0$ studied by Crystal Barrel, where the ρ resonance is not produced. We note only that the assignment 2^{++} has been recently questioned by Crystal Barrel [18], showing that the spin parity assignment is strongly dependent both on the initial states involved in the decay and on the model used in the analysis.

The second case we report refers to the annihilation at rest $p\bar{p} \to \pi^0\pi^0\pi^0$, which is at present under study at LEAR by the Crystal Barrel Collaboration [8]. In particular, we show the channel $p\bar{p}(1^{++}, 2^{++}) \to f_2(1270)\,\pi^0, f_2 \to \pi^0\pi^0$ from the $^3P_1, ^3P_2$ protonium states with f_2 and π in a relative p wave. The spin dis-

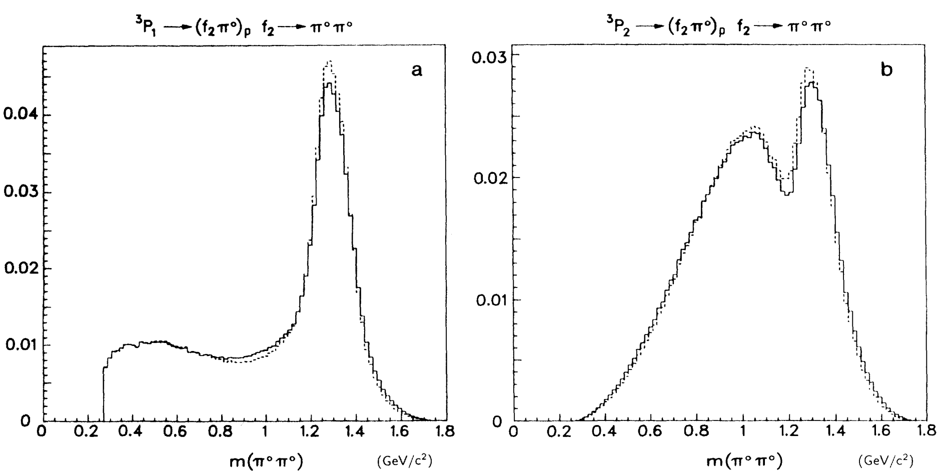

FIG. 3. $\pi^0\pi^0$ invariant mass for the reaction at rest $p\bar{p}(^3P_1) \to (f_2(1270)\pi^0)_p$, $f_2 \to \pi^0\pi^0$ (a) and $p\bar{p}(^3P_2) \to (f_2(1270)\pi^0)_p$, $f_2 \to \pi^0\pi^0$ (b). Solid and dashed lines are obtained with the covariant and Zemach noncovariant spin tensor formalisms, respectively.

tributions are given by (80) and (83), with a value of $z^2 \approx 0.152$ smaller than in the previous $\rho\pi$ case due to the higher f_2 mass. The $\pi^0\pi^0$ invariant masses are shown in Fig. 3. They are obtained with the weight (114), properly symmetrized as requested by the presence of three identical particles in the final state, with $m_0 = 1274$ and $\Gamma = 185$ MeV for the nominal mass and width of the $f_2(1270)$. The covariant tensor amplitudes are obtained by (63) and (65), where unnormalized tensors have been used. The Zemach noncovariant amplitudes are taken from [5,7,8].

Although the differences between the covariant and noncovariant ($z = 0$) distributions are not so large in these channels, they could be of some importance when high statistics data samples are analyzed with refined models.

VII. CONCLUSIONS

The results of this paper can be summarized into three main points.

Firstly, we have shown that the use of the covariant Cartesian tensors to describe the spin dynamics of a decay process leads to results different from those obtained with the so-called Zemach tensors, that are noncovariant. We have shown explicitly that the noncovariant approach is incorrect, because it does not consider the Lorentz boost from the produced resonance to the initial frame. This final boost cannot be neglected, because in a covariant theory the rotations do not commute with the translations.

Secondly, we have shown that, for all the cases considered, the decay amplitudes obtained with the covariant spin tensors are identical to those coming from the more usual helicity formalism when the helicity-coupling amplitude is made covariant with the method recently proposed by Chung [1]. This result is not surprising, because Chung's method relies on the use of covariant spin tensors and spinors in each helicity frame. However, the helicity amplitude is intrinsically noncovariant, whereas the tensor formalism is fully covariant, so that the found identity is not completely trivial. Because of these results, the use of the helicity frames appears as an unnecessary complication (at least when only massive particle of integer spin are involved in the decay), because the tensor formalism is fully covariant and in our opinion easier to use and to code.

Finally, we have shown that the covariant decay amplitudes give angular distributions in some cases drastically different from the noncovariant ones. This fact, although implicitly contained in the general paper by Chung [1], is explicitly shown here for the first time, since in [1] only the differences on the branching ratio determination are discussed, and no angular distribution is calculated.

This last point is probably the most interesting one, because it could have some important practical consequence. We have discussed this aspect in some detail in Sec. V for the low energy $p\bar{p}$ annihilation, but our conclusions apply to meson spectroscopy in general, in all the cases in which light resonances (around the ρ mass) are produced.

In a phenomenological analysis of real data, the decay amplitude is usually written in terms of many partial waves which contain the coherent contribution of several resonances. The resonance strengths and the relative phases between resonances are left as free parameters to be adjusted by fitting to the experimental distributions. Because of the large number of waves and free parameters, satisfying results (good χ^2) are obtained also with the use of noncovariant (Zemach or helicity) amplitudes. This is a very common situation, since, as already noted by Chung [1], *most, if not all partial waves analysis programs violate the covariance requirements.* However, one can hardly assign a reliable physical significance to best-fit parameters obtained with an *a priori* incorrect model.

The direct experimental test of covariance versus noncovariance is in principle possible, but in practice very difficult. One should search for channels where light and narrow resonances are produced incoherently in only one partial wave. An example is represented by the decay $p\bar{p} \to \phi\pi^0$ at rest, which comes by ${}^3S_1, {}^1P_1$ protonium states only. Starting from the ${}^1P_1$ state at rest, only the $l = 0$ relative wave in the final state contributes significantly. Since P wave protonium states can be selected by means of the x-ray tagging technique [19] or with the use of very low pressure hydrogen targets [16], one could test directly the covariant prediction (77) $W(1 \to 1 + 0) = 1 + z^2 \cos^2\theta$ [with $z \approx 0.7$ (see Fig. 1)] with the noncovariant one (the flat distribution). Unfortunately, the decay $p\bar{p} \to \phi\pi^0$ seems to be strongly depressed in P wave [20].

Nevertheless, since covariance must be a general requirement of any decay amplitude, we think that in meson spectroscopy the covariant spin formalisms should be employed more extensively.

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