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Three-loop free energy for high-temperature QED and QCD with fermions

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We compute the free energy density for gauge theories, with fermions, at high temperature and zero chemical potential. Specifically, we analytically compute the free energy through $O(g^4)$, which requires the evaluation of three-loop diagrams. This computation extends our previous result for pure gauge QCD.

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I. INTRODUCTION

The perturbative expansion of the free energy of high-temperature gauge theory has the form

$$F \sim T^4[c_0 + c_2 g^2 + c_3 g^3 + (c_4' \ln g + c_4) g^4 + O(g^5)]\,, \tag{1.1}$$

where the c_i are numerical coefficients (with some dependence on the choice of renormalization scale) and where we have assumed the temperature high enough that fermion masses can be ignored. In a previous work [1], we showed how to compute the coefficient c_4 of g^4 in pure, non-Abelian gauge theory from three-loop diagrams. We shall now incorporate fermions into the theory and so obtain a three-loop result for QED and real QCD. This computation is a mostly straightforward extension of our previous work, and so we refer the reader to that work for motivation and pedagogy. In fact, the basic calculations we need to do for fermions very closely parallel those we did previously for bosons, and our object in this paper will simply be to point out the relevant differences, catalog results for the basic building blocks of three-loop calculations, and present our final results.

In the next section, we fix our notation and conventions for coupling constants, group factors, and so forth. In Sec. II, we outline the basic integrals that are needed in order to compute fermionic contributions to the free energy. In Sec. III, we show how to derive analytic results for those basic integrals, though many of the details are left for appendices. Finally, in Sec. IV we present our result for the free energy and discuss its sensitivity to the choice of renormalization scale.

II. NOTATION AND CONVENTIONS

We'll consider gauge theories given by classical Euclidean Lagrangians of the form

$$\mathcal{L}_\mathrm{E} = \bar\psi\left(\partial_\mu - igA_\mu^a T^a\right)\gamma_\mu\psi + \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c)^2 + (\text{gauge fixing})\,, \tag{2.1}$$

where the T^a are the generators of a single, simple Lie group, such as U(1) or SU(3). To simplify presentation, we will not derive results for an arbitrary product of simple Lie groups such as SU(2)×U(1), but such cases could easily be handled simply by adjusting the overall group and coupling factors on the results we give for individual diagrams. d_A and C_A are the dimension and quadratic Casimir of the adjoint representation, with C_A given by

$$f^{abc} f^{dbc} = C_\mathrm{A}\delta^{ad}\,. \tag{2.2}$$

d_F is the dimension of the total fermion representation (e.g., 18 for six-flavor QCD), and S_F and $S_{2\mathrm{F}}$ are defined in terms of the generators T^a for the total fermion representation as

$$S_\mathrm{F} = \frac{1}{d_\mathrm{A}}\mathrm{tr}(T^2)\,, \qquad S_{2\mathrm{F}} = \frac{1}{d_\mathrm{A}}\mathrm{tr}\big[(T^2)^2\big]\,, \tag{2.3}$$

where $T^2 = T^a T^a$. For SU(N) with n_f fermions in the fundamental representation, the standard normalization of the coupling gives

$$d_\mathrm{A} = N^2 - 1\,, \qquad C_\mathrm{A} = N\,, \qquad d_\mathrm{F} = N n_\mathrm{f}\,,$$

$$S_\mathrm{F} = \frac{1}{2} n_\mathrm{f}\,, \qquad S_{2\mathrm{F}} = \frac{N^2-1}{4N} n_\mathrm{f}\,. \tag{2.4}$$

For U(1) theory, relabel g as e, and let the charges of the n_f fermions be $q_i e$. Then

$$d_\mathrm{A} = 1\,, \quad C_\mathrm{A} = 0\,, \quad d_\mathrm{F} = n_\mathrm{f}\,,$$

$$S_\mathrm{F} = \sum_i q_i^2\,, \quad S_{2\mathrm{F}} = \sum_i q_i^4\,. \tag{2.5}$$

We shall work in the Feynman gauge. We also work exclusively in the Euclidean (imaginary time) formulation of thermal field theory. We shall conventionally refer to four-momenta with capital letters K and to their components with lower-case letters: $K = (k_0, \mathbf{k})$. All four-momenta are Euclidean with discrete frequencies $k_0 = 2\pi nT$ for bosons and ghosts, and $k_0 = 2\pi\left(n+\frac{1}{2}\right)T$ for fermions. We regularize the theory by working in $d = 4-2\epsilon$ dimensions with the modified minimal subtraction ($\overline{\text{MS}}$) scheme, which corresponds to doing minimal subtraction (MS) and then changing the MS scale μ to the $\overline{\text{MS}}$ scale $\bar{\mu}$ by the substitution

$$\mu^2 = \frac{e^{\gamma_{\text{E}}}\bar{\mu}^2}{4\pi}. \tag{2.6}$$

To denote summation over discrete loop frequencies and integration over loop three-momenta, we use the shorthand notation

$$\sum\!\!\!\!\!\!\!\int_P \to \mu^{2\epsilon} T \sum_{p_0} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \tag{2.7}$$

for bosonic momenta and

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \to \mu^{2\epsilon} T \sum_{\{p_0\}} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \tag{2.8}$$

for fermionic momenta, where

$$\sum_{p_0} \to \sum_{p_0 = 2\pi nT}, \qquad \sum_{\{p_0\}} \to \sum_{p_0 = 2\pi(n+\frac{1}{2})T}. \tag{2.9}$$

We shall also sometimes use the notation

$$\sum\!\!\!\!\!\!\!\int_{P+\{P\}} \to \sum\!\!\!\!\!\!\!\int_P + \sum\!\!\!\!\!\!\!\int_{\{P\}}. \tag{2.10}$$

We handle the resummation of hard thermal loops [which is required to make perturbation theory well-behaved beyond $O(g^2)$] as we did in Ref. [1]. Specifically, we must improve our propagators by incorporating the Debye screening mass M for A_0, which is determined at leading order by the self-energy diagrams of Fig. 1:

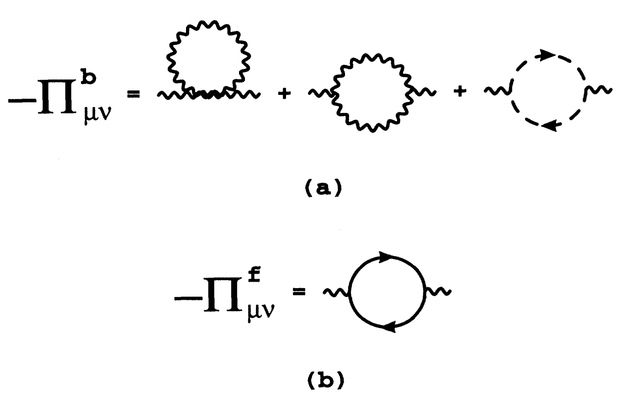

FIG. 1. The (a) bosonic and (b) fermionic contributions to the one-loop gluon self-energy.

