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Effective quark-antiquark potential for the constituent quark model

M. Baker
University of Washington, Seattle, Washington 98105

James S. Ball
University of Utah, Salt Lake City, Utah 84112

F. Zachariasen
California Institute of Technology, Pasadena, California 91125
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We use dual QCD to derive an effective potential to order (quark mass)$^{-2}$ for a constituent quark and antiquark. This is done by expanding the dual QCD Lagrangian to second order in the $q\bar{q}$ spins and velocities around the static central potential, in which the quarks are both spinless and stationary. The field equations are then used to eliminate the dual gluon fields and the Higgs fields of dual QCD in favor of quark variables for an arbitrary but slowly moving $q\bar{q}$ pair with a Dirac string of arbitrary shape connecting them. The result is a Lagrangian, and therefore, a potential, which depends only on the $q\bar{q}$ positions, velocities, and spins. Dual QCD contains only three parameters, which can be determined from the vacuum energy density, the string tension, and the strength of the Coulomb singularity of the central potential. The only free parameters in the spin- and velocity-dependent part of the effective potential are, therefore, the masses of the c and b quarks. When inserted into a Schrödinger equation these potentials provide a complete effective constituent quark theory which can be used to calculate $q\bar{q}$ energy levels in terms of the masses and the masses can thereby be fixed (agreement with experiment is excellent). The various potentials, spin-spin, spin-orbit, and spinless velocity dependent, can also in principle be compared to lattice calculations of the same quantities. For the spin-orbit case, for example, the agreement is good, although lattice results are not yet precise enough for a real comparison to be made. For the potentials proportional to the velocity squared lattice results do not yet exist. We also attempt to extend the use of these potentials to heavy-light quark-antiquark systems through use of the Salpeter equation and the Dirac equation. The results of this effort are described in two Appendixes.

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I. INTRODUCTION AND PRELIMINARIES

This paper is devoted to the calculation of the complete dynamical potential between an arbitrarily moving heavy-quark–antiquark pair. The potential is given as an explicit function of the quark coordinates and velocities, includes all spin effects, and is accurate to order (mass)$^{-2}$, or quark velocity squared. (This is as far as one can go with a calculation that does not allow for the explicit radiation of gluons, and is therefore the extent to which the concept of a potential has a meaning.) If the potential we obtain is used in a Schrödinger equation (or, if desired, in any other wave equation such as, for example, a Salpeter equation to take into account the relativistic kinetic energy of the quarks) then we have a complete dynamical theory of constituent quark-antiquark pairs.

Our calculation of the potential is based on the dual QCD Lagrangian [1,2]. This Lagrangian can be viewed as an effective theory of long-distance QCD, which incorporates in a very simple way most of what we believe to be true about real QCD. The dual field theory is relativistic, unitary, renormalizable, obeys non-Abelian dual gauge invariance, and automatically gives rise to confinement of color via a non-Abelian dual Meissner effect. The theory is described in terms of a dual color octet vector potential C_μ^a (the quanta of which are dual gluons) as well as a set of three octet scalar "Higgs fields" $B_i^a (i = 1, 2, 3; a = 1, \ldots, 8)$. The Higgs mechanism, through the scalar fields B, gives mass to all eight dual gluons, as well as to the Higgs fields themselves, thus confining color flux into Z_N flux tubes and providing a dual superconductivity explanation of confinement. (At sufficiently high temperature, as in ordinary superconductivity, the spontaneous symmetry breaking disappears; so does the dual gluon mass and with it confinement [3].)

The introduction of quarks into dual QCD requires some delicacy, analogous to the introduction of magnetic monopoles in ordinary electrodynamics. Each quark must have attached to it a Dirac string, and these strings can cause technical problems in calculations. In Appendix A we have described how these difficulties can be dealt with in electrodynamics, and how the electrodynamic interaction of a pair of oppositely charged particles of arbitrary mass can be described using the dual vector potential, with an arbitrarily moving Dirac string connecting the arbitrarily moving changes [4]. We can carry over these same techniques to dual QCD in order to describe a quark-antiquark pair, and this is in fact

what makes it possible to derive the dynamical quark-antiquark potential [to order $(\text{mass})^{-2}$].

The paper is organized as follows. We first write down the dual QCD Lagrangian for a quark and an antiquark connected by a Dirac string of arbitrary shape. We then solve the dual QCD field equations for the various fields and for stationary quarks; this gives us our zero order solution and the static quark-antiquark central potential [1]. We then expand the dual QCD Lagrangian in inverse powers of quark mass to order $(\text{mass})^{-2}$, and calculate the corrected dual QCD fields in terms of the quark positions, velocities, and spins. These results are then used to eliminate the fields from the problem, and to calculate the energy of the system in terms only of the quark variables. This gives rise to a quark-antiquark potential containing (in addition to the static central potential obtained in zero order) a spin-spin potential V_{spin}, a spin-orbit potential $V_{\text{spin orbit}}$, and a spin-independent potential V_{v^2} of order quark velocity squared. All of these potentials are explicitly given in terms of the quark variables. The parameters they depend on are only the constituent quark masses m_1 and m_2; the other three parameters in dual QCD [1] are fixed from the vacuum energy density and the behavior of the central potential at zero and infinite distance (this last parameter is of course just the string tension).

The calculation we describe here has been partially carried out before, in that the various spin- and velocity-dependent potentials have been evaluated assuming a particular motion of the quark-antiquark pair, namely, that they travel in circles [5–7]. Needless to say, our results here, which are valid for general quark motion, reduce to the old results when circular motion is imposed.

The potentials are used in the Schrödinger equation to predict the masses of heavy-quark–antiquark systems. The results of this are quite good. When extended to light-heavy systems, such as $s\bar{c}$ or $s\bar{b}$, however, the results are far worse. Clearly the relativistic effects associated with light quarks are very important. To explore this further we have also used our potentials in the Salpeter equation and in the Dirac equation. The results are mixed, and (particularly with the Salpeter equation) some ambiguities exist. (These calculations are described in Appendixes B and C.) We need a correct relativistic generalization of our potentials before we can deal with light quarks.

In dual QCD the dynamical field is C_μ^a, the vector potential dual to the ordinary vector potential A_μ^a. In the absence of quarks, the Lagrangian for C_μ^a is given by [1]

$$\mathcal{L} = 2\,\text{tr}[\tfrac{1}{2}(\mathbf{H}^2 - \mathbf{D}^2) + \tfrac{1}{2}(\mathcal{D}_\mu \mathbf{B})^2] - W(B)\ , \qquad (1.1)$$

where the trace is over color indices, and where

$$\mathcal{D}_\mu \mathbf{B} = \partial_\mu \mathbf{B} - ig[C_\mu, \mathbf{B}]\ . \qquad (1.2)$$

$\mathbf{D}$ and $\mathbf{H}$ are the non-Abelian generalizations of the color electric displacement and magnetic field:

$$\mathbf{D} = -\boldsymbol{\nabla} \times \mathbf{C} - \tfrac{1}{2} ig[\mathbf{C}, \times \mathbf{C}] \qquad (1.3)$$

and

$$\mathbf{H} = -\boldsymbol{\nabla} C_0 - \partial_0 \mathbf{C} - ig[\mathbf{C}, C_0]\ , \qquad (1.4)$$

where $C_\mu = \sum C_\mu^a \frac{1}{2}\lambda_a$ and $\frac{1}{2}\lambda_a$ are the generators of SU(3). In the infrared limit, the dual coupling constant $g = 2\pi/e$ where e is the ordinary Yang-Mills coupling constant. The quantity $\mathbf{B}$ represents the three scalar octets necessary to give mass to all color components of C_μ, $\mathbf{B} = (B_1, B_2, B_3)$. The function $W(\mathbf{B})$ is the counterterm needed for renormalization and plays the role of a Higgs potential; its explicit form is given in Ref. [1]. Since they couple to the dual potentials the scalar fields $\mathbf{B}$ carry color magnetic charge.

Our first use of Eq. (1.1) was to calculate the field configurations associated with quantized color electric flux tubes [8]. These, with one unit of quantized Z_3 flux, can be viewed as the fields between a static quark and antiquark at infinite separation. (Note that in the gauge chosen in Ref. [8], the Dirac strings (see below) attached to the quarks at $\pm\infty$ are taken to $\pm\infty$, respectively, and do not join the two quarks.)

For the calculation of flux tubes we make the simplest color ansatz that produces a closed set of nontrivial field equations [8]. The fields $\mathbf{D}$, $\mathbf{H}$, and $\mathbf{C}$ are all proportional to the color matrix $Y = \lambda_8/\sqrt{3}$. The field C_0 is zero. Two of the three B fields can be chosen equal: $B_1 = B_2 \equiv B$. B_1, B_2, and B_3 are chosen to be in the color directions $(\lambda_7, -\lambda_5, \lambda_2)$, respectively. For this choice the function $W(B)$ becomes

$$\begin{aligned} W(B) = &-\tfrac{16}{9}\lambda(-\tilde{F}_0^2)(B_1^2 + B_2^2 + B_3^2) \\ &+\tfrac{14}{3}\lambda(B_1^2 + B_2^2 + B_3^2)^2 \\ &+\tfrac{22}{3}\lambda(B_1^4 + B_2^4 + B_3^4)\ . \end{aligned}$$

There are two parameters, which we call λ and $-\tilde{F}_0^2$, [1] in the Higgs potential, so, including g, we have three parameters overall. These three parameters can be roughly determined from the solutions for the dual QCD flux tube and the zero and infinite distance limits of the central potential [8]. First, from the form of $W(\mathbf{B})$, we can compute the vacuum energy density to be $\epsilon_{\text{vac}} = -\lambda(\tilde{F}_0^2)^2/9$. Through the use of the trace anomaly ϵ_{vac} can be related to the magnetic condensate: $\epsilon_{\text{vac}} = -\frac{11}{32}G_2$. Next, we can calculate the string tension σ in the flux tube which is also the coefficient of the linear (in separation) term in the central potential. Finally we can obtain g from the value of $\alpha_s = \pi/g^2$ given by the Coulomb singularity in the central potential. Putting all of this together, and using the values $G_2 = (330\ \text{MeV})^4$, $\sigma = (427\ \text{MeV})^2$, and $\alpha_s = 0.39$ we obtain $\lambda = 1.61$, $g^2/\lambda = 5$, and $\sqrt{-\tilde{F}_0^2} = 420$ MeV. With these parameters the flux tube comes out to have a radius of about half a fermi. These rough values can serve as a guide to our more precise fits to the energy levels in the $c\bar{c}$ and $b\bar{b}$ systems, once we have calculated all the terms in the order $(\text{mass})^{-2}$ heavy quark potential.

We next wish to extend this to a system consisting of a classical heavy quark of charge e and an antiquark of charge $-e$ at finite separation having masses m_1 and m_2, and spins $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. The quark charge density must also lie in the Y color direction in order to absorb the flux of

D. Because quarks in our dual theory are like magnetic monopoles in ordinary electrodynamics, we must modify (1.3) so that Gauss' law is satisfied. This is achieved by adding a string field $\mathbf{D}_s$ to (1.3) [1].

The string field is chosen to connect the two quarks (a different gauge choice from that made in the flux tube case). It satisfies

$$\begin{aligned}\nabla \cdot \mathbf{D}_s &= \rho(\mathbf{x},t) \\ &= e[\delta^3(\mathbf{x}-\mathbf{x}_1(t)) - \delta^3(\mathbf{x}-\mathbf{x}_2(t))]Y \ , \end{aligned} \tag{1.5}$$

where $\mathbf{x}_{1,2}(t)$ are the positions of the two quarks at time t. Thus $\mathbf{D}_s$ must lie in the Y color direction as well. Explicitly, we have [4]

$$\mathbf{D}_s(\mathbf{x},t) = -e\int_{\tau_2}^{\tau_1} d\tau \frac{d\mathbf{y}(\tau,t)}{d\tau}\delta^3(\mathbf{x}-\mathbf{y}(\tau,t))Y \ , \tag{1.6}$$

where τ parametrizes position along the string $\mathbf{y}(\tau,t)$ and $\mathbf{y}(\tau_{1,2},t) = \mathbf{x}_{1,2}(t)$. The moving string induces a magnetization $\mathbf{H}_s$ as well and we can write [4]

$$\mathbf{H}_s = e\int_{\tau_2}^{\tau_1} d\tau \frac{d\mathbf{y}(\tau,t)}{dt} \times \frac{d\mathbf{y}(\tau,t)}{d\tau}\delta^3(\mathbf{x}-\mathbf{y}(\tau,t)) \ . \tag{1.7}$$

These two fields satisfy

$$\nabla \times \mathbf{H}_s - \partial_0 \mathbf{D}_s = \mathbf{j} \ , \tag{1.8}$$

where

$$\mathbf{j} = e[\mathbf{v}_1\delta^3(\mathbf{x}-\mathbf{x}_1(t)) - \mathbf{v}_2\delta^3(\mathbf{x}-\mathbf{x}_2(t))] \tag{1.9}$$

is the quark current. [Here $\mathbf{v}_{1,2} = (d/dt)\mathbf{x}_{1,2}(t)$.] There is also, due to the quark spin, a magnetization

$$\mathbf{M} = \left[\frac{e\boldsymbol{\sigma}_1}{2m_1}\delta^3(\mathbf{x}-\mathbf{x}_1) - \frac{e\boldsymbol{\sigma}_2}{2m_2}\delta^3(\mathbf{x}-\mathbf{x}_2)\right]Y \ , \tag{1.10a}$$

and due to the quark motion an induced polarization $\mathbf{P}$:

$$\mathbf{P} = \left[\mathbf{v}_1 \times \frac{e\boldsymbol{\sigma}_1}{2m_1}\delta^3(\mathbf{x}-\mathbf{x}_1) - \mathbf{v}_2 \times \frac{e\boldsymbol{\sigma}_2}{2m_2}\delta^3(\mathbf{x}-\mathbf{x}_2)\right]Y \ . \tag{1.10b}$$

Both $\mathbf{P}$ and $\mathbf{M}$ are in the Y color direction.

The electric displacement $\mathbf{D}$ and the magnetic field $\mathbf{H}$ satisfy Maxwell's equations (since all of these fields are in the same color direction, this sector of the problem, that is, the sector excluding $\mathbf{B}$, is Abelian):

$$\nabla \cdot \mathbf{D} = \rho - \nabla \cdot \mathbf{P} \ , \tag{1.11}$$

$$\nabla \times \mathbf{H} - \partial_0 \mathbf{D} = \mathbf{j} + \nabla \times \mathbf{M} \ . \tag{1.12}$$

The solutions to these equations are expressed in terms of the dual potentials through

$$\mathbf{D} = -\nabla \times \mathbf{C} + \mathbf{D}_s - \mathbf{P} \ , \tag{1.13a}$$

$$\mathbf{H} = -\nabla C_0 - \partial_0 \mathbf{C} + \mathbf{H}_s + \mathbf{M} \ . \tag{1.13b}$$

These solutions can be inserted into Eq. (1.1), and, as we shall see, we can retain the flux tube color structure of $\mathbf{B}$ and the fact that $B_1 = B_2 = B$.

