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Wormholes in the Brans-Dicke theory of gravitation

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(Received 5 July 1994)

It is shown that the static spherically symmetric solutions to the Brans-Dicke theory of gravitation give rise either to a naked singularity if the post Newtonian parameter $\gamma < 1$ or to a wormhole if $\gamma > 1$.

PACS number(s): 04.20.Gz, 04.50.+h

In recent years there has occurred a renewed interest in wormhole physics, since the analysis of classical traversable wormholes performed by Morris and Thorne [1] and by Morris, Thorne, and Yurtsever [2]. Most of the efforts have been concentrated on the possibility of constructing time machines and on the requirement of matter violating the weak energy condition (WEC) [3]. The standard type of work treats spacetimes that are solutions to the Einstein field equations of general relativity, alternative models of gravity having been explored with the aim of understanding the role of WEC violation in connection with wormholes [4,5]. To our knowledge, researches to investigate whether the Brans-Dicke scalar-tensor theory of gravitation [6] can describe spacetimes with a wormhole geometry have been done only in the dynamic but not in the static case [7]. It is, however, important to find phenomena for which the Brans-Dicke and the Einstein theory make qualitatively different predictions, so it seems worth raising the question of the existence of static Brans-Dicke wormholes, even though the size of the dimensionless coupling constant ω is consistent with $|\omega| \gtrsim 500$. The very large value of the parameter ω and the uncertainty on experimental data actually make the Brans-Dicke and Einstein theories both agree with the observational tests related to weak local gravitational fields, gravitational collapsed and cosmological situations. In this paper we shall discuss the static spherically symmetric solutions to the Brans-Dicke equations about a point source of mass M, after expressing them as a function of the post-Newtonian parameter γ given by $\gamma = (1+\omega)/(2+\omega)$. It will be shown that if $\gamma > 1$ these solutions can describe a two-way (no horizon) traversable wormhole, which would reduce to a one-way traversable Schwarzschild wormhole (black hole) in the limit $\gamma \to 1$. Of course, for the Brans-Dicke wormholes to have a physical meaning they should be perturbatively stable. Making this requirement would involve an analysis which is, however, beyond the scope of the present work; it seems nevertheless appropriate to recall that Schwarzschild wormholes are unstable under perturbations [8] and, since Brans-Dicke gravity differs only slightly from general relativity, one expects those perturbations to be important here as well.

The Brans-Dicke field equations are

$$G_{\alpha\beta} = \frac{8\pi}{\phi} T_{\alpha\beta} + \frac{\omega}{\phi^2}(\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2} g_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma}) + \frac{1}{\phi}(\phi_{;\alpha;\beta} - g_{\alpha\beta}\Box\phi) , \tag{1}$$

$$\Box\phi = \frac{8\pi}{3+2\omega} T . \tag{2}$$

The most general static spherically symmetric line element can be written as

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\mu(r)}dr^2 + e^{\lambda(r)} r^2 d\Omega^2 , \tag{3}$$

where $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$. In the gauge we select,

$$\lambda - \mu = \ln\left[1 - \frac{2\eta}{r}\right] , \tag{4}$$

the solutions are

$$e^{\nu(r)} = \left[1 - \frac{2\eta}{r}\right]^A , \tag{5a}$$

$$e^{\mu(r)} = \left[1 - \frac{2\eta}{r}\right]^B , \tag{5b}$$

$$e^{\lambda(r)} = \left[1 - \frac{2\eta}{r}\right]^{1+B} , \tag{5c}$$

$$\phi(r) = \phi_0\left[1 - \frac{2\eta}{r}\right]^{-(A+B)/2} . \tag{5d}$$

Here η, A, B are constants, $\phi_0 = (1/G_0)[2/(1+\gamma)]$ as defined in [6], and, moreover, the following constraint must hold:

$$1 - \gamma = \frac{(A+B)^2}{2(1+AB)} . \tag{6}$$

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Solutions in the form given by Eqs. (5) have already been found in Ref. [9]. For the specification of the constants it is convenient to write the line element in isotropic form. By means of the transformation

$$r = \bar{r}\left[1 + \frac{\eta}{2\bar{r}}\right]^2 , \tag{7}$$

we get

$$ds^2 = -\left[\frac{1-\eta/2\bar{r}}{1+\eta/2\bar{r}}\right]^{2A} dt^2 + \left[\frac{1-\eta/2\bar{r}}{1+\eta/2\bar{r}}\right]^{2B} [1+\eta/2\bar{r}]^4 (d\bar{r}^2 + \bar{r}^2 d\Omega^2) . \tag{8}$$