$$\begin{aligned} M^2\delta^{ab} &= \Pi^{ab}_{00}(0) = \Pi^{ab}_{\mu\mu}(0) \\ &= g^2\delta^{ab}\left[C_{\text{A}}(d-2)^2 \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2}\right. \\ &\qquad \left. -4S_{\text{F}}(d-2)\sum\!\!\!\!\!\!\!\int_{\{Q\}} \frac{1}{Q^2}\right]. \end{aligned} \tag{2.11}$$

This is accomplished by rewriting our Lagrangian density, in frequency space, as

$$\mathcal{L}_{\text{E}} = \left(\mathcal{L}_{\text{E}} + \tfrac{1}{2}M^2 A_0^a A_0^a \delta_{p_0}\right) - \tfrac{1}{2}M^2 A_0^a A_0^a \delta_{p_0}, \tag{2.12}$$

where δ_{p_0} is shorthand for the the Kronecker delta function $\delta_{p_0,0}$. Then we absorb the first A_0^2 term into our unperturbed Lagrangian $\mathcal{L}_0$ and treat the second A_0^2 term as a perturbation.

III. THE BASIC INTEGRALS

The most basic one-loop integrals that appear in high-temperature field theory are of the form

$$b_n \equiv \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^{2n}}, \qquad f_n \equiv \sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^{2n}}. \tag{3.1}$$

The bosonic form of these integrals needed for the calculation were reviewed in Ref. [1] and are given by

$$b_1 = \frac{T^2}{12}\left[1 + \epsilon\left(2\ln\frac{\bar{\mu}}{4\pi T} + 2\frac{\zeta'(-1)}{\zeta(-1)} + 2\right)\right] + O(\epsilon^2), \tag{3.2}$$

$$b_2 = \frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2\ln\frac{\bar{\mu}}{4\pi T} + 2\gamma_{\text{E}}\right] + O(\epsilon). \tag{3.3}$$

As noted in Ref. [2], the f_n are then easily determined by considering $f_n + b_n$ and then scaling the momenta $(p_0, \mathbf{p})$ by 2 so that it becomes proportional to b_n. One finds

$$f_n = (2^{2n+1-d} - 1)b_n. \tag{3.4}$$

In Ref. [1], we reviewed how three-loop diagrams contributing to the free energy can be reduced to some simple sum integrals at $O(g^4)$. The most basic was

$$I^{\text{bb}}_{\text{ball}} \equiv \sum\!\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2Q^2R^2(P+Q+R)^2} = \sum\!\!\!\!\!\!\!\int_P \left[\Pi^{\text{b}}(P)\right]^2, \tag{3.5}$$

which corresponds to the basketball diagram of scalar theory, depicted in Fig. 2(a). Π^{b} is defined by

$$\Pi^{\text{b}}(P) \equiv \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(P+Q)^2}, \tag{3.6}$$

and we have introduced the superscript b for Π to indicate that it is defined with a bosonic frequency sum. When fermions are included, the reduction of diagrams to a few simple integrals requires introducing some fermionic relatives of $I^{\text{bb}}_{\text{ball}}$:

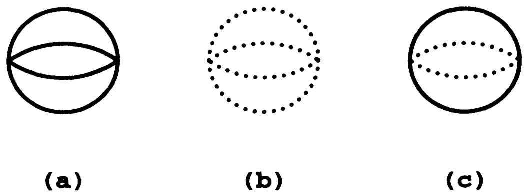

FIG. 2. The (a) bosonic, (b) fermionic, and (c) mixed scalar basketball diagrams. Solid (dotted) lines correspond to scalar propagators with bosonic (fermionic) frequencies.

$$I^{\rm ff}_{\rm ball} \equiv \sum\!\!\!\!\!\!\int_P \left[\Pi^{\rm f}(P)\right]^2 , \qquad (3.7)$$

$$I^{\rm bf}_{\rm ball} \equiv \sum\!\!\!\!\!\!\int_P \Pi^{\rm b}(P)\Pi^{\rm f}(P) , \qquad (3.8)$$

$$\Pi^{\rm f}(P) \equiv \sum\!\!\!\!\!\!\int_{\{Q\}} \frac{1}{Q^2(P+Q)^2} . \qquad (3.9)$$

These are depicted by Figs. 2(b) and 2(c).

Another basic integral encountered in the pure gauge theory case was the one associated with the scalar sunset diagram of Fig. 3(a), evaluated to leading order in masses:

$$I^{\rm b}_{\rm sun}(m_1,m_2,m_3)$$

$$\equiv \sum\!\!\!\!\!\!\int_{PQ} \frac{1}{(P^2+m_1^2)(Q^2+m_2^2)[(P+Q)^2+m_3^2]} . \qquad (3.10)$$

When fermions are included, one also needs

$$I^{\rm f}_{\rm sun} \equiv \sum\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2(P+Q)^2} = \sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\Pi^{\rm f}(P) , \qquad (3.11)$$

corresponding to Fig. 3(b). The above integral is infrared finite because fermionic Euclidean frequencies p_0 are never zero; so, unlike the bosonic case, the masses can be dropped at leading order in m/T. By using the same contour-trick argument that was used in Appendix F.1 of Ref. [1], one can easily show that

$$I^{\rm f}_{\rm sun} = 0 . \qquad (3.12)$$

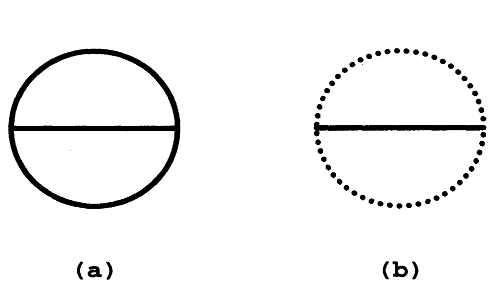

FIG. 3. The (a) bosonic and (b) fermionic setting sun diagrams.

However, it is sometimes convenient to also know the pieces of $I^{\rm f}_{\rm sun}$ corresponding to restricting the frequency sum in various ways, and these are discussed in Appendix E.

Finally, the pure gauge theory calculation required the integral corresponding to the bosonic piece of Fig. 4(l):

$$d_{\rm A} C_{\rm A}^2 g^4 I^{\rm bb}_{\rm QCD} \equiv \sum\!\!\!\!\!\!\int_P \frac{1}{P^4} {\rm tr}\,[\Delta\Pi^{\rm b}_{\mu\nu}(P)]^2 , \qquad (3.13)$$

where $\Pi^{\rm b}_{\mu\nu}$ is the bosonic contribution to the vector self-energy, given by Fig. 1(a), and the notation

$$\Delta\Pi_{\mu\nu}(P) \equiv \Pi_{\mu\nu}(P) - \Pi_{\mu\nu}(0)\delta_{p_0} \qquad (3.14)$$

has been used.