The commutation relations $[Y,\lambda_5] = -i\lambda_4$, $[Y,\lambda_7] = -i\lambda_6$, and $[Y,\lambda_2] = 0$ yields

$$\mathcal{D}_\mu \mathbf{B} = ((\lambda_7\partial_\mu - \lambda_6 g C_\mu)B, (-\lambda_5\partial_\mu + \lambda_4 g C_\mu)B, \lambda_2\partial_\mu B_3) \ . \tag{1.14}$$

The color traces are $2\,\mathrm{Tr}Y^2 = \frac{4}{3}$ and $2\,\mathrm{Tr}\lambda_a\lambda_b = 4\delta_{ab}$. These equations give the final form of $\mathcal{L}$ in terms of the coefficients C_μ, $\mathbf{H}$, and $\mathbf{D}$ of the color matrix Y and the functions B and B_3.

The resulting Lagrangian is, after evaluation of the color trace in Eq. (1.1),

$$\begin{aligned}\mathcal{L} = {}& \tfrac{2}{3}\mathbf{H}^2 - \tfrac{2}{3}\mathbf{D}^2 - 4g^2B^2(\mathbf{C}^2 - C_0^2) + 4B\nabla^2 B \\ & -4B\ddot{B} + 2B_3\nabla^2 B_3 - 2B_3\ddot{B}_3 - W(\mathbf{B}) \ , \end{aligned} \tag{1.15}$$

where the double overdot denotes the second derivative with respect to time. The nonquadratic terms in the fields reflect the underlying non-Abelian nature of the theory entering via the $(\mathcal{D}_\mu\mathbf{B})^2$ term in $\mathcal{L}$.

Note that it might appear from the commutation relations and (1.14) that we need to introduce components of $\mathbf{B}$ along the color directions λ_4 and λ_6. However, it is consistent to set these amplitudes equal to zero provided we choose $\nabla \cdot \mathbf{C} = 0$, as we do.

II. DERIVATIVE OF EQUATIONS FOR THE POTENTIAL

Given the expression for the Lagrangian density, Eq. (1.15), we may substitute the solutions (1.13) and (1.14) into it, to obtain the Lagrangian L. The first two terms of L are

$$\begin{aligned}\frac{2}{3}\int d^3\mathbf{x}(\mathbf{H}^2 - \mathbf{D}^2) = \frac{2}{3}\int d^3\mathbf{x}\{&(-\partial_0\mathbf{C} - \nabla C_0 + \mathbf{H}_s)^2 + 2\mathbf{M}\cdot(-\partial_0\mathbf{C} - \nabla C_0 + \mathbf{H}_s) + \mathbf{M}^2 - \mathbf{P}^2 \\ & -(-\nabla\times\mathbf{C} + \mathbf{D}_s)^2 + 2\mathbf{P}\cdot(-\nabla\times\mathbf{C} + \mathbf{D}_s)\} \ . \end{aligned} \tag{2.1}$$

Note that the cross terms

$$\int d^3\mathbf{x}(\mathbf{P}\cdot\mathbf{D}_s+\mathbf{M}\cdot\mathbf{H}_s)=-e\int_{\tau_2}^{\tau_1}d\tau\frac{d\mathbf{y}(\tau,t)}{d\tau}\cdot\left(\mathbf{P}(\mathbf{y},t)-\frac{d\mathbf{y}(\tau,t)}{dt}\times\mathbf{M}(\mathbf{y},t)\right)=0\ , \tag{2.2}$$

because of the fact that $\mathbf{P}(\mathbf{y})=d\mathbf{y}/dt\times\mathbf{M}(\mathbf{y})$. Furthermore, to second order in $(\text{mass})^{-1}$ we can omit the term $\mathbf{P}^2$. Thus altogether

$$\begin{aligned}L=\tfrac{2}{3}\int d^3\mathbf{x}\{&-(-\boldsymbol{\nabla}\times\mathbf{C}+\mathbf{D}_s)^2+2C_0\boldsymbol{\nabla}\cdot\mathbf{M}+(-\partial_0\mathbf{C}-\boldsymbol{\nabla}C_0+\mathbf{H}_s)^2+2\mathbf{P}\cdot(-\boldsymbol{\nabla}\times\mathbf{C})-2\mathbf{M}\cdot\partial_0\mathbf{C}+\mathbf{M}^2\\&+6g^2B^2(C_0^2-\mathbf{C}^2)+6(\partial_0B)^2-6(\boldsymbol{\nabla}B)^2+3(\partial_0B_3)^2-3(\boldsymbol{\nabla}B_3)^2-\tfrac{3}{2}W(B)\}\ .\end{aligned} \tag{2.3}$$

The classical field equations following from L are

$$\nabla^2C_0-\boldsymbol{\nabla}\cdot\mathbf{H}_s+\boldsymbol{\nabla}\cdot\dot{\mathbf{C}}-6g^2B^2C_0-\boldsymbol{\nabla}\cdot\mathbf{M}=0\ , \tag{2.4a}$$

$$\begin{aligned}-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{C})-\ddot{\mathbf{C}}+\boldsymbol{\nabla}\times\mathbf{D}_s&\\+\partial_0(-\boldsymbol{\nabla}C_0+\mathbf{H}_s)-6g^2B^2\mathbf{C}-\partial_0\mathbf{M}=0\ ,&\end{aligned} \tag{2.4b}$$

$$\nabla^2B-\ddot{B}-g^2(\mathbf{C}^2-C_0^2)B=\frac{1}{8}\frac{\partial W}{\partial B}\ , \tag{2.5a}$$

and

$$\nabla^2B_3-\ddot{B}_3=\frac{1}{4}\frac{\partial W}{\partial B_3}\ . \tag{2.5b}$$

The value of L for $\mathbf{M}=\mathbf{v}_1=\mathbf{v}_2=0$ is the negative of the central potential $V_0(R)$. We look for static solutions (i.e., $\dot{\mathbf{C}}=\dot{B}=\dot{B}_3=0$) with $C_0=0$ as well [1,9]. These solutions, which we denote $\bar{\mathbf{C}}$, $\bar{B}$, and $\bar{B}_3$, depend, of course, parametrically on the separation $R=|\mathbf{x}_1-\mathbf{x}_2|$ of the quark sources. Thus the energy $V_0(R)$ is simply

$$V_0(R)=-\int d^3\mathbf{x}\,\mathcal{L}_0(\bar{\mathbf{C}},\bar{B},\bar{B}_3)\ , \tag{2.6}$$

where

$$\begin{aligned}\mathcal{L}_0=\tfrac{2}{3}(\mathbf{D}_s-\boldsymbol{\nabla}\times\bar{\mathbf{C}})^2-4g^2\bar{B}^2\bar{\mathbf{C}}^2&\\+4\bar{B}\nabla^2\bar{B}+2\bar{B}_3\nabla^2\bar{B}_3-W(\bar{\mathbf{B}})\ ,&\end{aligned} \tag{2.7}$$

is the static spinless Lagrangian density obtained from (2.3) by setting $\mathbf{v}$, $\mathbf{M}$, C_0, and the time derivatives to zero. The field equations obeyed by $\bar{\mathbf{C}}$, $\bar{B}$, and $\bar{B}_3$ are immediately derived from (2.7), or equivalently from (2.4) and (2.5). Thus

$$-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\bar{\mathbf{C}})+\boldsymbol{\nabla}\times\mathbf{D}_s-6g^2\bar{B}^2\bar{\mathbf{C}}=0\ , \tag{2.8a}$$

$$\nabla^2\bar{B}-g^2\bar{\mathbf{C}}^2\bar{B}-\frac{1}{8}\frac{\partial W}{\partial\bar{B}}\ , \tag{2.8b}$$

$$\nabla^2\bar{B}_3=\frac{1}{4}\frac{\partial W}{\partial\bar{B}_3}\ . \tag{2.8c}$$

To solve these it is convenient to define [9]

$$\bar{\mathbf{C}}=\mathbf{c}+\mathbf{C}_D\ , \tag{2.9}$$

where $\mathbf{C}_D$ is the Dirac monopole potential of the two-quarks satisfying [4]

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{C}_D)+\boldsymbol{\nabla}\times\mathbf{D}_s=0\ . \tag{2.10a}$$

The solution to Eq. (2.10a) is

$$C_D=-\frac{e}{4\pi}\int_{\mathbf{x}_2}^{\mathbf{x}_1}d\mathbf{y}\times\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{y}|}\ . \tag{2.10b}$$

In cylindrical coordinates and for a straight line string joining the two quarks located on the z axis at $z=R_1$ and $z=-R_2$,

$$\mathbf{D}_s=e\{\theta(z-R_1)-\theta(z+R_2)\}\delta(x)\delta(y)\hat{e}_z \tag{2.10c}$$

and

$$\mathbf{C}_D\equiv\frac{e}{4\pi\rho}\left\{\frac{z-R_1}{\sqrt{\rho^2+(z-R_1)^2}}-\frac{z+R_2}{\sqrt{\rho^2+(z+R_2)^2}}\right\}\hat{e}_\varphi\ . \tag{2.10d}$$

The equation satisfied by $\mathbf{c}$ is easily seen to be

$$\nabla^2\mathbf{c}-6g^2\bar{B}^2(\mathbf{c}+\mathbf{C}_D)=0\ , \tag{2.11}$$

where as noted earlier we have made the gauge choice $\boldsymbol{\nabla}\cdot\mathbf{C}=\boldsymbol{\nabla}\cdot\mathbf{c}=0$. Equation (2.11) has the virtue, important for numerical solution, of not containing the singular quantity $\mathbf{D}_s$. Using the decomposition (2.9) we can rewrite Eq. (2.6) in the simpler form

$$\begin{aligned}V_0(R)=-\int d^3\mathbf{x}\{&\tfrac{2}{3}(\mathbf{D}_c-\boldsymbol{\nabla}\times\mathbf{c})^2\\&-4g^2\bar{B}^2(\mathbf{c}+\mathbf{C}_D)^2+4\bar{B}\nabla^2\bar{B}\\&+2\bar{B}_3\nabla^2\bar{B}_3-W(\bar{\mathbf{B}})\}\ ,\end{aligned} \tag{2.12}$$

where $\mathbf{D}_c$ is the Coulomb field of the two quarks:

$$\begin{aligned}\mathbf{D}_c&=\mathbf{D}_{c_1}+\mathbf{D}_{c_2}\\&=\frac{e}{4\pi}\frac{(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^3}-\frac{e}{4\pi}\frac{(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_2|^3}\ .\end{aligned} \tag{2.13}$$

The static solutions have the following features [9]. For $R\to0$, $\bar{\mathbf{C}}\to\mathbf{C}_D$ and $\mathbf{D}=\mathbf{D}_s-\boldsymbol{\nabla}\times\mathbf{C}$ becomes a pure dipole field. As R increases, $\mathbf{D}$ evolves from a pure dipole to a squashed (in the axial direction) dipole and then, as R increases still further, to a flux tube for very large R. This flux tube coincides with our original flux tube [8], except that this solution is in the gauge where the Dirac string joins the two quarks while our original solution was in the gauge where the strings went from the charges to $\pm\infty$.

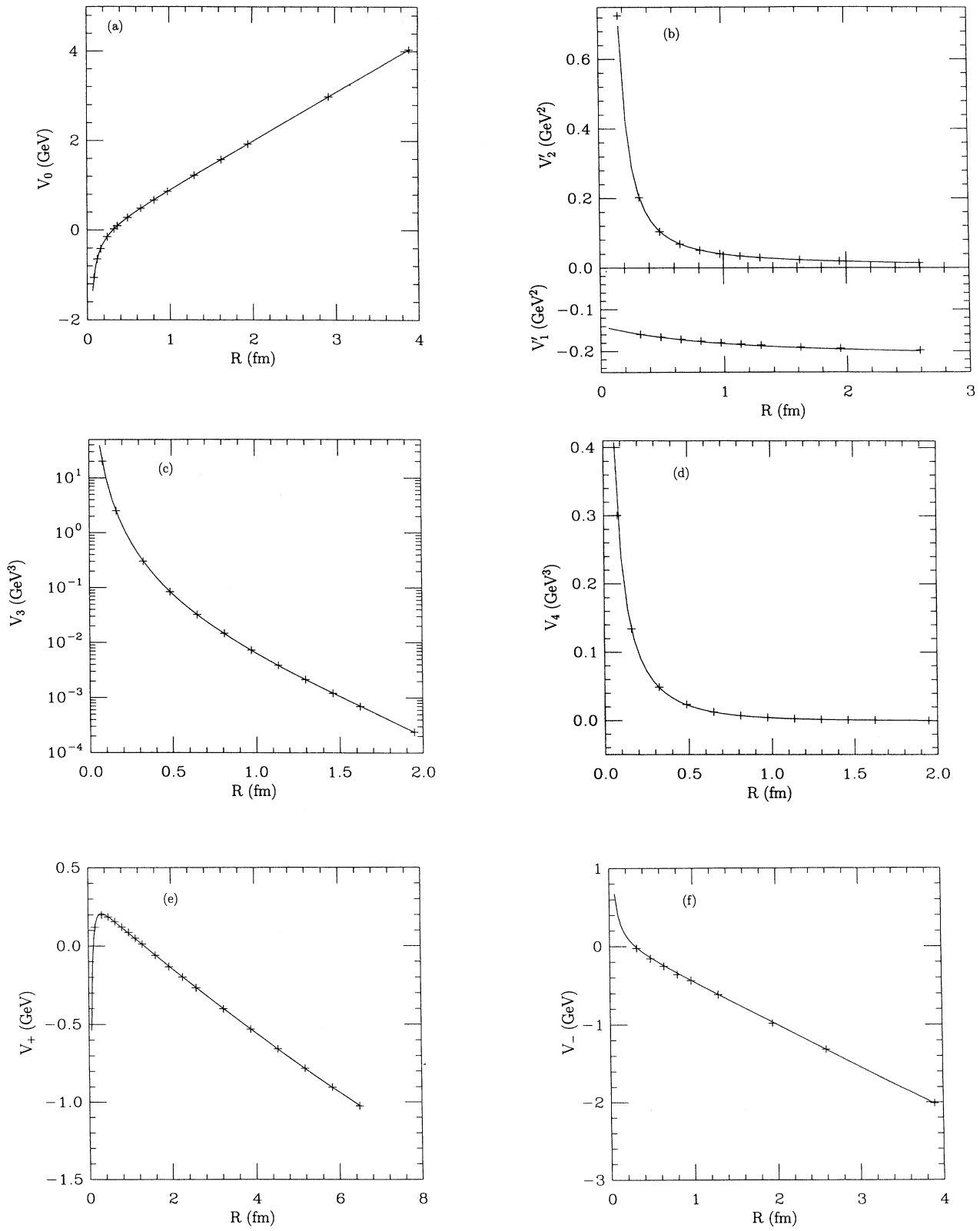

FIG. 1. (a) Analytic fit to the numerical results for the central potential $V_0(R)$. The solid line is the fit; the crosses are the numerically computed values. (b) The same for the spin-orbit potentials $V_1'(R)$ and $V_2'(R)$. (c) The same for the tensor force $V_3(R)$. (d) The same for the spin-spin potential $V_4(R)$. (e) The same for $V_+(R)$. (f) The same for $V_-(R)$. [What is actually plotted is $-V_0/2$ in accordance with Eq. (3C.10).] (g) $V_\parallel(R)$. [The plot is actually of $V_- + R\partial V_0/2\partial R$ as in Eq. (3.13).]