It is now immediately possible to relate our constants to the parameters β and γ which appear in the post-Newtonian metric

$$ds^2 \approx -\left[1 - 2\left(\frac{M}{\bar{r}}\right) + 2\beta\left(\frac{M}{\bar{r}}\right)^2\right]dt^2 + \left[1 + 2\gamma\left(\frac{M}{\bar{r}}\right)\right]\left(d\bar{r}^2 + \bar{r}^2 d\Omega^2\right) , \tag{9}$$

thus obtaining

$$A\eta = M, \quad \beta = 1, \quad \gamma = -\frac{B}{A} . \tag{10}$$

Finally, using Eq. (6) we have

$$A = \sqrt{\frac{2}{1+\gamma}} , \tag{11}$$

$$B = -\gamma\sqrt{\frac{2}{1+\gamma}} , \tag{12}$$

$$\eta = M\sqrt{\frac{1+\gamma}{2}} . \tag{13}$$

Leaving apart the value $\gamma = 1$ which corresponds to infinite $|\omega|$, it may be $-1 < \gamma < 1$ if $\omega > -\frac{3}{2}$ or alternatively it may be $\gamma > 1$ if $\omega < -2$. Both possibilities have to be considered, because the sign of $(\gamma - 1)$ is not yet experimentally fixed [10]. Moreover, there is no theoretical reason to restrict $(\gamma - 1)$ to, say, negative values. In the solutions we are discussing, the sign of $(\gamma - 1)$ implies by Eq. (5d) that the effective value of G, which replaces the gravitational constant G_0 and equals ϕ^{-1}, increases $(\gamma - 1 > 0)$ or decreases $(\gamma - 1 < 0)$ on approaching the source of mass M, and neither behavior seems *a priori* rejectable.

Turning to the question whether the Brans-Dicke spacetime can support a wormhole geometry, it is suitable to represent the metric (3) in the form

$$ds^2 = -e^{2\Phi(R)}dt^2 + \frac{dR^2}{1 - b(R)/R} + R^2 d\Omega^2 , \tag{14}$$

where $\Phi(R)$ and $b(R)$ are, respectively, known as the redshift function and the wormhole shape function, and are to be constrained by the properties enumerated in Ref. [1]. In our gauge the standard radial coordinate R is

$$R(r) = r\left[1 - \frac{2\eta}{r}\right]^{[1-\gamma\sqrt{2/(1+\gamma)}]/2} , \tag{15}$$

where η is given by Eq. (13).

It is apparent that if $\gamma < 1$ we must have $r \geq 2\eta$ and correspondingly $R \geq 0$ results. If instead $\gamma > 1$, we must have $r \geq r_0$, where the minimum allowed value r_0 for r is given by

$$r_0 = \eta\left[1 + \gamma\sqrt{\frac{2}{1+\gamma}}\,\right] \tag{16}$$

and correspondingly $R \geq R_0$ results, the value of R_0 being obtained by using Eqs. (15) and (16). It is immediately possible to verify that $r_0 > 2\eta$ and therefore $R_0 > 0$.

We shall denote by $r(R)$ the inverse of $R(r)$. It is then straightforward to obtain

$$2\Phi(R) = \sqrt{\frac{2}{1+\gamma}}\ln\left[1 - \frac{2\eta}{r(R)}\right] \tag{17}$$

and

$$\frac{b(R)}{R} = 1 - \frac{\left\{1 - [\eta/r(R)][1 + \gamma\sqrt{2/(1+\gamma)}]\right\}^2}{1 - 2\eta/r(R)} . \tag{18}$$

In the case $\gamma < 1$ one can identify a singularity at $R = 0$ where the invariant of curvature $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is divergent; the point $R = 0$ clearly constitutes a naked singularity. In the limit $\gamma \to 1$, when the Brans-Dicke equations reduce to the Einstein equations, the event horizon is placed at $R = 2M$ and only in this limit could one have a black hole solution, as pointed out by Hawking [11]. The occurrence of a naked singularity in the spherical vacuum solutions of the Brans-Dicke theory was already shown in the literature [12]; on the other hand, the same type of singularity is also found in the Einstein theory of gravitation with a massless scalar field (e.g., [13]) or in a charged dilaton gravity [14].