We shall need the same integral with the complete self-energy, which means we need

$$d_{\rm A} S_{\rm F}^2 g^4 I^{\rm ff}_{\rm QCD} \equiv \sum\!\!\!\!\!\!\int_P \frac{1}{P^4} {\rm tr}\,[\Delta\Pi^{\rm f}_{\mu\nu}(P)]^2 , \qquad (3.15)$$

$$d_{\rm A} C_{\rm A} S_{\rm F} g^4 I^{\rm bf}_{\rm QCD} \equiv \sum\!\!\!\!\!\!\int_P \frac{1}{P^4} {\rm tr}\,\Delta\Pi^{\rm b}_{\mu\nu}(P)\Delta\Pi^{\rm f}_{\mu\nu}(P) , \qquad (3.16)$$

where $\Pi^{\rm f}_{\mu\nu}$ is the fermionic contribution given by Fig. 1(b).

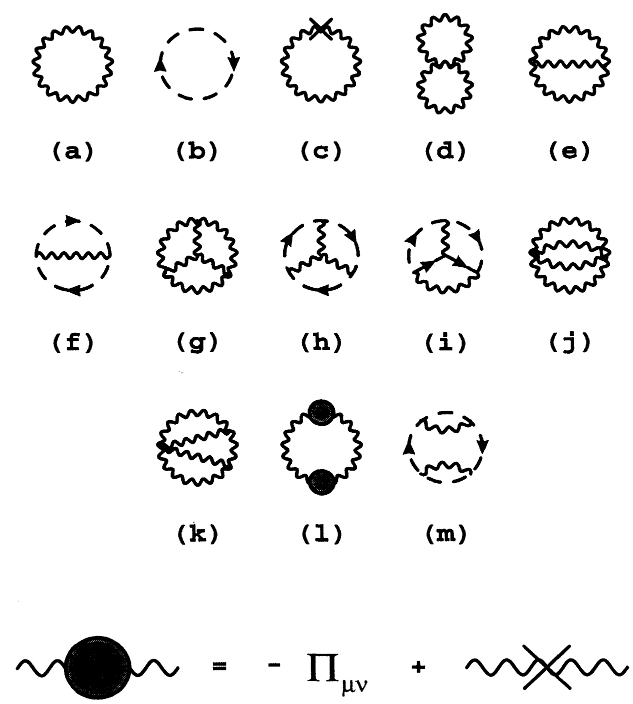

FIG. 4. Diagrams contributing to the free-energy in pure gauge theory. When fermions are added, we include the fermionic contribution to $\Pi_{\mu\nu}$ in diagram (l) and include the diagrams of Fig. 5. The crosses are the "thermal counter-terms" arising from the last term of (2.12), and the dashed lines are ghosts. We have not explicitly shown any zero-temperature counter-terms, and each diagram should be multiplied by the appropriate multiplicative renormalizations for vertices and propagators.

In the pure gauge theory case, all three-loop diagrams except Fig. 4(l) could be reduced to the basketball integral $I^{\rm bb}_{\rm ball}$ by the application of a few simple tricks. For instance, Fig. 4(i) is equal to

$$-\frac{1}{8}d_{\rm A}C_{\rm A}^2 g^4 \sum\!\!\!\!\!\!\int_{PQK} \frac{P\cdot(Q-K)\,(P-K)\cdot Q}{P^2Q^2K^2(P-Q)^2(Q-K)^2(K-P)^2} \tag{3.17}$$

and is reduced by (1) expanding numerator factors in terms of denominator factors to cancel factors between numerator and denominator, such as

$$P\cdot(Q-K) = \tfrac{1}{2}[(K-P)^2 - K^2 - (P-Q)^2 + Q^2], \tag{3.18}$$

(2) performing appropriate changes of variables to collect similar terms, and (3) using the identity

$$\sum\!\!\!\!\!\!\int_P \frac{P_\mu}{(P+Q)^2(P+K)^2} = -\frac{Q_\mu+K_\mu}{2} \times \sum\!\!\!\!\!\!\int_P \frac{1}{(P+Q)^2(P+K)^2}. \tag{3.19}$$

The last identity follows by changing variables $P\to -P-Q-K$,

$$\sum\!\!\!\!\!\!\int_P \frac{P_\mu}{(P+Q)^2(P+K)^2} = -\sum\!\!\!\!\!\!\int_P \frac{P_\mu}{(P+Q)^2(P+K)^2} -(Q_\mu+K_\mu)\sum\!\!\!\!\!\!\int_P \frac{1}{(P+Q)^2(P+K)^2}, \tag{3.20}$$

and then moving the first-term on the right-hand side over to the left-hand side. Unfortunately, this trick does not generalize to the case where $Q+K$ is fermionic instead of bosonic. If $Q+K$ is fermionic, then

$$\sum\!\!\!\!\!\!\int_P \frac{P_\mu}{(P+Q)^2(P+K)^2} = -\sum\!\!\!\!\!\!\int_{\{P\}} \frac{P_\mu}{(P+Q)^2(P+K)^2} -(Q_\mu+K_\mu)\sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{(P+Q)^2(P+K)^2}, \tag{3.21}$$

and there is no simple way to solve for the bosonic integral on the left-hand side. The failure of this trick

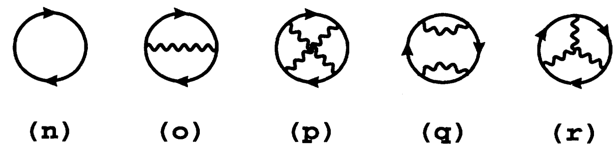

FIG. 5. Diagrams that must be added to Fig. 4 to include fermions in the calculation of the free energy.

requires us to introduce a new fundamental integral, as was done by Parwani and Corianò in Ref. [2]:[1]

$$H_3 \equiv \sum\!\!\!\!\!\!\int_{\{P\}QK} \frac{Q\cdot K}{P^2Q^2K^2(P+Q)^2(P+K)^2}. \tag{3.22}$$

If P were bosonic, this would be reducible by (3.19).

Figures 4 and 5 show all of the diagrams contributing to the free energy up to three loops. The reductions of all the three-loop diagrams to the basic integrals are given in Appendix A.