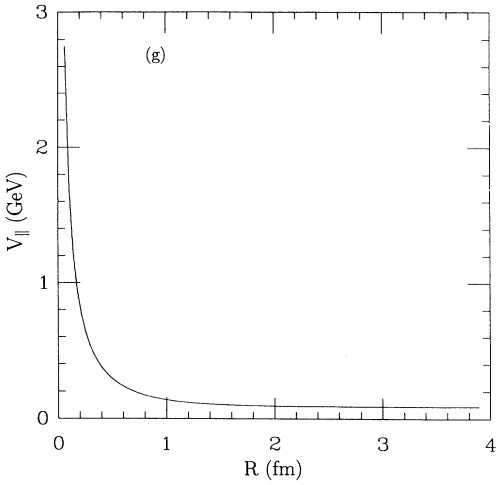

FIG. 1 *(Continued).*

As $R \to 0$, the central potential $V_0(R)$ obtained from Eq. (2.12) approaches a Coulomb potential while as $R \to \infty$ it becomes linear in R. An analytic fit to the central potential obtained from Eq. (2.12) using the numerical solutions of the static field equations in Eq. (2.8) and the values $g^2/\lambda = 5$ determined from the flux tube solution, is

$$V_0(R) = -\frac{4}{3}\frac{\alpha_s}{R}\exp\left[-0.511\left(\frac{\sigma}{\alpha_s}\right)^{1/2}R\right] + \sigma R - 0.646\sqrt{\sigma\alpha_s}\ , \tag{2.14}$$

where $\alpha_s = e^2/4\pi = \pi/g^2 = \pi/5\lambda$ and $\sigma = -1.0324\tilde{F}_0^2$ is the string tension. The exponential cutoff on the Coulomb term in Eq. (2.14) reflects the inability of the color electric field to penetrate into the dual superconducting vacuum. The potential is shown in Fig. 1(a).

We next want to calculate the $(\text{mass})^{-2}$ corrections to $V_0(R)$ produced by the spin and velocity of the quarks. It is easy to show that when the quark sources move, the static fields $\bar{\mathbf{C}}$, $\bar{B}$, and $\bar{B}_3$ move rigidly with the sources through first order in $\mathbf{v}$.

To determine L to second order it now suffices to replace $\mathbf{C}$, B, and B_3 by $\bar{\mathbf{C}}$, $\bar{B}$, and $\bar{B}_3$. This is because L is stationary with respect to variations around $\bar{\mathbf{C}}$, $\bar{B}$, and $\bar{B}_3$ since these satisfy the static field equations (2.8). Hence, there are no terms linear in the differences $\mathbf{C}-\bar{\mathbf{C}}$, $B-\bar{B}$, and $B_3-\bar{B}_3$, which are themselves already of second order.

The field equation for C_0 is (2.4a) with $\boldsymbol{\nabla}\cdot\mathbf{C}=0$:

$$-\nabla^2 C_0 + 6g^2\bar{B}^2 C_0 = -\boldsymbol{\nabla}\cdot\mathbf{H}_s - \boldsymbol{\nabla}\cdot\mathbf{M}\ . \tag{2.15}$$

Using this and a certain amount of algebra we can obtain from (2.3) the second-order Lagrangian, which turns out to be

$$L_2 = \int d^3\mathbf{x}\left\{\tfrac{2}{3}(-\partial_0\bar{\mathbf{C}}+\mathbf{H}_s)^2 + \tfrac{4}{3}(-\mathbf{P}\cdot\boldsymbol{\nabla}\times\bar{\mathbf{C}} - \mathbf{M}\cdot\partial_0\mathbf{C}) + \tfrac{2}{3}M^2 + \tfrac{2}{3}C_0(\boldsymbol{\nabla}\cdot\mathbf{M}+\boldsymbol{\nabla}\cdot\mathbf{H}_s) + 4(\partial_0\bar{B})^2 + 2(\partial_0\bar{B}_3)^2\right\}. \tag{2.16}$$

From this we can extract the three second-order potentials V_{spin}, $V_{\text{spin orbit}}$, and V_{v^2} (the term independent of spin but quadratic in the quark velocities). We note that the V's are by definition the negative of the appropriate parts of the Lagrangian. Setting $\mathbf{v}$ and the time derivatives equal to zero in Eq. (2.16) gives

$$V_{\text{spin}} = -\tfrac{2}{3}\int d^3\mathbf{x}\{C_0\boldsymbol{\nabla}\cdot\mathbf{M} + \mathbf{M}^2\}\ . \tag{2.17}$$

The $\mathbf{M}^2$ term here is evidently just part of the perturbation theory, or one gluon exchange, result; $C_0\boldsymbol{\nabla}\cdot\mathbf{M}$ contains the nonperturbative effects of the theory as well as the rest of perturbation theory.

Similarly the linear terms in $\mathbf{v}$ give

$$V_{\text{spin orbit}} = \int d^3\mathbf{x}\{-\tfrac{4}{3}(\mathbf{P}\cdot\boldsymbol{\nabla}\times\mathbf{C} + \mathbf{M}\cdot\partial_0\mathbf{C}) + \tfrac{4}{3}C_0\boldsymbol{\nabla}\cdot\mathbf{H}_s\}\ . \tag{2.18}$$

To this we must add by hand the contribution of the Thomas precession,

$$V_{\text{spin orbit}}^{\text{Thomas}} = -\left(\frac{\boldsymbol{\sigma}_1\cdot\mathbf{L}}{4m_1^2} + \frac{\boldsymbol{\sigma}_2\cdot\mathbf{L}}{4m_2^2}\right)\frac{1}{R}\frac{\partial V_0(R)}{\partial R}\ , \tag{2.19}$$

to account for the relativistic effect of spin.

In order to evaluate the nonperturbative contributions to these two potentials it is convenient to decompose C_0 into

$$C_0 \equiv c_0 + C_M\ , \tag{2.20}$$

where C_M satisfies

$$\nabla^2 C_M = -\boldsymbol{\nabla}\cdot\mathbf{M}\ . \tag{2.21}$$

Explicitly,

$$C_M = \frac{e}{4\pi}\left\{\frac{\boldsymbol{\sigma}_1\cdot(\mathbf{x}-\mathbf{x}_1)}{2m_1|\mathbf{x}-\mathbf{x}_1|^3} - \frac{\boldsymbol{\sigma}_2\cdot(\mathbf{x}-\mathbf{x}_2)}{2m_2|\mathbf{x}-\mathbf{x}_2|^3}\right\}\ . \tag{2.22}$$

Equation (2.15) then becomes an equation for c_0:

$$\nabla^2 c_0 - 6g^2B^2(c_0 + C_M) = 0\ . \tag{2.23}$$

The calculation of V_{spin} and $V_{\text{spin orbit}}$ thus requires the numerical solution of (2.23) for c_0, inserting the solution [via Eq. (2.20)] into Eqs. (2.17) and (2.18), and adding the Thomas term.

The contribution of C_M in Eqs. (2.17) and (2.18) for V_{spin} and $V_{\text{spin orbit}}$ is simply the Breit-Fermi expres-

sion for these two potentials. The function c_0 provides the nonperturbative contribution resulting from the non-Abelian nature of original Lagrangian, and thereby expresses the effects of confinement.

Finally we read off the expression for V_{v^2} by setting $\mathbf{M}=0$ in the expression for L_2 and in the field equation (2.4a). This yields

$$V_{v^2} = -\int d^3\mathbf{x}\{\tfrac{2}{3}(-\partial_0\mathbf{C}+\mathbf{H}_s)^2+\tfrac{2}{3}C_0\boldsymbol{\nabla}\cdot\mathbf{H}_s +4(\partial_0 B)^2+2(\partial_0 B_3)^2\}\ , \tag{2.24}$$

where now, from (2.4a), and because we use the gauge $\boldsymbol{\nabla}\cdot\mathbf{C}=0$,

$$\nabla^2 C_0 - 6g^2\bar{B}^2 C_0 = \boldsymbol{\nabla}\cdot\mathbf{H}_s\ . \tag{2.25}$$

We can define c_0' by

$$C_0 = c_0' + C_{0D}\ , \tag{2.26}$$

where C_{0D} satisfies

$$\nabla^2 C_{0D} = \boldsymbol{\nabla}\cdot\mathbf{H}_s\ . \tag{2.27}$$

The solution of (2.27) is [4]

$$C_{0D} = \frac{3}{4\pi}\int_{\tau_2}^{\tau_1} d\tau \frac{d\mathbf{y}(\tau,t)}{d\tau}\times\frac{d\mathbf{y}(\tau,t)}{dt}\cdot\frac{[\mathbf{x}-\mathbf{y}(\tau,t)]}{|\mathbf{x}-\mathbf{y}|^3}\ , \tag{2.28}$$

and Eq. (2.25) becomes

$$\nabla^2 c_0' - 6g^2\bar{B}^2(c_0'+C_{0D}) = 0\ . \tag{2.29}$$

C_{0D} for a straight line string becomes

$$C_{0D} = \frac{\mathbf{v}_1+\mathbf{v}_2}{2}\cdot\mathbf{C}_D + \frac{d\mathbf{R}}{dt}\cdot\hat{e}_\varphi\left\{\frac{e}{4\pi\rho R}\left[\frac{\rho^2+z(z-R/2)}{\sqrt{\rho^2+(z-R/2)^2}}-\frac{\rho^2+z(z+R/2)}{\sqrt{\rho^2+(z+R/2)^2}}\right]\right\}\ . \tag{2.30}$$

We recall that we have written $\bar{\mathbf{C}}=\mathbf{c}+\mathbf{C}_D$ in the course of solving the central potential problem. We furthermore note the identity

$$\mathbf{H}_{\rm BS} = \mathbf{H}_s - \partial_0\mathbf{C}_D - \boldsymbol{\nabla}C_{0D}\ , \tag{2.31}$$

where $\mathbf{H}_{\rm BS}$ is the Biot-Savart field, which is

$$\mathbf{H}_{\rm BS} = \frac{e}{4\pi}\left\{\frac{\mathbf{v}_1\times(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^3}-\frac{\mathbf{v}_2\times(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_2|^3}\right\}\ . \tag{2.32}$$

Using Eqs. (2.29) and (2.31) and some algebra we can then show that (2.24) can be rewritten in the form

$$V_{v^2} = -\int d^3\mathbf{x}\{\tfrac{2}{3}(\mathbf{H}_{\rm BS}^2+(\partial_0\mathbf{c})^2-2\mathbf{H}_{\rm BS}\cdot\partial_0\mathbf{c}) -\tfrac{2}{3}\mathbf{H}_s\cdot\boldsymbol{\nabla}c_0'+4(\partial_0 B)^2+2(\partial_0 B_3)^2\}\ . \tag{2.33}$$

In the following section we shall use these forms, namely, (2.17), (2.18), and (2.33) to explicitly compute all of the second-order potentials.

III. EVALUATION OF THE POTENTIALS

We have already evaluated the central potential $V_0(R)$ and given an analytic fit to the solution obtained from numerically solving the field equation for $\mathbf{c}$ and numerically evaluating the integral giving V_0. Our result is Eq. (2.14). (This calculation is also described in Refs. [1,15].)

In our earlier papers on $V_{\rm spin}$ [5] and $V_{\rm spin\ orbit}$ [6] we have given the derivation of these potentials for circular motion of the quarks. It turns out that the radial quark motion included in this paper does not affect these potentials, so our earlier results remain valid [5,6]. The only potential which needs discussion is, therefore, V_v^2, because here the radial motion changes the circular orbit results [7] drastically.

The general form for V_{v^2}, for arbitrary quark motion, is

$$\begin{aligned} V_{v^2} = {}& \frac{[\mathbf{R}\times(\mathbf{v}_1-\mathbf{v}_2)]^2}{4R^2}V_+(R) \\ &+\frac{[\mathbf{R}\times(\mathbf{v}_1+\mathbf{v}_2)]^2}{4R^2}V_-(R) \\ &+\frac{[\mathbf{R}\cdot(\mathbf{v}_1+\mathbf{v}_2)]^2}{4R^2}V_\parallel(R) \\ &+\frac{[R\cdot(\mathbf{v}_1-\mathbf{v}_2)]^2}{4R^2}V_L(R)\ . \end{aligned} \tag{3.1}$$

Note that in the system where the center of mass of the quark-antiquark is at rest, $\mathbf{v}_1=\mathbf{p}/m_1, \mathbf{v}_2=-\mathbf{p}/m_2$, so

$$\frac{[\mathbf{R}\times(\mathbf{v}_1-\mathbf{v}_2)]^2}{4R^2} = \frac{L^2}{4R^2}\left(\frac{1}{m_1}+\frac{1}{m^2}\right)^2 \tag{3.2}$$

and

$$\frac{[\mathbf{R}\times(\mathbf{v}_1+\mathbf{v}_2)]^2}{4R^2} = \frac{L^2}{4R^2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)^2\ , \tag{3.3}$$

where $\mathbf{L}=\mathbf{R}\times\mathbf{p}$ is the orbital angular momentum. This is the reason for the notation V_+ and V_-. The calculations of V_+, $V_-V_\parallel$, and V_L from Eq. (2.33) are long and tedious. The results are

$$V_+ = -\frac{2\alpha_s}{3R} + \frac{4e}{3R}\int_0^\infty \rho^2 d\rho \int_{-\infty}^\infty dz \frac{c}{[\rho^2+(z-R/2)^2]^{5/2}}\left(2\rho^2+2z^2-\frac{Rz}{2}-\frac{R^2}{4}\right)$$
$$+\frac{8e}{3R}\int_0^{R/2} z\,dz \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho c_0'|_{\rho=0}$$
$$-\frac{2\pi}{R^2}\int_0^\infty \rho\,d\rho\int_{-\infty}^\infty dz\left\{\frac{4}{3}\left[L\left(\rho\frac{\partial c}{\partial z}-z\frac{\partial c}{\partial\rho}\right)^2+\frac{r^2c^2}{\rho^2}\right]+8\left(\rho\frac{2\bar{B}}{\partial z}-z\frac{\partial\bar{B}}{\partial\rho}\right)^2+4\left(\rho\frac{\partial\bar{B}_3}{\partial z}-z\frac{\partial\bar{B}_3}{\partial\rho}\right)^2\right\}, \tag{3.4}$$

$$V_- = \frac{2\alpha_s}{3R}+2e\int_0^\infty \rho^2 d\rho\int_{-\infty}^\infty dz\frac{(z-R/2)c}{[\rho^2+(z-R/2)^2]^{5/2}}-\frac{2e}{3}\int_0^{R/2} dz\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho c|_{\rho=0}$$
$$-2\pi\int_0^\infty \rho\,d\rho\int_{-\infty}^\infty dz\left\{\frac{1}{3}\left[\left(\frac{\partial c}{\partial\rho}\right)^2+\frac{c^2}{\rho^2}\right]+2\left(\frac{\partial\bar{B}}{\partial\rho}\right)^2+\left(\frac{\partial\bar{B}_3}{\partial\rho}\right)^2\right\}, \tag{3.5}$$

$$V_\parallel = \frac{4\alpha_s}{3R}-\frac{4e}{3}\int_0^\infty \rho^2 d\rho\int_{-\infty}^\infty dz\frac{\partial c/\partial z}{[\rho^2+(z-R/2)^2]^{3/2}}$$
$$-4\pi\int_0^\infty \rho\,d\rho\int_{-\infty}^\infty dz\left[\frac{1}{3}\left(\frac{\partial c}{\partial z}\right)^2+2\left(\frac{\partial\bar{B}}{\partial z}\right)^2+\left(\frac{\partial\bar{B}_3}{\partial z}\right)^2\right], \tag{3.6}$$

and finally

$$V_L = -\frac{4}{3}\frac{\alpha_s}{R}+\frac{8e}{3}\int_0^\infty \rho\,d\rho\int_{-\infty}^\infty dz\frac{\partial c/\partial R}{[\rho^2+(z-R/2)^2]^{3/2}}-16\pi\int_0^\infty \rho\,d\rho\int_{-\infty}^\infty dz\left[\frac{1}{3}\left(\frac{\partial c}{\partial R}\right)^2+2\left(\frac{\partial\bar{B}}{\partial R}\right)^2\right.$$
$$\left.+\left(\frac{\partial\bar{B}_3}{\partial R}\right)^2\right]. \tag{3.7}$$