The remaining case to be discussed, i.e., $\gamma > 1$, will bring into evidence the existence of static wormhole solutions to the Brans-Dicke equations. To this end let us first consider the redshift function $\Phi(R)$. We notice that since now $R \geq R_0 > 0$ there is no horizon, and $\Phi(R)$ is finite everywhere. As to the shape function $b(R)$, it satisfies the condition $b(R)/R \leq 1$ and $b(R)/R \to 0$ as $R \to \infty$. Moreover, it results that $b(R)/R = 1$ when R reaches its minimum value R_0. The functions $\Phi(R)$ and $b(R)$ thus meet all the requirements needed to describe what can be called a Brans-Dicke wormhole.

The line element for an equatorial slice through the wormhole at a fixed instant of time is

$$ds^2 = \left[1 + \left(\frac{dz}{dR}\right)^2\right]dR^2 + R^2 d\varphi^2 , \tag{19}$$

where the embedding function $z(R)$ is a solution of

$$\frac{dz}{dR} = \pm\left[\frac{R}{b(R)} - 1\right]^{-1/2} . \tag{20}$$

At the value $R = R_0$ (the wormhole throat) Eq. (20) is divergent, which means that the embedded surface is vertical there. Spatial geometry is better studied by introducing the proper radial coordinate

$$l(R) = \pm \int_{R_0}^{R} \frac{dR}{[1 - b(R)/R]^{1/2}} , \tag{21}$$

which can now be written

$$\begin{aligned} l(R) = \pm \frac{2\eta}{(1-B/2)} \Bigg[& \left(1 - \frac{2\eta}{r(R)}\right)^{1-B/2} \\ & \times F\left(2, 1 - \frac{B}{2}; 2 - \frac{B}{2}; 1 - \frac{2\eta}{r(R)}\right) \\ & - \left(1 - \frac{2\eta}{r_0}\right)^{1-B/2} F\left(2, 1 - \frac{B}{2}; 2 - \frac{B}{2}; 1 - \frac{2\eta}{r_0}\right) \Bigg] , \end{aligned} \tag{22}$$

where F is the hypergeometric function and B is given by Eq. (12).

Let us finally examine whether the wormhole discussed above requires a WEC violating energy-momentum tensor to support it. Equation (1) can suitably be rewritten as

$$G_{\alpha\beta} = \frac{8\pi}{\phi} [T_{\alpha\beta} + (T_\phi)_{\alpha\beta}] , \tag{23}$$

where $(T_\phi)_{\alpha\beta}$ is composed of two terms: the former comes from the energy-momentum stress of the scalar field and the latter results from the presence of second derivatives in the action for the Brans-Dicke theory. From

$$G_{00} = \frac{(1-\gamma)(1+2\gamma)}{(1+\gamma)} \frac{\eta^2}{r^4(R)} \left[1 - \frac{2\eta}{r(R)}\right]^{2[\sqrt{2/(1+\gamma)}-1]} \tag{24}$$

it follows that $G_{00} < 0$, because we are considering the case $\gamma > 1$. This fact, since the coupling $8\pi\phi^{-1}$ is nonnegative, implies that the WEC is violated. The connection between wormhole existence and WEC violation has been discussed by several authors [1,2,15–17].

The reason why the absence of horizons in Brans-Dicke wormholes implies WEC violation may have the following physical explanation [1]. A traversable wormhole requires that light rays which enter it at one mouth and emerge from the other have cross-sectional areas initially decreasing and then increasing. This conversion can be produced by gravitational repulsion which acts on the light rays passing near the throat provided in that region resides a negative energy density, as is effectively guaranteed by the Brans-Dicke scalar field ϕ in the case $\gamma > 1$.

In the Einstein theory of general relativity the properties required for the functions $\Phi(R)$ and $b(R)$ cause such constraints on the matter stress tensor as to make necessary the occurrence of exotic matter, especially in the wormhole throat, where the absence of a horizon is required. In the Brans-Dicke theory of gravitation the role of exotic matter is instead played, if $\gamma > 1$ (or $\omega < -2$), by the scalar field ϕ and therefore, via Mach's principle, by the mass distribution in the Universe.

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