IV. RESULTS FOR INTEGRALS

A. The fermionic basketball $I^{\rm ff}_{\rm ball}$

The derivation of $I^{\rm ff}_{\rm ball}$ closely parallels that of the bosonic $I^{\rm bb}_{\rm ball}$ in Ref. [1] with the main difference being that the bosonic sum identity

$$\sum_{q_0} e^{-|q_0|r}e^{-|p_0+q_0|r} = (\coth\bar r + |\bar p_0|)e^{-|p_0|r} \tag{4.1}$$

is replaced by

$$\sum_{\{q_0\}} e^{-|q_0|r}e^{-|p_0+q_0|r} = ({\rm csch}\,\bar r + |\bar p_0|)e^{-|p_0|r}, \tag{4.2}$$

where p_0 represents bosonic frequencies and

$$\bar r \equiv 2\pi T r, \qquad \bar p_0 \equiv p_0/2\pi T. \tag{4.3}$$

This has the effect of simply replacing occurrences of coth (and its small r expansion) in the bosonic derivation by csch (and its small r expansion). So, for instance,

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[1] We have adopted their notation, H_3, for this integral. Their H_1 and H_2 correspond to our $I^{\rm bb}_{\rm ball}$ and $I^{\rm ff}_{\rm ball}$, respectively, and their H_4 is discussed in Appendix F.

$$\Pi^{f(T)}(P) = \frac{T}{(4\pi)^2} \int d^3r \, \frac{1}{r^2} e^{i\mathbf{p}\cdot\mathbf{r}} \left(\operatorname{csch}\bar{r} - \frac{1}{\bar{r}}\right) e^{-|p_0|r} + O(\epsilon) , \tag{4.4}$$

where $\Pi^{f(T)}$ is the finite-temperature contribution to Π^f. Similarly,

$$\begin{aligned}
\sum\!\!\!\!\!\!\int_P \left\{ [\Pi^{f(T)}(P)]^2 - \left(\frac{2}{P^2} \sum\!\!\!\!\!\!\int_{\{Q\}} \frac{1}{Q^2}\right)^2 \right\} &= \frac{T^4}{32\pi^2} \int_0^\infty d\bar{r}\, \bar{r}^{-2} \left[\left(\operatorname{csch}\bar{r} - \frac{1}{\bar{r}}\right)^2 - \left(\frac{-\bar{r}}{6}\right)^2\right] (\coth\bar{r} - 1) \\
&\quad + \frac{T^4}{32\pi^2} \int_0^\infty d\bar{r}\, \bar{r}^{-2} \left(\operatorname{csch}\bar{r} - \frac{1}{\bar{r}}\right)^2 + O(\epsilon) \\
&= \frac{1}{(4\pi)^2} \left(\frac{T^2}{12}\right)^2 \left[2\frac{\zeta'(-3)}{\zeta(-3)} - 2\gamma_E + \frac{73}{15} - \frac{6}{5}\ln 2\right] + O(\epsilon) .
\end{aligned} \tag{4.5}$$

The integrations in the last equation were performed using the method of Appendix C. Putting the above result together with (3.2), (3.3), and (3.4) gives

$$\sum\!\!\!\!\!\!\int [\Pi^{f(T)}]^2 = \frac{1}{(4\pi)^2} \left(\frac{T^2}{12}\right)^2 \left[\frac{1}{\epsilon} + 6\ln\frac{\bar{\mu}}{4\pi T} + 2\frac{\zeta'(-3)}{\zeta(-3)} + 4\frac{\zeta'(-1)}{\zeta(-1)} + \frac{133}{15} - \frac{26}{5}\ln 2\right] + O(\epsilon) . \tag{4.6}$$

The remaining terms needed to evaluate I^{ff}_{ball} are discussed in Appendix D. The final result for the fermionic basketball is

$$I^{ff}_{ball} = \frac{1}{(4\pi)^2} \left(\frac{T^2}{12}\right)^2 \left[\frac{3}{2\epsilon} + 9\ln\frac{\bar{\mu}}{4\pi T} - 3\frac{\zeta'(-3)}{\zeta(-3)} + 12\frac{\zeta'(-1)}{\zeta(-1)} + \frac{173}{20} - \frac{63}{5}\ln 2\right] + O(\epsilon) . \tag{4.7}$$

This agrees with the numerical result of Ref. [2].

B. The mixed basketball I^{bf}_{ball}

As has been noted by Parwani and Corianò [2], I^{bf}_{ball} can be written in terms of I^{bb}_{ball} and I^{ff}_{ball} by the trick of rescaling momenta $(p_0, \mathbf{p})$ by a factor of 2:

$$\begin{aligned}
I^{bb}_{ball} &= \sum\!\!\!\!\!\!\int_{PQKR} \frac{\delta(P+Q+K+R)}{P^2Q^2K^2R^2} \\
&= 2^{3d-11} \sum\!\!\!\!\!\!\int_{P+\{P\}} \sum\!\!\!\!\!\!\int_{Q+\{Q\}} \sum\!\!\!\!\!\!\int_{K+\{K\}} \sum\!\!\!\!\!\!\int_{R+\{R\}} \frac{\delta(P+Q+K+R)}{P^2Q^2K^2R^2} \\
&= 2^{3d-11} \left[I^{bb}_{ball} + 6 I^{bf}_{ball} + I^{ff}_{ball}\right] ,
\end{aligned} \tag{4.8}$$

so that

$$I^{bf}_{ball} = -\frac{1}{6}\left(1 - 2^{11-3d}\right) I^{bb}_{ball} - \frac{1}{6} I^{ff}_{ball} . \tag{4.9}$$

To simplify the notation above, we have used the shorthand

$$\delta(P+Q+K+R) \equiv \mu^{-2\epsilon} \frac{1}{T} \delta_{p_0+q_0+k_0+r_0} (2\pi)^{d-1} \delta^{(d-1)}(\mathbf{p}+\mathbf{q}+\mathbf{k}+\mathbf{r}) . \tag{4.10}$$

C. The new integral H_3

We now turn to the integral H_3 of (3.22), which is the one integral that is not directly analogous to a previous bosonic calculation. However, our attack on H_3 is inspired by our derivation of $I^{bb}_{qcd} \sim \sum\!\!\!\!\!\!\int P^{-4}(\Delta\Pi^b_{\mu\nu})^2$ in Ref. [1], where we noted that the orthogonality of $\Pi^b_{\mu\nu}$ to P_μ lead to useful algebraic simplifications. We will therefore rewrite H_3 in an analogous form. First note that H_3 is of the form

$$H_3 = \sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2} \left[A_\mu(P)\right]^2 , \tag{4.11}$$

where

$$A_\mu(P) \equiv \sum\!\!\!\!\!\!\int_Q \frac{Q_\mu}{Q^2(P+Q)^2}\,. \tag{4.12}$$

Our method is to replace A_μ by something that is orthogonal to P_μ. So define

$$I_3 \equiv \sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2}\left[J_\mu(P)\right]^2 , \tag{4.13}$$

where

$$J_\mu(P) \equiv \sum\!\!\!\!\!\!\int_Q \frac{(2Q+P)_\mu}{Q^2(P+Q)^2} - \frac{P_\mu}{P^2}\left(\sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2} - \sum\!\!\!\!\!\!\int_{\{Q\}} \frac{1}{Q^2}\right). \tag{4.14}$$

It is easy to verify using our standard reduction tricks that $P\cdot J{=}0$ and that

$$H_3 = \frac{1}{4}I_3 + \frac{1}{4}I^{\rm bf}_{\rm ball} + \frac{1}{4}(b_1-f_1)^2 f_2 - \frac{1}{2}(b_1-f_1)I^{\rm f}_{\rm sun}\,. \tag{4.15}$$

Now focus on I_3. The orthogonality of J to P and Lorentz invariance imply that J_μ has the form

$$J_\mu(P) = \left(n_\mu - \frac{n\cdot P}{P^2}P_\mu\right) f(P)\,, \tag{4.16}$$

where $n_\mu = (1,\mathbf{0})$ is the four-velocity of the thermal bath.