The first terms in these equations are the contributions due to single gluon exchange. As before, an analytic fit to the numerical output is made:

$$V_+ = -\frac{2\alpha_s}{3R}\exp\left[-1.14\left(\frac{\sigma}{\alpha_s}\right)^{1/2}R\right]-0.208\sigma R$$
$$+1.12\sqrt{\sigma\alpha_s}\,, \tag{3.8}$$

$$V_- = -\tfrac{1}{2}V_0 \text{ (see below)}\,, \tag{3.9}$$

$$V_\parallel = V_- + \frac{R}{2}\frac{\partial V_0}{\partial R} \text{ (see below)}\,, \tag{3.10}$$

$$V_L = -\frac{4\alpha_s}{3R}\exp\left[-0.685\left(\frac{\sigma}{\alpha_s}\right)^{1/2}R\right]$$
$$+0.0885\sqrt{\sigma\alpha_s}\,. \tag{3.11}$$

It is important to note that relativistic invariance of the Lagrangian density imposes the following exact relations between the various terms in V_{v^2} and V_0, namely, [12]

$$V_- = -\tfrac{1}{2}V_0 \tag{3.12}$$

and

$$V_\parallel = V_- + \frac{R}{2}\frac{\partial V_0}{\partial R}\,. \tag{3.13}$$

IV. COMPARISON TO EXPERIMENT

The numerical procedures we have employed to solve the differential equations for the fields, and then to evaluate the integrals over these fields to obtain the various potentials, are the same ones we have used before. They are described in detail in Refs. [1,5,8]. The only changes are in V_{v^2}, and in the differential equation for c_0'. However, since C_{0D} is not singular, no special treatment is required, and the numerical solution for c_0' is straightforward. The evaluation of the integrals in V_{v^2} are, therefore, straightforward too.

Two kinds of tests can be made of our results. First is a comparison with general conditions such as the Gromes relation [10], Eqs. (3.12) and (3.13), and with lattice results. Second is a fit to the energy levels of the $c\bar{c}$ and $b\bar{b}$ quark systems to determine our parameters in detail. Once these activities are finished we can use our potentials to predict unobserved states of $c\bar{c}$ and $b\bar{b}$, as well as levels in mixed quark systems such as $c\bar{b}$. Finally, if we are very optimistic and willing to extend our results to the not very heavy strange quark, we can also calculate the $b\bar{s}$ and $c\bar{s}$ systems to compare with what data exist concerning these.

We begin with the Gromes relation [10]. This applies to the spin-orbit potentials, and on quite general grounds requires that

$$V_1'(R) = V_2'(R) - \frac{\partial V_0(R)}{\partial R}\,. \tag{4.1}$$

Our numerical results agree with this to the accuracy of

our numerical calculation, and the analytic fits to V_1', V_2', and V_0 satisfy it exactly.

We note that, from Eq. (5.2c), $V_2'(R) \to 0$ as R $\to \infty$. Hence, for large R, Eq. (4.1) gives

$$V_1'(R) = -\frac{\partial V_0(R)}{\partial R} \ . \tag{4.2}$$

This relation is valid for a potential produced by scalar particle exchange, so the long-range part of our spin-orbit potentials mimics scalar particle exchange. Nevertheless, V_1' has nothing to do with one particle exchange; it is completely nonperturbative. V_2', on the other hand, mimics one vector gluon exchange.

Next we may compare our computed V_0, V_-, and $V_\parallel$ with the general relations (3.12) and (3.13). Again we find agreement to within the accuracy of our numerical calculations.

There have been a number of lattice gauge calculations of the spin orbit potentials which can be summarized as follows: V_2' is similar to one vector gluon exchange, while V_1' is long range. We show in Fig. 2 a comparison of lattice results [11] with our predictions for V_1' and V_2'. Clearly the general trend is in agreement, but a real test will require more accurate lattice results. [Incidentally, in the figures for V_1' and V_2' we have used values for g (or $\alpha_s = \pi/g^2$), and σ obtained from our fit to $c\bar{c}$ and $b\bar{b}$ levels described below.]

In Fig. 3 we plot the potential $V_3(R)$ against R, as well as the "perturbative" result due to single vector gluon exchange. As we see, there is a major difference at large R, in that our result falls off much more rapidly than the "perturbative" result. This again reflects the inability of the color electric field to penetrate the dual QCD superconducting vacuum.

Figure 4 shows the same plot for $V_4(R)$. Here the "perturbative" result is simply a delta function.

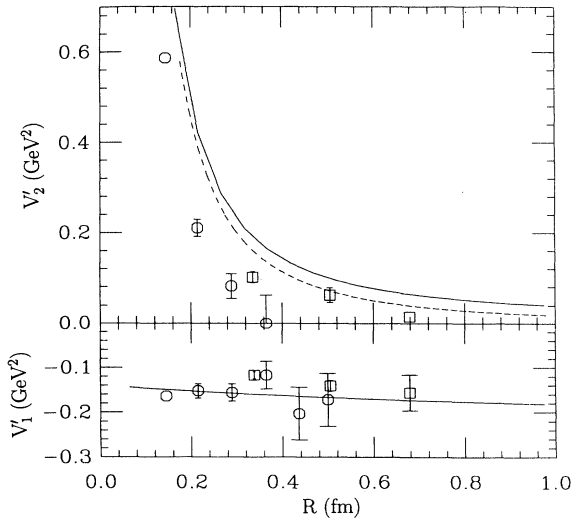

FIG. 2. Comparison of our computed spin-orbit potentials (solid lines) with lattice calculations [10] (indicated by the data points). The dotted line shows the single gluon exchange result for $V_2'(R)$; $V_1'(R)$ is zero for single gluon exchange.

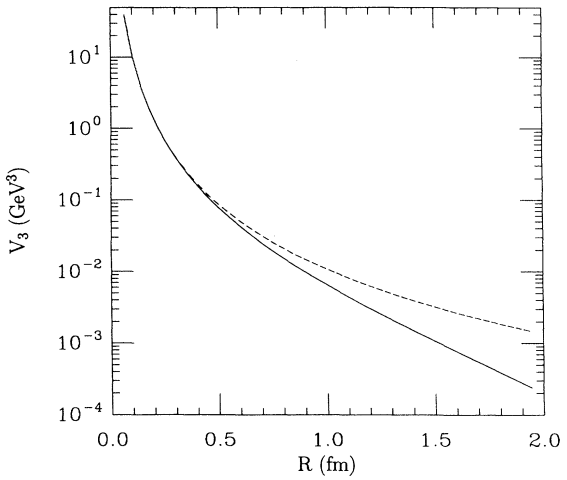

FIG. 3. The tensor force potential $V_3(R)$ (solid line) and the perturbative gluon exchange contribution (dotted line).

Our next step is to compute the masses of heavy-quark–antiquark pairs by inserting our potentials (appropriately symmetrized to guarantee Hermiticity) into the Schrödinger equation. The masses are calculated as a function of our various parameters and the differences between the calculated and experimental value are minimized.

In all our fits to the masses of the $c\bar{c}$ and $b\bar{b}$ systems we have chosen $g'^2 \equiv g^2/\lambda = 5$. Varying this parameter requires a complete solution to the dual QCD field equations for $\bar{\mathbf{C}}$, $\bar{B}$, and $\bar{B}_3$. Since these are nonlinear partial differential equations, it is not practical to include g' as a parameter in our minimization search. We have, however, redone the complete calculation for $g'^2 = 2$ and $g'^2 = 10$. Over this range the final results showed very little dependence on g'^2 with the fit for $g'^2 = 5$ being slightly better than for the other values. The dual QCD

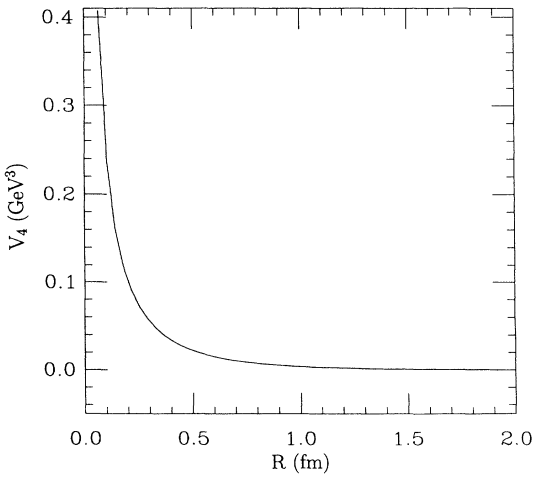

FIG. 4. The same plot as in Fig. 4 for $V_4(R)$. Here the single gluon exchange contribution is a delta function at the origin.

parameters λ and $\sqrt{-\tilde{F}_0^2}$ change in such a way as to keep the physical quantities α_s and the string tension essentially constant. As result, the exact value of g'^2 does not affect the fit and we can safely fix it to be 5, which was determined from our flux tube considerations [8]. This means that we set $\alpha_s = \pi/5\lambda$ in carrying out the fitting.

The procedure for obtaining a best fit to the energy levels of the known $c\bar{c}$ and $b\bar{b}$ states is as follows. We define an effective χ^2 as

$$\chi^2 = \sum \left(\frac{(\text{experiment} - \text{theory})}{(0.01 \times \text{experiment})} \right)^2 . \qquad (4.3)$$

This would be the actual χ^2 if the experimental statistical error was in fact 1% or equivalently what might be expected to be equal to the number of degrees of freedom if the theory was good to 1%. Our remaining four parameters α_s, σ, and the two-quark masses m_c and m_b are then varied to minimize the effective χ^2. Our procedure is the following: Using our central potential we solve the Schrödinger equation to determine the eigenvalues and the wave functions for the necessary orbital angular momentum states. The spin- and angular-momentum-dependent potentials are then used perturbatively to calculate the energies of the individual states and the χ^2 is evaluated. The four parameters are then varied to minimize χ^2. It should be emphasized that these are only parameters and the dependence of the potentials on these parameters is completely determined by dual QCD.

Once the best fit parameters are determined we can predict the unobserved energy levels. Our best fit to the 17 observed states is given in Table I. The resulting χ^2 is 1.85, corresponding to an average least-square error of 0.03%. The largest error occurs for the $\Upsilon(4S)$, where our result is too heavy by 42 MeV.

Our best fit parameters are (i) $\alpha_s = 0.3702$, (ii) $\sigma = 0.1994$ GeV^2, (iii) $m_c = 1.340$ GeV, and (iv) $m_b = 4.770$ GeV. These parameters are only slightly changed from our flux tube fits [8]. They result in a flux tube radius of $0.5f$, and a scaling parameter $\sqrt{-\lambda\tilde{F}_0^2} = 572$ MeV.

TABLE I. Predicted and experimental masses of all observed $c\bar{c}$ and $b\bar{b}$ states below threshold. (See Ref. [28].)

State	Pred. mass (GeV)	Exper. mass (GeV)
$\eta_c(1S)$	2.968	2.980
$\psi(1S)$	3.118	3.097
$\psi(2S)$	3.696	3.686
$\chi_{c_0}(1P)$	3.437	3.415
$\chi_{c_1}(1P)$	3.498	3.511
$\chi_{c_2}(1P)$	3.540	3.556
$h_c(1P)$	3.511	3.526 [19]
$\Upsilon(1S)$	9.464	9.460
$\Upsilon(2S)$	9.998	10.023
$\Upsilon(3S)$	10.340	10.355
$\Upsilon(4S)$	10.622	10.580
$\chi_{b_0}(1P)$	9.866	9.860
$\chi_{b_1}(1P)$	9.897	9.892
$\chi_{b_2}(1P)$	9.921	9.913
$\chi_{b_0}(2P)$	10.221	10.232
$\chi_{b_1}(2P)$	10.246	10.255
$\chi_{b_2}(2P)$	10.266	10.268

TABLE II. Predicted masses of the unobserved $c\bar{c}$ and $b\bar{b}$ states.

State	Pred. mass (GeV)
$\eta_c(2S)$	3.589
$(^1D_2)c\bar{c}$ $(n=1)$	3.826
$(^3D_1)c\bar{c}$ $(n=1)$	3.814
$\eta_b(1S)$	9.314
$\eta_b(2S)$	9.931
$\eta_b(3S)$	10.288
$\eta_b(4S)$	10.577
$(^1P_1)b\bar{b}$ $(n=1)$	9.906
$(^1P_1)b\bar{b}$ $(n=2)$	10.254
$(^1D_2)b\bar{b}$ $(n=1)$	10.155
$(^1D_2)b\bar{b}$ $(n=2)$	10.450
$\chi_{b_0}(3P)$	10.519
$\chi_{b_1}(3P)$	10.542
$\chi_{b_2}(3P)$	10.561
$(^3D_1)b\bar{b}$ $(n=1)$	10.147
$(^3D_2)b\bar{b}$ $(n=1)$	10.153
$(^3D_3)b\bar{b}$ $(n=1)$	10.158
$(^3D_1)b\bar{b}$ $(n=2)$	10.442
$(^3D_2)b\bar{b}$ $(n=2)$	10.448
$(^3D_3)b\bar{b}$ $(n=2)$	10.453

The predicted energy levels for the as yet unobserved (nearly) stable states of these systems are shown in Table II.

Finally, we have also computed the $^1S_0$ and $^3S_1$ states of the $c\bar{s}$ and $b\bar{s}$ systems. (These are known as D_s, D_s^*, B_s, and B_s^*, respectively.) Our results for these fits are shown in Table III, and use a value $m_s = 350$ MeV. This mass is very low, and it is not surprising that the fit in this case is much poorer than for the $c\bar{c}$ and $b\bar{b}$ systems. It is possible that using the Salpeter equation or the Dirac equation rather than the Schrödinger equation to compute the levels involving s quarks will help; this will be discussed in Appendixes B and C.

The potentials can be used to predict low-lying states of the, as yet, unobserved $c\bar{b}$ system. For the $^1S_0$ (called $\eta_{c\bar{b}}$) and $^3S_1$ states of $c\bar{b}$ we find masses of 6.285 GeV and 6.377 GeV, respectively.