Then

$$J_\mu(P) = \frac{P^2}{p^2}\left(n_\mu - \frac{p_0}{P^2}P_\mu\right) j_0(P) \tag{4.17}$$

and

$$I_3 = \sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{p^2}\left[j_0(P)\right]^2 . \tag{4.18}$$

The next simplification occurs by noting that J_μ vanishes at zero temperature because

$$\int \frac{d^dQ}{(2\pi)^d}\frac{(2Q+P)_\mu}{Q^2(P+Q)^2} = 0 \tag{4.19}$$

by antisymmetry under $Q \to -(Q{+}P)$. The large P behavior of $j_0(P)$ is therefore the large P behavior of its finite-temperature piece $j_0^{(T)}(P)$, which is $O(1/P^3)$ because the $O(1/P)$ behavior of the individual pieces cancels:

$$\left(\sum\!\!\!\!\!\!\int_{\{P\}} \frac{(2Q+P)_\mu}{Q^2(P+Q)^2}\right)^{(T)} \longrightarrow \frac{P_\mu}{P^2}\left(\sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2} - \sum\!\!\!\!\!\!\int_{\{Q\}} \frac{1}{Q^2}\right)$$

as fermionic P $\to\infty$. (4.20)

As a result, I_3 is both UV and IR finite as $\epsilon{\to}0$. So we can set $d=4$ and evaluate j_0 for fermionic p_0 in terms of the massless scalar propagator Δ:

$$\begin{aligned} j_0(P) &= T\sum_{q_0}\int d^3r\, e^{i\mathbf{p}\cdot\mathbf{r}}\Delta(q_0,\mathbf{r})\,\Delta(p_0+q_0,\mathbf{r})\,(2q_0+p_0) - \frac{T^2}{8}\frac{p_0}{P^2} + O(\epsilon) \\ &= \frac{T^2}{(4\pi)^2}\sum_{q_0}\int d^3r\,\frac{1}{r^2}e^{i\mathbf{p}\cdot\mathbf{r}}e^{-|q_0|r}e^{-|p_0+q_0|r}(2q_0+p_0) - \frac{T^2}{8}\frac{p_0}{P^2} + O(\epsilon) \\ &= \frac{T^2}{16\pi}\int d^3r\,\frac{1}{r^2}e^{i\mathbf{p}\cdot\mathbf{r}}\left\{\partial_{\bar r}\left[(\mathrm{csch}\bar r - \coth\bar r)e^{-|p_0|r}\right] + \frac{1}{2}e^{-|p_0|r}\right\}\mathrm{sgn}p_0 - \frac{T^2}{8}\frac{p_0}{P^2} + O(\epsilon) \\ &= -\frac{T}{8\pi}\int_0^\infty dr\left(\partial_r\frac{\sin pr}{pr}\right)\left(\mathrm{csch}\bar r - \coth\bar r + \tfrac{1}{2}\bar r\right)e^{-|p_0|r}\mathrm{sgn}p_0 + O(\epsilon)\,, \end{aligned} \tag{4.21}$$

where $\mathrm{sgn}p_0$ means the sign (±1) of p_0. Plugging into (4.18), doing the fermionic p_0 sum, and using the identity

$$\frac{1}{(2\pi)^3}\int\frac{d^3p}{p^2}\frac{\sin pr}{pr}\frac{\sin ps}{ps} = \frac{1}{4\pi}\left[\frac{1}{r}\theta(r-s) + \frac{1}{s}\theta(s-r)\right] \tag{4.22}$$

to do the p integration then yields

$$\begin{aligned} I_3 &= \frac{T^4}{128\pi^2}\int_0^\infty d\bar r\,\bar r^{-2}\left(\mathrm{csch}\bar r - \coth\bar r + \tfrac{1}{2}\bar r\right)^2\mathrm{csch}\bar r + O(\epsilon) \\ &= \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\left[\frac{9}{2}\frac{\zeta'(-3)}{\zeta(-3)} - 9\frac{\zeta'(-1)}{\zeta(-1)} + \frac{9}{2}\gamma_{\rm E} + \frac{9}{2} + \frac{117}{10}\ln 2\right] + O(\epsilon)\,. \end{aligned} \tag{4.23}$$

Again, we have used the methods of Appendix C to do the integrations. Using (4.15) to relate I_3 to H_3 finally gives

$$H_3 = \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\left[\frac{3}{8\epsilon}+\frac{9}{4}\ln\frac{\bar\mu}{4\pi T}+\frac{3}{2}\frac{\zeta'(-3)}{\zeta(-3)}-\frac{3}{2}\frac{\zeta'(-1)}{\zeta(-1)}+\frac{9}{4}\gamma_{\rm E}+\frac{361}{160}+\frac{57}{10}\ln 2\right]+O(\epsilon)\,. \qquad (4.24)$$

This agrees, within errors, with the numerical results of Ref. [2].[2]

V. RESULTS AND DISCUSSION

The evaluation of the final basic integrals $I^{\rm ff}_{\rm qcd}$ and $I^{\rm bf}_{\rm qcd}$ closely parallel the derivation of $I^{\rm bb}_{\rm qcd}$ in Ref. [1], and we leave the details for Appendix F. Combining all the results for individual graphs collected in Appendix A, our final result for the free energy is

$$\begin{aligned}F = d_{\rm A}T^4\frac{\pi^2}{9}\Bigg\{&-\frac{1}{5}\left(1+\frac{7d_{\rm F}}{4d_{\rm A}}\right)+\left(\frac{g}{4\pi}\right)^2\left(C_{\rm A}+\tfrac{5}{2}S_{\rm F}\right)\\&-\frac{16}{\sqrt3}\left(\frac{g}{4\pi}\right)^3\left(C_{\rm A}+S_{\rm F}\right)^{\frac32}-48\left(\frac{g}{4\pi}\right)^4C_{\rm A}(C_{\rm A}+S_{\rm F})\ln\left(\frac{g}{2\pi}\sqrt{\frac{C_{\rm A}+S_{\rm F}}{3}}\right)\\&+\left(\frac{g}{4\pi}\right)^4C_{\rm A}^2\left[\frac{22}{3}\ln\frac{\bar\mu}{4\pi T}+\frac{38}{3}\frac{\zeta'(-3)}{\zeta(-3)}-\frac{148}{3}\frac{\zeta'(-1)}{\zeta(-1)}-4\gamma_{\rm E}+\frac{64}{5}\right]\\&+\left(\frac{g}{4\pi}\right)^4C_{\rm A}S_{\rm F}\left[\frac{47}{3}\ln\frac{\bar\mu}{4\pi T}+\frac{1}{3}\frac{\zeta'(-3)}{\zeta(-3)}-\frac{74}{3}\frac{\zeta'(-1)}{\zeta(-1)}-8\gamma_{\rm E}+\frac{1759}{60}+\frac{37}{5}\ln 2\right]\\&+\left(\frac{g}{4\pi}\right)^4S_{\rm F}^2\left[-\frac{20}{3}\ln\frac{\bar\mu}{4\pi T}+\frac{8}{3}\frac{\zeta'(-3)}{\zeta(-3)}-\frac{16}{3}\frac{\zeta'(-1)}{\zeta(-1)}-4\gamma_{\rm E}-\frac{1}{3}+\frac{88}{5}\ln 2\right]\\&+\left(\frac{g}{4\pi}\right)^4S_{\rm 2F}\left[-\frac{105}{4}+24\ln 2\right]+O(g^5)\Bigg\}\,. \qquad (5.1)\end{aligned}$$

Evaluated numerically for QCD with $n_{\rm f}$ quark flavors, this is

$$\begin{aligned}F = -\frac{8\pi^2T^4}{45}\Bigg\{&1+\tfrac{21}{32}n_{\rm f}-0.09499\,g^2\left(1+\tfrac{5}{12}n_{\rm f}\right)+0.12094\,g^3\left(1+\tfrac16 n_{\rm f}\right)^{3/2}\\&+g^4\Bigg[0.08662\left(1+\tfrac16 n_{\rm f}\right)\ln\left(g\sqrt{1+\tfrac16 n_{\rm f}}\right)-0.01323\left(1+\tfrac{5}{12}n_{\rm f}\right)\left(1-\tfrac{2}{33}n_{\rm f}\right)\ln\frac{\bar\mu}{T}\\&+0.01733-0.00763\,n_{\rm f}-0.00088\,n_{\rm f}^2\Bigg]+O(g^5)\Bigg\}\,. \qquad (5.2)\end{aligned}$$

For QED with $n_{\rm f}$ massless charged fermions with charges $q_i e$, the free energy is

$$\begin{aligned}F = -\frac{\pi^2T^4}{45}\Bigg\{&1+\tfrac74 n_{\rm f}-0.07916\,e^2\textstyle\sum q_i^2+0.02328\,e^3\left(\sum q_i^2\right)^{3/2}\\&+e^4\left[\left(-0.00352+0.00134\ln\frac{\bar\mu}{T}\right)\left(\textstyle\sum q_i^2\right)^2+0.00193\textstyle\sum q_i^4\right]+O(e^5)\Bigg\}\,. \qquad (5.3)\end{aligned}$$

Our QED result agrees, within errors, with the purely numerical derivation of Ref. [2]. Ref. [2] also gives the QED result for the $O(e^5)$ piece.

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[2]We have also made a more precise numerical test of our analytic methods by computing (4.18) by brute force: we did the p integration and Euclidean p_0 sum numerically and used the contour trick to get an integral form for $j_0(p_0,p)$, which we also evaluated numerically. The p_0 sum converges quite quickly, and summing $\bar p_0$ up to $\pm 5/2$ gave agreement with our analytic result to 0.05%.

As in Ref. [1], we can now investigate whether the perturbative expansion of the QCD free energy is well-behaved for physically-realized values of couplings. Figure 6 shows the result for six-flavor QCD when $\alpha_s(T)$=0.1 (which corresponds to scales of order a few 100 GeV). The free energy is plotted vs. the choice of renormalization scale $\bar{\mu}$. We have taken

$$\frac{1}{g^2(\bar{\mu})} \approx \frac{1}{g^2(T)} - \beta_0 \ln\frac{\bar{\mu}}{T} + \frac{\beta_1}{\beta_0} \ln\left(1 - \beta_0 g^2(T) \ln\frac{\bar{\mu}}{T}\right) , \tag{5.4}$$

where

$$\beta_0 = \frac{1}{(4\pi)^2}\left(-\tfrac{22}{3}C_{\rm A} + \tfrac{8}{3}S_{\rm F}\right) ,$$

$$\beta_1 = \frac{1}{(4\pi)^4}\left(-\tfrac{68}{3}C_{\rm A}^2 + \tfrac{40}{3}C_{\rm A}S_{\rm F} + 8S_{2\rm F}\right) . \tag{5.5}$$

If the expansion is well behaved, the result for F should become more independent of $\bar{\mu}$ as higher-order corrections are included. Instead, we see that it does not. In Ref. [1], we argued that the g^3 term involves different physics than the g^2 term, and that it should perhaps be treated separately when discussing the behavior of the series. Ideally, one should calculate the free energy through g^5, which is the first order that compensates for the μ dependence of the g^3 term. With our present results, however, we can at least follow Ref. [1] and plot results where (1) we artificially exclude the g^3 term, or (2) we improve our $O(g^4)$ result with the $g^5 \ln\mu$ term required by renormalization-group invariance:

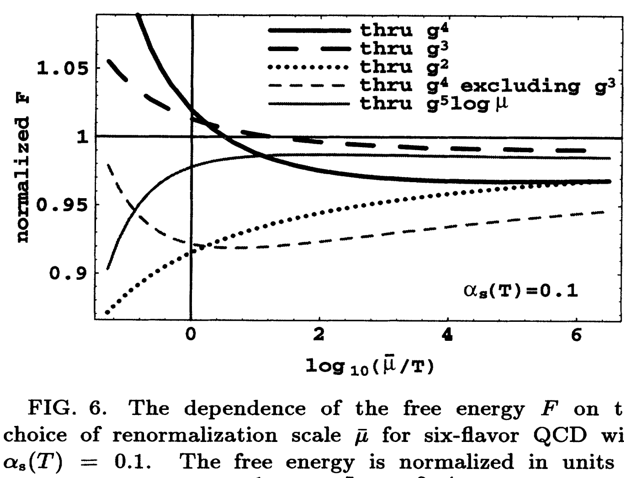

FIG. 6. The dependence of the free energy F on the choice of renormalization scale $\bar{\mu}$ for six-flavor QCD with $\alpha_s(T)$ = 0.1. The free energy is normalized in units of the ideal gas result $-(\frac{1}{45}d_{\rm A} + \frac{7}{180}d_{\rm F})\pi^2T^4$. The thick solid, dashed, and dotted lines are the results for F including terms through g^4, g^3, and g^2, respectively. The light solid curve is the g^4 result plus the $g^5\ln(\bar{\mu}/T)$ term required by renormalization group invariance. The light dashed curve is the g^4 result minus the g^3 term.