The dual QCD region of applicability is in the IR or long-range behavior and, therefore, provided no information about short-range effects. From what is known about QCD, one would expect the coupling constant to run at short distances reflecting the fact that the theory is asymptotically free. We have investigated what such a modification might do to our fits to the data by using a

TABLE III. Predicted masses of observed $c\bar{s}$ and $b\bar{s}$ states. Note that the mass of B_s^* minus the mass of B_s is known experimentally to be 47 MeV. The experimental masses are from Ref. [28].

$c\bar{s}$		$b\bar{s}$	
Theory	Expt.	Theory	Expt.
$^1S_0$ 2.343 GeV	1.969	5.788	5.359→ 5.409
$^3S_1$ 2.500	2.536	5.843	5.406→ 5.456

modified formula for the coupling constant for the central potential:

$$\alpha_s(R) = \begin{cases} \alpha_s, & R > R_0 \ , \\ \dfrac{\alpha_s}{1+\frac{R-R_0}{R_0}\beta \ln(R/R_0)}, & R < R_0 \ . \end{cases} \tag{4.4}$$

β was calculated from the β function for QCD for no flavors and R_0 was arbitrarily fixed at 1/GeV. The resulting χ^2, obtained from fitting the data with this modification, was 1.25, an improvement over using the long-distance potential everywhere.

V. SUMMARY AND CONCLUSION

The entire heavy quark potential to order $(\text{mass})^{-2}$ can in general be written in the form [12]

$$\begin{aligned} V = V_0(R) &+ \left(\frac{\boldsymbol{\sigma}_1\cdot\mathbf{L}}{4m_1^2}+\frac{\boldsymbol{\sigma}_2\cdot\mathbf{L}}{4m_2^2}\right)\frac{1}{R}[V_1'(R)+2V_1'(R)] + \frac{(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{L}}{4m_1m_2}\frac{1}{R}V_2'(R) + \frac{1}{4m_1m_2}\left(\frac{\boldsymbol{\sigma}_1\cdot\mathbf{R}\boldsymbol{\sigma}_2\cdot\mathbf{R}}{R^2}\right. \\ &\left.-\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{3}\right)V_3(R) + \frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{12m_1m_2}V_4(R) + \frac{L^2}{4R^2}\left[\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2 V_+(R) + \left(\frac{1}{m_1}-\frac{1}{m_2}\right)^2 V_-(R)\right] \\ &+\frac{[\mathbf{R}\cdot(\mathbf{v}_1+\mathbf{v}_2)]^2}{4R^2}V_\parallel(R) + \frac{[\mathbf{R}\cdot(\mathbf{v}_1-\mathbf{v}_2)]^2}{4R^2}V_L(R) \ . \end{aligned} \tag{5.1}$$

Within the classical approximation to dual QCD the nine potentials appearing in Eq. (5.1) have been evaluated as functions of R, the quark-antiquark separation, by first solving numerically the differential equations (2.11), (2.29), and (2.39) for the functions $\mathbf{c} = c(\rho,z)\hat{e}_\varphi$, $c_0(\rho,z)$, and $c_0'(\rho,z)$, respectively, and then inserting these solutions into the integrals (2.12), (2.17), (2.18), and (2.33). The potentials have an overall power of R (determined by dimensions) and are otherwise functions of $\alpha_s, g^2/\lambda$, which we fix to be 5, and the string tension σ. Analytic fits are made to the potentials computed in this way; we restate these here for convenience:

$$V_0(R) = -\frac{4\alpha_s}{3R}\exp\left[-0.511\left(\frac{\sigma}{\alpha_s}\right)^{1/2}R\right] + \sigma R - 0.646\sqrt{\sigma\alpha_s} \ , \tag{5.2a}$$

$$V_1'(R) = \frac{4\alpha_s}{3R}0.511\sqrt{\frac{\sigma}{\alpha_s}}(1-e^{-0.511\sqrt{\sigma/\alpha_s}R}) - \sigma \ , \tag{5.2b}$$

$$V_2'(R) = \frac{4\alpha_s}{3R}\left(0.511\sqrt{\frac{\sigma}{\alpha_s}} + \frac{e^{-0.511\sqrt{\sigma/\alpha_s}R}}{R}\right) \ , \tag{5.2c}$$

$$V_3(R) = 12\pi\alpha_s e^{-0.281\sqrt{\sigma/\alpha_s}R}\left(\frac{0.101}{R^3}+\frac{0.0417\sqrt{\sigma/\alpha_s}}{R^2}-\frac{4.2\times10^{-3}\frac{\sigma}{\alpha_s}}{R}\right) \ , \tag{5.2d}$$

$$V_4(R) = \frac{32\pi\alpha_s}{3}\delta^3(\mathbf{R}) + 12\pi\alpha_s e^{-0.413\sqrt{\sigma/\alpha_s}R}\left[\frac{0.019\frac{\sigma}{\alpha_s}}{R} - 1.7\times10^{-3}\left(\frac{\sigma}{\alpha_s}\right)^{3/2}\right] \ , \tag{5.2e}$$

$$V_+(R) = -\frac{2\alpha_s}{3R}e^{-1.14\sqrt{\frac{\sigma}{\alpha_s}}R} - 0.208\sigma R + 1.12\sqrt{\sigma\alpha_s} \ , \tag{5.2f}$$

$$V_-(R) = -\frac{1}{2}V_0(R) \ , \tag{5.2g}$$

$$V_\parallel(R) = V_-(R) + \frac{R}{2}\frac{\partial V_0(R)}{\partial R} \ , \tag{5.2h}$$

$$V_L(R) = -\frac{4\alpha_s}{3R}e^{-0.685\sqrt{\frac{\sigma}{\alpha_s}}R} + 0.0885\sqrt{\sigma\alpha_s} \ . \tag{5.2i}$$

The fits and their comparison to the results of the numerical calculations are shown in Figs. 1(a)–1(g) and are evidently excellent.

As to the comparison of these potentials with experiment we first comment that $V_0(R)$ agrees extremely well with phenomenological central potentials, as shown in Fig. 5. Second we have remarked that the spin-orbit potentials $V_1'(R)$ and $V_2'(R)$ satisfy, to the limit of our numerical accuracy, the Gromes relation [10]

$$V_1'(R) = V_2'(R) - \frac{\partial}{\partial R}V_0(R)$$

which is derived using the general QCD expressions for V_0, V_1', and V_2' together with invariance under Lorentz boosts. Third as seen in Fig. 2, our spin-orbit potentials are in good agreement with lattice calculations of these [11]; in particular it is worth noting that single gluon exchange predicts for $V_1'(R)$ zero while both our result and the lattice result show a nearly constant negative value.

To obtain the energy levels of the $c\bar{c}$ and $b\bar{b}$ systems, the analytic fits to seven potentials (the V_- and V_{11} terms vanish for equal quark masses) are used perturbatively in the Schrödinger equation describing the central-quark–

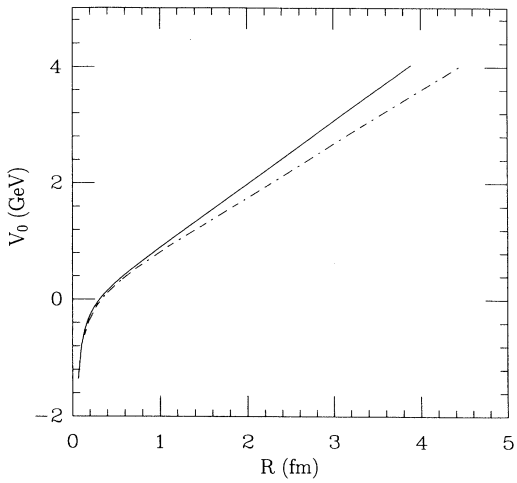

FIG. 5. Comparison of our computed central potential $V_0(R)$ with a phenomenological potential (from Ref. [13]). The difference in these potentials is due to the fact that our fits require a larger string tension.

antiquark interaction, and the energy levels are evaluated numerically. The results depend on the quark masses m_b and m_c, and the parameters λ, σ, and α_s, related to the dual coupling constant, in the infrared limit by $\alpha_s = \pi/g^2$. Values for these last three parameters are roughly known *a priori* from our previous fits which lead us to choose $g^2/\lambda = 5$ (as remarked earlier, choosing $g^2/\lambda = 10$ or 2 changes the fits little). The other two parameters are varied together with m_b and m_c to provide a best fit to the known $c\bar{c}$ and $b\bar{b}$ levels, as described in Sec. IV and shown in Table I. The resulting best fit parameters are given in Sec. IV. The values of λ and $\sqrt{-\tilde{F}_0^2}$ are quite compatible with the estimates obtained from vacuum energy density computed from the trace anomaly together with the value of the magnetic condensate [14], and the string tension. We found these parameters to be $\lambda \sim 1.61$ and $\sqrt{-\tilde{F}_0^2} \sim 420$ MeV, as mentioned in Sec. I.

Predicted values, using these parameters, for as yet unobserved levels in the $c\bar{c}$ and $b\bar{b}$ system are given in Table II; the $^1S_0$ (or $\eta_{c\bar{b}}$) and $^3S_1$ states of $c\bar{b}$ have also been computed (using the unequal mass potentials), and come out to be 6.276 GeV and 6.365 GeV, respectively.

In the literature on phenomenological heavy quark potentials several "improvements" have sometimes been made. We next explain why we have chosen not to incorporate any of these in our potentials.

The first "improvement" is to include the so-called Lüscher term [15]. This term, which consists of a contribution of $\pi^2/12R$ to $V_0(R)$, results from quantum excitations of a Lagrangian describing a string of flux joining the two quarks [16]. It is evidently valid only when the quark separation greatly exceeds the flux tube width, if then [17]. This is the case only at separations so large that the R part of $V_0(R)$ completely dominates any $1/R$ terms. What the Lüscher term turns into for smaller R, where, if it existed in the same form, it might influence the fits, is completely unknown. We, therefore, see no valid reason for including it in $V_0(R)$.

Finally, there are various relativistic corrections, configuration mixing, and other phenomena sometimes included in heavy quarkonium level fitting. All of these effects (except the Thomas term, which we have incorporated) go beyond the $(\text{mass})^{-2}$ order of approximation employed here. In the spirit of sticking to a consistent, well-defined, systematic approximation, we have, therefore, chosen to ignore all such effects.

In conclusion, we may ask if any other of our predictions can be tested. One obvious thing is to find further levels in quarkonium, and to identify $c\bar{b}$ states. A perhaps more interesting and more sensitive test would be to improve lattice calculations of all eight (nine in the unequal mass case) of our potentials. As we have seen, lattice calculations do exist for the two spin-orbit potentials $V_1'(R)$ and $V_2'(R)$, and these in fact agree with our predictions for these potentials rather well. We would like to see similar comparison for the other potentials.

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APPENDIX A: ELECTRODYNAMICS WITH DUAL POTENTIALS

Dirac [18] showed that Maxwell's equations could be extended to include both electrically and magnetically charged particles by connecting the magnetically charged particles to strings. In the absence of magnetically charged particles one can apply Dirac's method to ordinary electrodynamics by connecting electrically charged particles to strings. In this formulation Maxwell's equations become equations for dual potentials C_μ whose sources are the polarization currents produced by the Dirac strings. The potentials themselves depend upon the location of the strings but they yield the same string independent electromagnetic fields as the usual procedure.

If in addition the dual potentials C_μ are minimally coupled to Higgs fields, these fields necessarily carry magnetic charge. Such a theory describes the motion of electrically charged particles connected by Dirac strings in a dual superconductor. If extended to non-Abelian gauge theory [7] it becomes a concrete realization of the Mandelstam 't Hooft [19,20] picture of color confinement as a manifestation of dual superconductivity. We have found that in order to understand this mechanism for confinement it is very helpful to first have a clear picture of how dual potentials work in ordinary electrodynamics.

Therefore, in this appendix we present an elementary discussion of Dirac's method applied to electrodynamics and work out some simple examples. Our discussion is entirely pedagogical and contains nothing new although some specific results obtained here may not be readily accessible elsewhere.

Consider first a sourceless linear dielectric medium. Then Maxwell's equations

$$\begin{array}{ccc} \text{(a)} & \text{(b)} & \text{(c)} \\ \nabla\cdot\mathbf{D}=0 & \nabla\cdot\mathbf{B}=0 & \mathbf{D}=\epsilon\mathbf{E} \\ \nabla\times\mathbf{H}-\partial_0\mathbf{D}=0 & \nabla\times\mathbf{E}+\partial_0\mathbf{B}=0 & \mathbf{B}=\mu\mathbf{H}\ , \end{array} \tag{A1}$$

can be solved by introducing vector potentials in either of two ways. The conventional choice is to write

$$\mathbf{B}\equiv\nabla\times\mathbf{A},\quad \mathbf{E}\equiv-\partial_0\mathbf{A}-\nabla A_0\ , \tag{A2a}$$

in which case Eqs. (A1b) become kinematical identities and the dynamics is contained in Eqs. (A1a). The vector $A^\mu=(A_0,\mathbf{A})$ is called the vector potential. The alternate (dual) choice is to write

$$\mathbf{D}=-\nabla\times\mathbf{C},\quad \mathbf{H}=-\partial_0\mathbf{C}-\nabla C_0\ , \tag{A2b}$$

in which case Eqs. (A1a) are kinematical identities and Eq. (A1b) contain the dynamics. The vector $C^\mu=(C_0,\mathbf{C})$ is called the dual vector potential.

Let us first use C_μ to solve the source-free Maxwell equations (A1) in order to get accustomed to using the dual potential. We first write Eqs. (A2b) in covariant form by defining

$$G_{0k}=H_k,\quad G_{ij}=\epsilon_{ijk}D_k\ , \tag{A3}$$

so that Eqs. (A2b) take the form

$$G_{\mu\nu}=\partial_\mu C_\nu-\partial_\nu C_\mu\ . \tag{A4}$$

In a relativistic medium $\epsilon=1/\mu$. Then using Eq. (A3) we can write the constitutive equations as

$$E_i=\frac{\mu}{2}\epsilon_{ijk}G_{jk},\quad B_i=\mu G_{0i}\ , \tag{A5}$$

and Maxwell's equations (A1b) as

$$\partial^\alpha\mu G_{\alpha\beta}=0\ . \tag{A6}$$

Equation (A6) for C_μ have the same form as the usual Maxwell equations for A_μ, obtained from (A1a), with the replacement $\mu\to\epsilon$, and they are solved in the same way. Equation (A5) then gives the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ in terms of C_μ.

Electric current sources $j_\mu=(\rho,\mathbf{j})$ appear only in Eqs. (A1a) and not in Eqs. (A1b). Hence in the presence of electric currents, Eqs. (A1b) remain valid and are still kinematic identities in terms of A^μ. Equations (A1a), in contrast, are no longer identities in terms of C^μ. However, Dirac has shown how to generalize Eqs. (A2b) in order to satisfy Eqs. (A1a) with dual potentials C^μ even in the presence of electric currents.