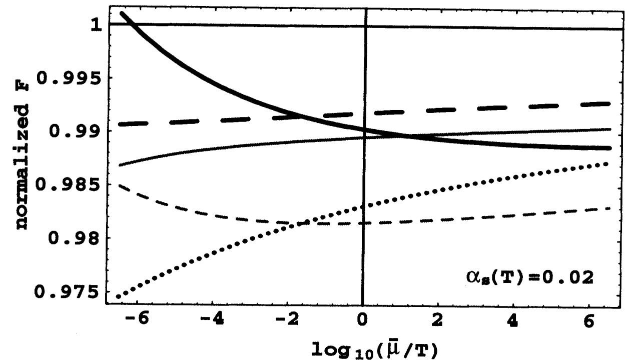

FIG. 7. The same as Fig. 6 but for $\alpha_s(T)$ = 0.02.

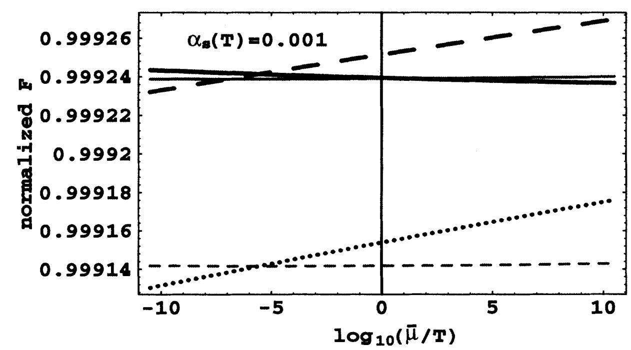

FIG. 8. The same as Fig. 6 but for $\alpha_s(T)$ = 0.001.

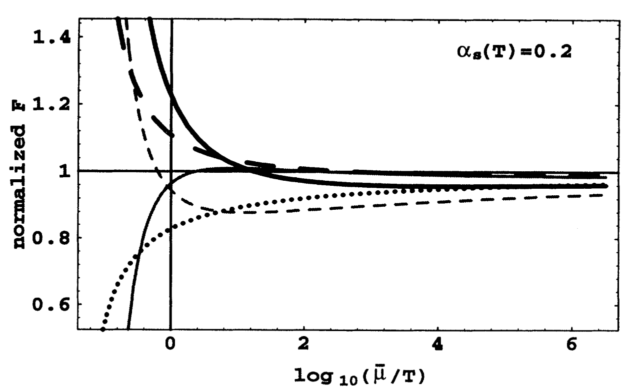

FIG. 9. The same as Fig. 6 but for $\alpha_s(T)$ = 0.2 and n_f = 5.

$$\Delta F = d_A T^4 \frac{\pi^2}{9}\left\{ -\frac{16}{\sqrt{3}}\left(\frac{g}{4\pi}\right)^5 (C_A + S_F)^{\frac{3}{2}} (11C_A - 4S_F) \times \ln \frac{\bar{\mu}}{4\pi T}\right\}. \tag{5.6}$$

Either modification can be seen to somewhat improve the behavior of the perturbative expansion of Fig. 6. For comparison with our previous results for the pure gauge case in Ref. [1], Figs. 7 and 8 show similar plots for the smaller couplings $\alpha_s(T)=0.02$ and $\alpha_s(T)=0.001$, where the perturbative expansion becomes progressively better behaved, as it should. For those readers who might be interested in the behavior of the expansion at scales of order several GeV, Fig. 9 shows our results for the case $\alpha_s(T) = 0.2$.

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APPENDIX A: RESULTS FOR INDIVIDUAL GRAPHS

Writing $F=\mu^{-2\epsilon} d_A \mathcal{F}$ and ignoring terms of $O(\epsilon)$, the diagrams of Fig. 4 are given in Appendix A of Ref. [1] except that

$$-\mathcal{F}^c = -\frac{1}{8\pi}(M_1^2 + M_2^2 + M_3^2 + M_4^2) MT\,, \tag{A1c}$$

$$-\mathcal{F}^l = \frac{1}{4} g^4 \left[C_A^2 I_{QCD}^{bb} + 2C_A S_F I_{QCD}^{bf} + S_F^2 I_{QCD}^{ff}\right] + O(g^5)\,. \tag{A1l}$$

The diagrams of Fig. 5 are given by

$$-\mathcal{F}^n = \frac{2d_F}{d_A} \sum\!\!\!\!\!\!\int_{\{P\}} \ln P^2\,, \tag{A1n}$$

$$-\mathcal{F}^o = S_F g^2 Z_g^2 \left[(d-2) f_1 (2b_1 - f_1) + \delta_3\right]\,, \tag{A1o}$$

$$\delta_3 = M^2 \sum\!\!\!\!\!\!\int_P \frac{\delta_{p_0}}{P^2}\Pi^f(P) - 4 I_{resum}^f + \frac{1}{8\pi}\frac{M_4^2 MT}{S_F g^2 Z_g^2}\,,$$

$$-\mathcal{F}^p = \left(-\tfrac{1}{2} S_F C_A + S_{2F}\right) g^4 (d-2) \times \left[\frac{(6-d)}{2} I_{ball}^{ff} - 2\epsilon I_{ball}^{bf}\right]\,, \tag{A1p}$$

$$-\mathcal{F}^q = S_{2F} g^4 (d-2)^2 \left[-I_{ball}^{bf} + 2H_3 - f_2 (f_1 - b_1)^2\right]\,, \tag{A1q}$$

$$-\mathcal{F}^r = C_A S_F g^4 (d-2) I_{ball}^{bf}\,. \tag{A1r}$$

The resummation of the gauge boson line in diagram (o) is similar to the resummation of diagram (e), discussed in Ref. [1]. In particular,

$$I_{resum}^f \equiv \sum\!\!\!\!\!\!\int_{P\{Q\}} \left[\frac{\delta_{p_0}}{P^2+M^2} - \frac{\delta_{p_0}}{P^2}\right]\left[\frac{q_0^2}{Q^2(P+Q)^2} - \frac{q_0^2}{Q^4}\right] = O(g^3, \epsilon)\,. \tag{A2}$$

The last equality follows from a derivation similar to the bosonic case treated in Ref. [1], and means that I_{resum}^f can be ignored at the order under consideration.