When charged particles are present Eqs. (A1a) become

$$\begin{array}{cc} \text{Gauss' law} & \text{Ampere's law} \\ \nabla\cdot\mathbf{D}=\rho & \nabla\times\mathbf{H}=\mathbf{j}+\frac{\partial\mathbf{D}}{\partial t}. \end{array} \tag{A7}$$

Suppose that the total charge $Q=\int\rho\,d\mathbf{x}=0$. (If $Q\neq0$, then there will be Dirac strings extending to infinity, but nothing essential will be changed.) Then we can always find a polarization vector $\mathbf{P}$ and a magnetization vector $\mathbf{M}$ so that

$$\rho=-N\cdot\mathbf{P},\quad \mathbf{j}=\nabla\times\mathbf{M}+\frac{\partial\mathbf{P}}{\partial t}\ , \tag{A8}$$

where $\mathbf{P}$ is the dipole moment per unit volume and $\mathbf{M}$ is the magnetic moment per unit volume. Inserting Eq. (A8) into (A7) we obtain

$$\nabla\cdot(\mathbf{D}+\mathbf{P})=0,\quad \nabla\times(\mathbf{H}-\mathbf{M})-\frac{\partial(\mathbf{D}+\mathbf{P})}{\partial t}=0\ .$$

Hence,

$$\mathbf{D}=-\nabla\times\mathbf{C}-\mathbf{P},\quad \mathbf{H}=-\nabla C_0-\frac{\partial\mathbf{C}}{\partial t}+\mathbf{M}\ . \tag{A9}$$

Equation (A7) then become kinematical identities and Eqs. (A1b) contain the dynamics as before. Using the definitions (A3) of $G_{\mu\nu}$ we can write Eqs. (A9) in the covariant form

$$G_{\mu\nu}=\partial_\mu C_\nu-\partial_\nu C_\mu+G^s_{\mu\nu}\ , \tag{A10}$$

where the tensor $G^s_{\mu\nu}$ has components

$$G^s_{0k}=M_k,\quad G^s_{ij}=-\epsilon_{ijk}P_k\ , \tag{A11}$$

and we have now specialized to the case where $\mathbf{P}$ and $\mathbf{M}$ arise from Dirac strings connecting the charged particles; hence the superscript s on $G^s_{\mu\nu}$. Equation (A10) is just the generalization of Eq. (A4) to account for the presence of charged particles. Equation (A11) shows that $G^s_{\mu\nu}$ is the dual of the polarization tensor. Equation (A1b) and (A1c) are unchanged so that Eqs. (A5) and (A6) remain the same as does the definition (A3) of $G_{\mu\nu}$. The effect of the charged particles is to change the relation (A4) between $G_{\mu\nu}$ and C_μ to (A10) where $G^s_{\mu\nu}$ is determined in terms of ρ and $\mathbf{j}$ by solving Eq. (A8) for $\mathbf{P}$ and $\mathbf{M}$.

Substituting Eq. (A10) into Eq. (A6) we obtain

$$\partial^\alpha\mu(\partial_\alpha C_\beta-\partial_\beta C_\alpha)=-\partial^\alpha\mu G^s_{\alpha\beta}\ , \tag{A12}$$

which determines the dual potentials C_μ in terms of $G^s_{\mu\nu}$. Equation (A12) provide an alternate form of Maxwell's equations which are completely equivalent to the usual form expressed in terms of the vector potential A_μ: namely,

$$\partial^\alpha\epsilon(\partial_\alpha A_\beta-\partial_\beta A_\alpha)=j_\beta\ . \tag{A13}$$

All text books on electricity and magnetism could be rewritten using only dual potentials C_μ satisfying Eq. (A12) and the same electromagnetic forces between charged particles would be obtained. The potentials themselves, however, could be completely different. For

example, in a dielectric medium having a wave number dependent dielectric constant $\epsilon(q) \to 0$ as $q^2 \to 0$ (corresponding to antiscreening at large distances), the potentials A_μ determined from Eq. (A13) would be singular at large distances, while the dual potentials C_μ satisfying Eq. (A12) with $\mu = 1/\epsilon \to \infty$ as $q^2 \to 0$ would be screened at large distances. Use of the potentials A_μ to describe this system would introduce singularities which do not appear in the dual potentials C_μ. Hence the dual potentials are the natural choice to describe a medium with long-range antiscreening.

Note that for $\mu = \epsilon = 1$, $\mathbf{B} = \mathbf{H}$, $\mathbf{D} = \mathbf{E}$ and substituting Eqs. (A9) in (A1b) gives the equation for the dual potentials in three-dimensional notation:

$$\nabla \cdot (-\nabla C_0 - \partial_0 \mathbf{C}) = -\nabla \cdot \mathbf{M} , \tag{A14}$$

$$\nabla \times (-\nabla \times \mathbf{C}) + \partial_0(-\partial_0 \mathbf{C} - \nabla C_0) = \nabla \times \mathbf{P} - \frac{\partial \mathbf{M}}{\partial t} . \tag{A15}$$

These equations are identical to Eq. (A12) with $\mu = 1$. They have the same form as the equations for A_μ, the ordinary vector potentials in a polarizable medium with $\mathbf{P}$ and $\mathbf{M}$ interchanged. For example, $-\nabla \cdot \mathbf{M}$ is the source of C_0. However Eqs. (A14) and (A15) describe the electrodynamics of electrically charged particles moving in the vacuum and $\mathbf{P}$ and $\mathbf{M}$ are the polarization and magnetization, respectively, of the Dirac strings attached to these particles, as we shall now see.

We now apply these results to the case of two particles of charge $e(-e)$ moving along trajectories $\mathbf{x}_1(t)[\mathbf{x}_2(t)]$ in free space with $\mu = \epsilon = 1$. Then

$$\rho(\mathbf{x}, t) = e[\delta^3(\mathbf{x} - \mathbf{x}_1(t)) - \delta^3(\mathbf{x} - \mathbf{x}_2(t))] \tag{A16}$$

and

$$\mathbf{j}(\mathbf{x}, t) = e[\mathbf{v}_1 \delta^3(\mathbf{x} - \mathbf{x}_1(t)) - \mathbf{v}_2 \delta^3(\mathbf{x} - \mathbf{x}_2(t))] , \tag{A17}$$

where $\mathbf{v}_i = d\mathbf{x}_i/dt$, $i = 1, 2$. We must find a polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ satisfying Eq. (A8) with ρ and $\mathbf{j}$ given by Eqs. (A16) and (A17). The solution of this problem was given by Dirac [18]. Let $\mathbf{y}(\sigma, t)$ be any line $L(t)$ connecting $\mathbf{x}_2(t)$ and $\mathbf{x}_1(t)$, i.e., $\mathbf{y}(\sigma_1, t) = \mathbf{x}_1(t)$, $\mathbf{y}(\sigma_2, t) = \mathbf{x}_2(t)$, $\sigma_2 \le \sigma \le \sigma_1$. [See Fig. 6(a).] On each element $d\mathbf{y}$ of L place a dipole moment $d\mathbf{p} = e\, d\mathbf{y}$. It is evident from Fig. 6(a) that the charge and current density produced by the sum of these dipoles is that due to the pair of moving oppositely charged particles, namely, Eqs. (A16) and (A17). To obtain (A16) formally we note that the dipole moment per unit volume $\mathbf{P}$ is

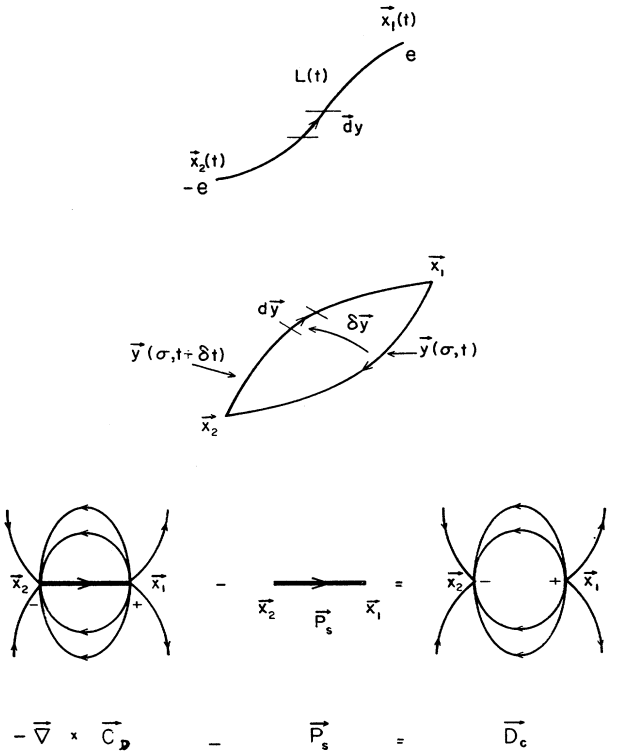

FIG. 6. (a) Dirac string $L(t)$ connecting oppositely charged particles. (b) Closed contour describing path of line integral on right-hand side of Eq. (A22). (c) Diagram representing string cancellation mechanism of Eq. (A32).

$$\mathbf{P}(x) = e \int_{\mathbf{x}_2(t)}^{\mathbf{x}_1(t)} d\mathbf{y}\, \delta(\mathbf{x} - \mathbf{y}) = e \int_{\sigma_2}^{\sigma_1} d\sigma \frac{\partial \mathbf{y}(\sigma, t)}{\partial \sigma} \delta(\mathbf{x} - \mathbf{y}(\sigma, t)) . \tag{A18}$$

Then

$$-\nabla \cdot \mathbf{P} = -e \int_{\mathbf{x}_2(t)}^{\mathbf{x}_1(t)} d\mathbf{y} \cdot \nabla_x \delta(\mathbf{x} - \mathbf{y}) = \rho(\mathbf{x}) . \tag{A19}$$

Furthermore since the line element $d\mathbf{y}$ is moving with velocity $\mathbf{v} = (\partial/\partial t)\mathbf{y}(\sigma, t)$, the string L in Fig. 6(a) has a magnetization

$$\mathbf{M} = e \int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{y} \times \frac{\partial \mathbf{y}}{\partial t} \delta(\mathbf{x} - \mathbf{y}) = e \int_{\sigma_2}^{\sigma_1} d\sigma \frac{\partial \mathbf{y}}{\partial \sigma} \times \frac{\partial \mathbf{y}}{\partial t} \delta(\mathbf{x} - \mathbf{y}(\sigma, t)) . \tag{A20}$$

Next we show explicitly that Eq. (A18) for $\mathbf{P}$ and (A20) for $\mathbf{M}$ give via Eq. (A8) the current density. From Eq. (A18) we have

$$\frac{\partial \mathbf{P}}{\partial t} = e\frac{\partial}{\partial t}\int_{\mathbf{x}_2(t)}^{\mathbf{x}_1(t)} d\mathbf{y}(\sigma,t)\delta(\mathbf{x}-\mathbf{y}(\sigma,t))$$
$$= e\left[\frac{d\mathbf{x}_1}{dt}\delta(\mathbf{x}-\mathbf{x}_1(t)) - \frac{d\mathbf{x}_2}{dt}\delta(\mathbf{x}-\mathbf{x}_2(t))\right] + \frac{e}{\delta t}\left[\int_{\mathbf{x}_2(t)}^{\mathbf{x}_1(t)} d\mathbf{y}(\sigma,t+\delta t)\delta(\mathbf{x}-\mathbf{y}(\sigma,t+\delta t))\right.$$
$$\left. - \int_{\mathbf{x}_2} \mathbf{x}_1 d\mathbf{y}(\sigma,t)\delta(\mathbf{x}-\mathbf{y}(\sigma,t))\right]$$
$$= \mathbf{j}(\mathbf{x},t) + \frac{e}{\delta t}\oint d\mathbf{y}\,\delta(\mathbf{x}-\mathbf{y}) \ . \tag{A21}$$

The first term on the right-hand side of Eq. (A21) arises from differentiating with respect to the end points with the path fixed. The line integral in Eq. (A21) is over a closed contour running from $\mathbf{x}_2(t)$ to $\mathbf{x}_1(t)$ along the path $\mathbf{y}(\sigma, t+\delta t)$ and returning to $\mathbf{x}_2(t)$ along $\mathbf{y}(\sigma,t)$. [See Fig. 6(b).] We denote $\mathbf{y}(\sigma,t+dt) - \mathbf{y}(\sigma,t) = \delta\mathbf{y}$ and the element of area $d\mathbf{y}\times\delta\mathbf{y} \equiv d\mathbf{S}$. Then by Stokes' theorem

$$\frac{e}{\delta t}\oint d\mathbf{y}\,\delta(\mathbf{x}-\mathbf{y}) = -\frac{e}{\delta t}\int d\mathbf{S}\times\boldsymbol{\nabla}_y\delta(\mathbf{x}-\mathbf{y})$$
$$= e\int d\mathbf{y}\times\frac{\delta\mathbf{y}}{\delta t}\times\boldsymbol{\nabla}_x\delta(\mathbf{x}-\mathbf{y})$$
$$= -\boldsymbol{\nabla}\times\mathbf{M} \ . \tag{A22}$$

Equations (A21) and (A22) yield Eq. (A8) with $\mathbf{j}$ given by Eq. (A17) as asserted.

Equations (A14) and (A15) with $\mathbf{P}$ and $\mathbf{M}$ given by Eqs. (A18) and (A24), respectively, determine C_μ. To obtain the explicit form of the covariant version of these equations we note that

$$G^s_{\mu\nu} = -e\epsilon_{\mu\nu\alpha\beta}\int_{\tau_2}^{\tau_1} d\tau\int_{\sigma_2}^{\sigma_1} d\sigma\frac{\partial y^\alpha}{\partial\tau}\frac{\partial y^\beta}{\partial\sigma}\delta^4(x-y) \ , \tag{A23}$$

where

$$x^\mu = (t,\mathbf{x}), \quad y^\mu = (y^0,\mathbf{y}) \ ,$$

and

$$d\tau = \sqrt{(dy^0)^2 - (d\mathbf{y})^2} \ .$$

Equation (A23) is the standard covariant form for the Dirac string field $G^s_{\mu\nu}$ [18]. To show that expressions (A11) and (A23) for $G^s_{\mu\nu}$ are the same first set $\mu = 0$ and $\nu = k$ in Eq. (A23):

$$G^s_{0k} = -e\epsilon_{kmn}\int_{\tau_2}^{\tau_1} d\tau\int_{\sigma_2}^{\sigma_1} d\sigma\frac{\partial y^m}{\partial\tau}\frac{\partial y^n}{\partial\sigma}\delta^3(\mathbf{x}-\mathbf{y})\delta(y_0-t)$$
$$= e\int_{\sigma_2}^{\sigma_1}\left(\frac{\partial\mathbf{y}}{\partial\sigma}\times\frac{\partial\mathbf{y}}{\partial t}\right)_k\delta^3(\mathbf{x}-\mathbf{y}) = M_k \ . \tag{A24a}$$

Next set $\mu = i$ and $\nu = j$ in Eq. (A23):

$$G^s_{ij} = -e\epsilon_{ij0k}\int_{\tau_1}^{\tau_2} d\tau\int_{\sigma_2}^{\sigma_1}\frac{\partial y^0}{\partial\tau}\frac{\partial y^k}{\partial\sigma}\delta^3(\mathbf{x}-\mathbf{y})\delta(y_0-t)$$
$$= -e\epsilon_{ijk}\int_{\sigma_2}^{\sigma_1} d\sigma\frac{\partial y^k}{\partial\sigma}\delta^3(\mathbf{x}-\mathbf{y}) = -\epsilon_{ijk}P_k \ . \tag{A24b}$$

Thus Eq. (A23) is just the covariant version of Eqs. (A18) and (A20).

Equation (A12) with $\mu = 1$, and $G^s_{\mu\nu}$ given by Eq. (A23) is the covariant form of Eqs. (A14) and (A15) determining the dual potential produced by a pair of oppositely charged particles moving in the vacuum. The resulting C_μ will depend upon the location of the string, but this dependence will drop out in the expression for the electromagnetic field tensor $G_{\mu\nu}$. We will show below how Eqs. (A14) and (A15) with $\mathbf{P}$ and $\mathbf{M}$ given by Eqs. (A18) and (A20) produce the usual expressions for the electric and magnetic fields of slowly moving particles.

Finally we note that the equation of motion (1.12) with $\mu = 1$, namely,

$$\partial^\mu G_{\mu\nu} = 0 \ , \tag{A25}$$

can be obtained from a Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = \tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} = \tfrac{1}{2}(\mathbf{H}^2 - \mathbf{D}^2) \ . \tag{A26}$$

We will see in Sec. V that this Lagrangian gives not only the field equations but also the particle equations of motion.

To understand better how dual potentials work we will solve Eqs. (A14) and (A15) for slowly moving particles. First consider charges at rest. Then $\mathbf{M} = 0$ and $\mathbf{P}$ is time independent, and Eq. (A14) becomes $\nabla^2 C_0 = 0$ (i.e., $C_0 = 0$), and Eq. (A15) reduces to

$$\boldsymbol{\nabla}\times(-\boldsymbol{\nabla}\times\mathbf{C}_D) = \boldsymbol{\nabla}\times\mathbf{P} \ , \tag{A27}$$

where we have denoted the static solution $\mathbf{C} = \mathbf{C}_D$ (for Dirac). Equation (A27) has the form of the equation of the vector potential due to a polarization current pro-

duced by the superposition (A18) of dipoles. Thus, $\mathbf{C}_D$ is just the vector potential produced by a superposition of point dipoles of strength $-e\,d\mathbf{y}$ distributed uniformly along the string [see Eq. (A18)], $\mathbf{C}_D$ is given by [21]

$$\mathbf{C}_D(\mathbf{x}) = -\frac{e}{4\pi}\int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{y}\times\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\ . \tag{A28}$$

Then

$$\begin{aligned}-\nabla\times\mathbf{C}_D &= \frac{e}{4\pi}\left\{\int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{y}\,\nabla\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} - \int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{y}\cdot\nabla\frac{\mathbf{x}-\mathbf{y}}{|x-\mathbf{y}|^3}\right\}\\ &= \frac{e}{4\pi}\left\{\int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{y}\,4\pi\delta(\mathbf{x}-\mathbf{y}) + \frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3} - \frac{\mathbf{x}-\mathbf{x}_2}{|\mathbf{x}-\mathbf{x}_2|^3}\right\}\\ &= \mathbf{P}+\mathbf{D}_C\ ,\end{aligned} \tag{A29}$$

where

$$\mathbf{D}_C \equiv \mathbf{D}_{\text{Coulomb}} = \frac{e}{4\pi}\left(\frac{\mathbf{x}-\mathbf{x}_1}{|x-\mathbf{x}_1|^3} - \frac{\mathbf{x}-\mathbf{x}_2}{|\mathbf{x}-\mathbf{x}_2|^3}\right)\ , \tag{A30}$$

so that

$$\mathbf{D} = -\nabla\times\mathbf{C}_D - \mathbf{P} = \mathbf{D}_C\ . \tag{A31}$$

The above elementary derivation of Coulomb's law indicates that it really is not too much larger to work with dual potentials and strings then to work with ordinary potentials and localized charges. The string cancellation mechanism in Eq. (A31) is depicted in Fig. 6(c) in which we have taken the string to be a straight line connecting $\mathbf{x}_2$ and $\mathbf{x}_1$. We see that $-\nabla\times\mathbf{C}_D$ gives a divergence free field distribution. The singular field passing through the line L is canceled by the singular polarization $\mathbf{P}$, leaving a Coulomb field with a source at $\mathbf{x}_1$ and a sink at $\mathbf{x}_2$.

Next, let us solve Eqs. (A14) and (A15) to first order in $\mathbf{v}_1$ and $\mathbf{v}_2$ and to zero order in the accelerations $\dot{\mathbf{v}}_1$ and $\dot{\mathbf{v}}_2$. First look at Eq. (A15). We choose the gauge $\nabla\cdot\mathbf{C}=0$ (note $\nabla\cdot\mathbf{C}_D=0$). Then Eq. (A14) becomes

$$-\nabla^2 C_{0D} = -\nabla\times\mathbf{M}\ , \tag{A32}$$

where we have denoted the solution $C_0 = C_{0D}$. Equation (A32) has the form of the equation for the scalar potential due to a polarization charge produced by the superposition (A20) of dipoles. Hence C_{0D} is just the scalar potential produced by a superposition of point dipoles of strength $e\,d\mathbf{y}\times\dot{\mathbf{y}}$ distributed uniformly along the string; i.e., C_{0D} is [21]

$$C_{0D} = \frac{e}{4\pi}\int_{\mathbf{x}_2(t)}^{\mathbf{x}_1(t)} (d\mathbf{y}\times\dot{\mathbf{y}})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\ . \tag{A33}$$

From Eqs. (A18) and (A33) we see that the time derivative of $\mathbf{M}$ and the time derivative of $\mathbf{M}$ and $\mathbf{C}_{0D}$ do not contain terms linear in the velocities. The same is true for $\partial_0^2\mathbf{C}_D$ calculated from Eq. (A31) with $\mathbf{x}_1\to\mathbf{x}_1(t)$, $\mathbf{x}_2\to\mathbf{x}_2(t)$. Hence to first order in the velocities Eq. (A15) reduces to Eq. (A27), and so $\mathbf{C}=\mathbf{C}_D$ and $\mathbf{D}=\mathbf{D}_C$.

To calculate $\mathbf{H}$ we use Eq. (A9) with $\mathbf{C}=\mathbf{C}_D$ and $C_0 = C_{0D}$. We first calculate

$$-\frac{\partial}{\partial t}\mathbf{C}_D = \frac{e}{4\pi}\frac{\partial}{\partial t}\int_{\mathbf{x}_2(t)}^{\mathbf{x}_1(t)} d\mathbf{y}\times\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\ . \tag{A34}$$

The evaluation of the right-hand side of Eq. (A34) parallels that of Eq. (A21) and we obtain

$$-\frac{\partial}{\partial t}\mathbf{C}_D = \mathbf{H}_{\text{BS}} + \frac{e}{\delta t}\oint d\mathbf{y}\times\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\ , \tag{A35}$$

where the term

$$\mathbf{H}_{\text{BS}} \equiv \frac{e}{4\pi}\left\{\mathbf{v}_1\times\frac{\mathbf{x}-\mathbf{x}_1(t)}{|\mathbf{x}-\mathbf{x}_1(t)|^3} - \mathbf{v}_2\times\frac{\mathbf{x}-\mathbf{x}_2(t)}{|\mathbf{x}-\mathbf{x}_2(t)|^3}\right\}\ , \tag{A36}$$

arises from time differentiation of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ in Eq. (A34) leaving the path fixed. The line integral in (A35), over the same contour occurring in Eq. (A19), arises from moving the string keeping the end points fixed. Paralleling Eq. (A20) we then apply Stokes' theorem to obtain

$$\begin{aligned}\frac{e}{\delta t}\oint\frac{d\mathbf{y}}{4\pi}\times\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} &= -\frac{e}{\delta t}\int\frac{d\mathbf{S}\times\nabla_y}{4\pi}\times\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\\ &= e\int_{x_2(t)}^{\mathbf{x}_1(t)}\left(d\mathbf{y}\times\frac{\delta\mathbf{y}}{\delta t}\right)\left(\nabla_y\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\right) + \nabla_x\frac{e}{4\pi}\int d\mathbf{y}\times\frac{\delta\mathbf{y}}{\delta t}\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\\ &= -\mathbf{M}+\nabla C_{0D}\ .\end{aligned} \tag{A37}$$

Equations (A35) and (A37) then yield

$$\mathbf{H} = -\frac{\partial\mathbf{C}_D}{\partial t} - \nabla C_{0D} + M = \mathbf{H}_{\text{BS}}\ . \tag{A38}$$

Thus we see that the Biot-Savart magnetic field $\mathbf{H}_{\text{BS}}$ comes from the time derivative of the limits $\mathbf{x}_2(t)$ and $\mathbf{x}_1(t)$ in the integral for $\mathbf{C}_D$, Eq. (A34). The remaining string dependent part of $\partial\mathbf{C}_D/\partial t$ cancels the contribu-

tions to $\mathbf{H}$ coming from C_{0D} and $\mathbf{M}$.

We now use the solutions for $\mathbf{C}_D$, $\mathbf{C}_{0D}$, $\mathbf{D}_C$, and $\mathbf{H}_{\rm BS}$ to eliminate the fields in the Lagrangian L given by

$$L = \int d\mathbf{x}\,\mathcal{L} = \frac{1}{2}\int d\mathbf{x}(\mathbf{H}^2 - \mathbf{D}^2)\ , \tag{A39}$$

to second order in the velocities of the charged particles. The Lagrangian L, defined as the integral over the Lagrangian density (A26) then becomes a function $L = L(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2)$ only of the positions and velocities of the charged particles. To higher order in the velocities one cannot eliminate the field degrees of freedom in L because of the presence of radiation.

For particles at rest we have $\mathbf{C} = \mathbf{C}_D$, $C_0 = 0$, $\mathbf{H} = 0$, $\mathbf{D} = \mathbf{D}_C$, and

$$L(\mathbf{v}_1 = \mathbf{v}_2 = 0) = -\int d\mathbf{x}\frac{1}{2}\mathbf{D}_C^2 = \frac{e^2}{4\pi|\mathbf{x}_1 - \mathbf{x}_2|}\ , \tag{A40}$$

where the self-energy has been subtracted. To first order in the velocities, $\mathbf{C} = \mathbf{C}_D$ and $\mathbf{D} = \mathbf{D}_C$ given by Eq. (A30) with $\mathbf{x}_1 \to \mathbf{x}_1(t), \mathbf{x}_2 \to \mathbf{x}_2(t)$. In other words the static field configuration follows adiabatically the motion of the charged particles. Furthermore since the Lagrangian L is stationary about static solutions of the field equations we have

$$-\int d\mathbf{x}\frac{1}{2}\mathbf{D}^2 = \frac{e^2}{4\pi|\mathbf{x}_1(t) - \mathbf{x}_2(t)|}\ ,$$

valid to second order in the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$.

All the velocity dependence in L then comes from $\int \mathbf{H}^2$ which to second order in the velocities is

$$\frac{1}{2}\int d\mathbf{x}\,\mathbf{H}^2 = \frac{1}{2}\int d\mathbf{x}(\mathbf{H}_{\rm BS})^2$$

$$= -\frac{1}{2}\frac{e^2}{4\pi R}\left[\mathbf{v}_1\cdot\mathbf{v}_2 + \frac{\mathbf{v}_1\cdot\mathbf{R}\mathbf{v}_2\cdot\mathbf{R}}{R^2}\right]\ , \tag{A41}$$

where the self-energies have again been subtracted out and where $\mathbf{R} = \mathbf{x}_1(t) - \mathbf{x}_2(t)$. Hence, we obtain, for the second-order Lagrangian L (first obtained by Darwin [23]),

$$L(\mathbf{x}_1, \mathbf{x}_2; \mathbf{v}_1, \mathbf{v}_2)$$

$$= \frac{1}{2}\int d\mathbf{x}[\mathbf{H}_{\rm BS}^2 - \mathbf{D}_C^2]$$

$$= \frac{e^2}{4\pi R} - \frac{1}{2}\frac{e^2}{4\pi R}\left[\mathbf{v}_1\cdot\mathbf{v}_2 + \frac{\mathbf{v}_1\cdot\mathbf{R}\mathbf{v}_2\cdot\mathbf{R}}{R^2}\right]\ . \tag{A42}$$

As a final remark we connect the notation of this paper to that used in our previous work [7] on QCD where we have introduced string fields $\mathbf{D}_s$ and $\mathbf{H}_s$ defined as

$$\mathbf{D}_s \equiv -\mathbf{P},\quad \mathbf{H}_s \equiv \mathbf{M}\ , \tag{A43}$$

so that Eqs. (A9) take the form

$$\mathbf{D} = -\boldsymbol{\nabla}\times\mathbf{C} + \mathbf{D}_s,\quad \mathbf{H} = -\boldsymbol{\nabla}C_0 - \frac{\partial\mathbf{C}}{\partial t} + \mathbf{H}_s\ . \tag{A44}$$

The fields $\mathbf{D}_s$ and $\mathbf{H}_s$ then cancel the string contributions to $-\boldsymbol{\nabla}\times\mathbf{C}$ and $-\boldsymbol{\nabla}C_0 - \partial\mathbf{C}/\partial t$ yielding fields $\mathbf{D}$ and $\mathbf{H}$ free of string singularities. For slowly moving particles this mechanism is explicitly exhibited by Eqs. (A29) and (A37).

The action S describing the electromagnetic interactions of a particle of charge e and mass m_1 with a particle of charge $-e$ and mass m_2 is

$$S = -m_1\int_{t_2}^{t_1}\sqrt{1-\mathbf{v}_1^2}\,dt - m_2\int_{t_2}^{t_1}\sqrt{1-\mathbf{v}_2^2}\,dt$$

$$+\int d^4x\,\mathcal{L}\ , \tag{A45}$$

where $\mathcal{L}$ is given by Eqs. (A26) and (A10). Varying C_μ in the action S gives the field equation (A25). To obtain the equations of motion for the particles and for the string which connects them we vary the string coordinates: $y^\lambda \to y^\lambda + \delta y^\lambda$ and correspondingly vary the particle positions $x_1^\lambda \to x_1^\lambda + \delta x_1^\lambda$, $x_2^\lambda \to x_2^\lambda + \delta x_2^\lambda$ such that

$$\delta x_1^\lambda(\tau) = \delta y^\lambda(\sigma_1, \tau),\quad \delta x_2^\lambda(\tau) = \delta y^\lambda(\sigma_2, \tau)\ . \tag{A46}$$

Denote

$$\frac{\partial y^\lambda}{\partial\sigma} = y'^\lambda,\quad \frac{\partial y^\lambda}{\partial\tau} = \dot{y}^\lambda\ . \tag{A47}$$

Then

$$\delta\left(-\frac{1}{4}\int d^4x\,G^{\mu\nu}G_{\mu\nu}\right) = -\frac{1}{2}\int d^4x\,G^{\mu\nu}(x)\delta G^s_{\mu\nu}(x)\ , \tag{A48}$$

where

$$\delta G^s_{\mu\nu} = -e\epsilon_{\mu\nu\lambda\sigma}\int_{\tau_2}^{\tau_1}d\tau\int_{\sigma_2}^{\sigma_1}d\sigma\{[\delta\dot{y}'^\lambda y'^\alpha + \dot{y}^\lambda\delta y'^\alpha]\delta(x-y) + \dot{y}^\lambda y'^\alpha\partial_{y\beta}\delta(x-y)\delta y^\beta\}\ . \tag{A49}$$

Hence,

$$
\begin{aligned}
-\frac{1}{2}\int dx\, G^{\mu\nu}\delta G^s_{\mu\nu} &= \frac{e}{2}\epsilon_{\mu\nu\lambda\alpha}\int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma\, G^{\mu\nu}(y)(\delta\dot{y}^\lambda y'^\alpha + \dot{y}^\lambda \delta y'^\alpha) + \dot{y}^\lambda y'^\alpha \partial_\beta G^{\mu\nu}(y)\delta y^\beta \\
&= \frac{e}{2}\epsilon_{\mu\nu\lambda\alpha}\int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \Bigg\{ -\delta y^\lambda \frac{d}{d\tau}(G^{\mu\nu} y'^\alpha) - \delta y^\alpha \frac{d}{d\sigma}(G^{\mu\nu}\dot{y}^\lambda) + \delta y^\beta \dot{y}^\lambda y'^\alpha \partial_\beta G^{\mu\nu} \\
&\quad + \frac{d}{d\tau}(\delta y^\lambda G^{\mu\nu} y'^\alpha + \frac{d}{d\sigma}(\delta y^\alpha G^{\mu\nu}\dot{y}^\lambda) \Bigg\} \\
&= e\frac{\epsilon_{\mu\nu\lambda\alpha}}{2}\int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \Bigg\{ \delta y^\lambda y'^\alpha \dot{y}^\beta \partial_\beta G^{\mu\nu} - \delta y^\alpha \dot{y}^\lambda y'^\beta \partial_\beta G^{\mu\nu} + \delta y^\beta \dot{y}^\lambda y'^\alpha \partial_\beta G^{\mu\nu} \\
&\quad + \frac{d}{d\sigma}(\delta y^\alpha G^{\mu\nu}\dot{y}^\lambda) \Bigg\} \\
&= \frac{e}{2}\int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \Bigg\{ (\delta y^\lambda y'^\alpha \dot{y}^\beta)(-\epsilon_{\mu\nu\lambda\alpha}\partial_\beta G^{\mu\nu} - \epsilon_{\mu\nu\beta\lambda}\partial_\alpha G^{\mu\nu} + \epsilon_{\mu\nu\beta\alpha}\partial_\lambda G^{\mu\nu}) \\
&\quad + \epsilon_{\mu\nu\lambda\alpha}\frac{d}{d\sigma}(\delta y^\alpha G^{\mu\nu}\dot{y}^\lambda) \Bigg\} ,
\end{aligned}
\tag{A50}
$$

where we have used the fact that the variations of δy^λ vanish at τ_2 and τ_1.

Next we use the identity

$$
-(\epsilon_{\mu\nu\lambda\alpha}\partial_\beta + \epsilon_{\mu\nu\beta\lambda}\partial_\alpha + \epsilon_{\mu\nu\alpha\beta}\partial_\lambda)G^{\mu\nu} = 2\epsilon_{\mu\lambda\alpha\beta}\partial_\nu G^{\nu\mu} \tag{A51}
$$

and obtain

$$
-\frac{1}{2}dx\, G^{\mu\nu}\delta G^s_{\mu\nu} = e\int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma\, \delta y^\lambda g'^\alpha \dot{y}^\beta \epsilon_{\mu\lambda\alpha\beta}\partial_\nu G^{\nu\mu} + \frac{e}{2}\epsilon_{\mu\nu\lambda\alpha}\int_{\tau_2}^{\tau_1} d\tau(\delta x_1^\alpha G^{\mu\nu}(x_1)\dot{x}_1^\lambda - \delta x_2^\alpha G^{\mu\nu}(x_2)\dot{x}_2^\lambda) . \tag{A52}
$$

We must add to the above variation that of the particle action S_P:

$$
\delta S_P = \delta\int_{\tau_2}^{\tau_1} d\tau \left[-m_1\sqrt{\dot{x}_1^2} - m_2\sqrt{\dot{x}_2^2}\right] = \int_{\tau_2}^{\tau_1} d\tau(-m_1\dot{x}_{1\alpha}\delta\dot{x}_1^\alpha - m_2\dot{x}_{2\alpha}\delta\dot{x}_2^\alpha) = \int_{\tau_2}^{\tau_1}(m_1\ddot{x}_{1\alpha}\delta x_1^\alpha + m_2\ddot{x}_{2\alpha}\delta x_2^\alpha) . \tag{A53}
$$

The total change in the action δS due to a change in particle coordinates is

$$
\begin{aligned}
\delta S &= \delta S_P + \delta\left(-\frac{1}{4}\int dx\, G_{\mu\nu}G^{\mu\nu}\right) \\
&= \int_{\tau_2}^{\tau_1} d\tau \left[\delta x_1^\alpha\left(m_1\ddot{x}_{1\alpha} + \frac{e}{2}\epsilon_{\mu\nu\lambda\alpha}G^{\mu\nu}(x_1)\dot{x}_1^\lambda\right) + \delta x_2^\alpha\left(m_2\ddot{x}_{2\alpha} - \frac{e}{2}\epsilon_{\mu\nu\lambda\alpha}G^{\mu\nu}(x_2)\dot{x}_2^\lambda\right)\right] .
\end{aligned}
\tag{A54}
$$

In proceeding from (A52) to (A54) we used the field equation (A25) which eliminates the string contribution to the variation of the action in Eq. (A52).

If we had introduced further interactions [7] of the C_μ so that $\partial_\mu G^{\mu\nu} \neq 0$, then there would have been additional variations of the action arising from the first term in Eq. (A52). In that case Hamilton's principle $\delta S = 0$ gives

$$
e\int_{\sigma_2}^{\sigma_1} d\sigma\, y'^\alpha y'^\beta \epsilon_{\mu\lambda\alpha\beta}\partial_\nu G^{\nu\mu}(y) = 0 , \tag{A55}
$$

in addition to the Lorentz force equations

$$
m_1\ddot{x}_{1\alpha} = \frac{e}{2}\epsilon_{\alpha\mu\nu\lambda}G^{\mu\nu}(x_1)\dot{x}_1^\lambda
$$

and

$$
m_2\ddot{x}_{2\alpha} = \frac{e}{2}\epsilon_{\alpha\mu\nu\lambda}G^{\mu\nu}(x_2)\dot{x}_2^\lambda , \tag{A56}
$$

following from Eq. (A54). Thus Eq. (A55) provides a boundary condition along the strings upon the current $\partial_\mu G^{\nu\mu}$.

Our purpose in writing the appendix is pedagogical, though of course, motivated by our interest in using dual potentials in QCD. We have seen how normal classical electrodynamics can be handled completely in terms of dual potentials, and that the use of these potentials gives the solution to the conventional Maxwell equations for the electric and magnetic fields and leads to the usual Lorentz force law for the motion of charged particles. While dual potentials provide a somewhat awkward way to solve electrodynamics when charges are present, the are nevertheless the natural variables to describe a dielectric medium with long-distance antiscreening. This

is the underlying reason for their utility in describing long-range QCD.

APPENDIX B: USE OF THE SALPETER EQUATION

There are several important reasons for seeking a relativistic treatment of quark-antiquark bound states. First of all, for quarks moving in a Coulomb potential $v/c \simeq \alpha$ where α is the coefficient of the $1/R$ singularity, this number is about 0.5 for our potential, making even the $b\bar{b}$ states quite relativistic. This certainly causes some doubt as to the reliability of calculations that employ the Schrödinger equation, and quantities that depend on short-range behavior are particularly suspect. A second reason is the need to treat lower mass quarks where relativistic effects will be even more important.

The Salpeter equation

$$\left(\sqrt{p^2+m_1^2}+\sqrt{p^2+m_2^2}+V\right)\Psi = E\Psi \tag{B1}$$

represents a simple generalization to that the quark kinematics relativistically. We have used the method proposed by Fulcher [24] employing a finite number of basis functions to obtain a matrix representation of the operator on the left of Eq. (B1). The eigenvalues and eigenvectors are easily calculated for our central potential.

The next step in the calculation of quarkonium energy levels is the perturbative calculation of the splitting produced by the potentials given in Eq. (5.2). Because of the δ function that appears in V_4 this is not a trivial step.

It has been shown by Durand *et al.* [25] that the Salpeter wave functions behaves as $1/R^{1-\gamma}$ for a potential that behaves as α/R near the origin, where

$$\gamma = \frac{\alpha}{2}\tan\left(\frac{1}{2}\pi\gamma\right) . \tag{B2}$$

Note that as $\alpha \to 4/\pi$, γ goes to zero. For larger values of α, the solution of Eq. (B2) shifts to another branch of the tangent and no normalizable wave function exists. Fortunately, our value of $\frac{4}{3}\alpha_s$ is about 0.5 so that a solution exists but $\gamma \simeq 0.8$. If one uses a running coupling constant to weaken the singularity at the origin, the power behavior disappears, but Durand [26] has shown that a logarithmic singularity remains and $\Psi(0)$ is still infinite.

We have developed a method to regularize the δ-function terms. Recall that the δ function in V_4 is the result of taking the nonrelativistic limit of the Dirac equation in which the vector potential of the nucleus is treated as a perturbation. The hyperfine matrix element is

$$\Delta E \sim \int d^3r\, \bar{\Psi}\alpha\cdot\mathbf{A}\Psi , \tag{B3}$$

where

$$\mathbf{A} = \frac{\boldsymbol{\mu}\times\mathbf{r}}{r^3} , \tag{B4}$$

where $\boldsymbol{\mu}$ is the nuclear magnetic moment. Writing this matrix element in terms of the large g and small f components of the Dirac spinor we obtain

$$\Delta E \sim \int dr\, f(r)g(r) . \tag{B5}$$

Note that this is finite even though, like the Salpeter wave function, the Dirac wave functions are singular at the origin. If we eliminate the small component we find

$$\Delta E \sim -\int dr \frac{2m}{E+m-V} g(r)\frac{dg}{dr} , \tag{B6}$$

where V is the Coulomb potential. In this form it is clear that V in the denominator is regulating the singularity at the origin. Taking the nonrelativistic limit we obtain

$$\Delta E \sim -\int dr\, g(r)\frac{dg}{dr} = -\frac{1}{2}g(0)^2 . \tag{B7}$$

This would be a problem if it were not for the fact that g now is the Schrödinger wave function which is well behaved at the origin. This is exactly the same term that the δ-function potential produces. Evidently the correct replacement for the δ function for a relativistic wave function is

$$\delta^3(\mathbf{r}) \to -\frac{4m}{E+m-V}\Psi(r)\frac{d\Psi}{dr} . \tag{B8}$$

This prescription has the further advantage that no new parameters have been introduced.

The fit to the heavy quark data using the Salpeter equation and the potential with a running coupling is shown in Table IV. The value of χ^2 was 2.6, significantly worse that our Schrödinger results ($\chi^2 = 1.25$).

For heavy-light quark–antiquark systems the Salpeter equation predicts the mass of the D, D^*, B, and B^* to be all too high by between 300 and 500 MeV. These values

TABLE IV. Salpeter equation fit to masses of all observed $c\bar{c}$ and $b\bar{b}$ states below threshold. Parameters are $\alpha_s = 0.2876$, $\sigma = 0.2285$ GeV^2, $m_c = 1.286$ GeV, and $m_b = 4.701$ GeV. $R_0 = 0.5$ fixed.

State	Pred. mass (GeV)	Exper. mass (GeV)
$\eta_c(1S)$	2.967	2.980
$\psi(1S)$	3.126	3.097
$\psi(2S)$	3.700	3.686
$\chi_{c_0}(1P)$	3.427	3.415
$\chi_{c_1}(1P)$	3.497	3.511
$\chi_{c_2}(1P)$	3.543	3.556
$h_c(1P)$	3.510	3.526
$\Upsilon(1S)$	9.511	9.460
$\Upsilon(2S)$	10.004	10.023
$\Upsilon(3S)$	10.350	10.355
$\Upsilon(4S)$	10.642	10.580
$\chi_{b_0}(1P)$	9.854	9.860
$\chi_{b_1}(1P)$	9.878	9.892
$\chi_{b_2}(1P)$	9.896	9.913
$\chi_{b_0}(2P)$	10.213	10.232
$\chi_{b_1}(2P)$	10.236	10.255
$\chi_{b_2}(2P)$	10.254	10.268

are the lowest that can be obtained, and occur for a light quark mass of 425 MeV, also a rather high number.

APPENDIX C: USE OF THE DIRAC EQUATION

In a recent paper Mur *et al.* [27] give theoretical arguments for the use of the Dirac equation to describe heavy-light quark–antiquark systems in QCD. We begin with the investigation of states of a light quark bound to a c or b quark, and determine the eigenvalues and eigenvectors of the Dirac equation in the limit that the heavy mass is infinite. The hyperfine splitting will then be calculated perturbatively, using the heavy quark mass obtained in our Schrödinger fit. The radial Dirac equation in matrix form for the small and large components is

$$\begin{pmatrix} V_v + m + V_s & -\frac{d}{dr} - \frac{l+1}{r} \\ \frac{d}{dr} - \frac{l}{r} & V_v - m - V_s \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix} = E \begin{pmatrix} g \\ f \end{pmatrix} , \tag{C1}$$

where V_s is the scalar potential and V_v is the fourth component of a four-vector. The conventional view is that the confining potential is a scalar and that the Coulomb term is a vector. For our potentials the fact that the string tension appears in V_1' tends to support this division, although in this application we really have no choice because there is no solution to the Dirac equation for a linear potential in V_v.

Each of the four matrices on the left of Eq. (C1) is expressed as an $N \times N$ matrix using the same basis function as in the solution of the Salpeter equation. The resulting eigenvalues and eigenvectors are then used to calculate the hyperfine splitting. For a pure Coulomb potential ($\alpha = 0.5$ and $m = 1$) using a 20×20 representation for the individual matrices, we obtained 0.866 028 for the ground-state energy to be compared to 0.866 025 for the exact result. For the linear potential we compared our results with those of Mur *et al.* [27] and found excellent agreement.

TABLE V. Predicted masses of B, B^*, D, and D^* mesons from the Dirac equation [28]. m_u is the light quark.

$m_u = 97$ MeV					
State	Theory	Expt.	State	Theory	Expt.
B	5.304 GeV	5.279	D	1.880	1.869
B^*	5.347	5.325	D^*	1.995	2.010

In the calculation of the masses of the heavy-quark light-quark systems we first adjusted the light-quark mass (the only free parameter in this calculation) to give the correct center of masses of the s-wave states and then calculated the splitting using this mass and the heavy-quark mass. The results are not very good.

A simple correction to the infinite heavy-quark mass limit, described above, is to introduce a nonrelativistic heavy-quark kinetic energy term [29]. Doing this significantly improves the results, as shown in Table V. The four masses of D, D^*, B, and B^* can be fitted with our theoretical χ^2 of 1.3, using just one parameter—the light quark mass—which comes out to be 97 MeV, a very reasonable value. This calculation is done neglecting the velocity dependence potential, and all but the perturbative part of the spin-spin potential, since this is already included in the Dirac equation.

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