The multiplicative renormalization constant used for the coupling is given by

$$g_{bare} = Z_g g \mu^\epsilon = \left[1 - \frac{11}{6}\frac{C_A g^2}{(4\pi)^2\epsilon} + \frac{2}{3}\frac{S_F g^2}{(4\pi)^2 \epsilon} + O(g^4)\right] g\mu^\epsilon\,. \tag{A3}$$

The vector mass M is given by (2.11), and M_4^2 is the piece of M^2 due to the fermion contribution of Fig. 1(b).

APPENDIX B: LARGE P BEHAVIOR OF $\Pi^f(P)$

In Appendix B of Ref. [1], we derived the large P behavior of the bosonic $\Pi^{b(T)}(P)$ to be

$$\Pi^{b(T)}(P) = 2J_{-1}^b \frac{T^2}{P^2} + 8J_1^b \frac{T^4}{P^6}\left(\frac{p^2}{d-1} - p_0^2\right) + O(T^6/P^6)\,, \tag{B1}$$

where

$$J_\alpha^b = \left(\frac{4\pi\mu^2}{T^2}\right)^\epsilon \frac{\Gamma(3-2\epsilon+\alpha)\zeta(3-2\epsilon+\alpha)}{4\pi^{3/2}\Gamma\left(\frac{3}{2}-\epsilon\right)}\,. \tag{B2}$$

The easiest way to get a similar expression for the fermionic $\Pi^{f(T)}$ is to note that, by scaling momenta (q_0, q) by a factor of 2, one gets

$$\sum\!\!\!\!\!\!\int_{Q+\{Q\}} \frac{1}{Q^2(P+Q)^2} = 2^{5-d} \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2(2P+Q)^2}\,, \tag{B3}$$

and so

$$\Pi^f(P) = 2^{5-d}\Pi^b(2P) - \Pi^b(P)\,. \tag{B4}$$

Applying this identity to (B1) then gives the same formula,

$$\Pi^{f(T)}(P) = 2J_{-1}^f \frac{T^2}{P^2} + 8J_1^f \frac{T^4}{P^6}\left(\frac{p^2}{d-1} - p_0^2\right) + O(T^6/P^6)\,, \tag{B5}$$

for the fermionic case but with

$$J_\alpha^f = \left(2^{2-d-\alpha} - 1\right) J_\alpha^b\,. \tag{B6}$$

APPENDIX C: NEW INTEGRALS OF HYPERBOLIC FUNCTIONS

In Appendix C of Ref. [1], we showed how to evaluate convergent integrals of the form

$$I = \int_0^\infty dx \left(\sum_{m,n} c_{mn} x^m \coth^n x + \sum_m d_m x^m e^{-a_m x} \right) \tag{C1}$$

by regulating the individual terms by introducing an extra factor of x^δ and taking $\delta \to 0$ at the end. In the current work, we need to extend our set of integrals to

$$I = \int_0^\infty dx \left(\sum_{m,n,p} c_{mnp} x^m \coth^n x \operatorname{csch}^p x + \sum_m d_m x^m e^{-a_m x} \right). \tag{C2}$$

To handle this, we need in addition to our previous basic regulated integrals

$$\int_0^\infty dx\, x^z = 0\,, \tag{C3}$$

$$\int_0^\infty dx\, x^z \coth x = 2^{-z}\Gamma(z+1)\,\zeta(z+1)\,, \tag{C4}$$

$$\int_0^\infty dx\, x^z e^{-ax} = a^{-1-z}\Gamma(1+z)\,, \tag{C5}$$

the new integral

$$\int_0^\infty dx\, x^z \operatorname{csch} x = (2-2^{-z})\Gamma(z+1)\,\zeta(z+1)\,. \tag{C6}$$

In addition, we need to generalize our previous recursion relation to

$$\int_0^\infty dx\, x^z \coth^n x \operatorname{csch}^p x = \int_0^\infty dx \left[\frac{z}{p+n-1} x^{z-1} \coth^{n-1} x \operatorname{csch}^p x + \frac{n-1}{p+n-1} x^z \coth^{n-2} x \operatorname{csch}^p x \right] \tag{C7}$$

and also use

$$\int_0^\infty dx\, x^z \coth^n x \operatorname{csch}^p x = \int_0^\infty dx \left[x^z \coth^{n+2} x \operatorname{csch}^{p-2} x - x^z \coth^n x \operatorname{csch}^{p-2} x \right]. \tag{C8}$$

APPENDIX D: COMPLETION OF THE CALCULATION OF $I_{\rm ball}^{\rm ff}$

In this section, we complete the evaluation of the fermionic basketball integral. The derivation directly parallels the bosonic case treated in Appendix D of Ref. [1], with hyperbolic cotangents becoming hyperbolic cosecants as we discussed earlier. We therefore refer the reader to Ref. [1] and shall here simply present the differences for a selected few intermediate results.

As in the bosonic case, we write

$$\sum\!\!\!\!\!\!\!\int_P \Pi^{{\rm f}(T)}\Pi^{(0)} = I_{\rm a}^{\rm f} + I_{\rm b}^{\rm f} + I_{\rm c}^{\rm f}\,, \tag{D1}$$

where

$$I_{\rm a}^{\rm f} \equiv \sum\!\!\!\!\!\!\!\int_P \left[\Pi^{(0)}(P) - \frac{1}{(4\pi)^2\epsilon} \right] \left[\Pi^{{\rm f}(T)}(P) - \Pi_{\rm UV}^{{\rm f}(T)}(P) \right], \tag{D2}$$

$$I_{\rm b}^{\rm f} \equiv \frac{1}{(4\pi)^2\epsilon} \sum\!\!\!\!\!\!\!\int_P \left[\Pi^{{\rm f}(T)}(P) - \Pi_{\rm UV}^{{\rm f}(T)}(P) \right], \tag{D3}$$

$$I_{\rm c}^{\rm f} \equiv \sum\!\!\!\!\!\!\!\int_P \Pi^{(0)}(P)\Pi_{\rm UV}^{{\rm f}(T)}(P)\,, \tag{D4}$$

and

$$\Pi_{\rm UV}^{{\rm f}(T)}(P) \equiv 2J_{-1}^{\rm f}\frac{T^2}{P^2} + (1-\delta_{p_0})8J_1^{\rm f}\frac{T^4}{P^6}\left(\frac{p^2}{d-1} - p_0^2\right). \tag{D5}$$

The zero-temperature piece $\Pi^{(0)}$ of $\Pi^{\rm f}$ is the same as the bosonic case. One finds

$$I_{\rm a}^{\rm f} = \frac{T^4}{(4\pi)^2}\frac{1}{2}\int_0^\infty \frac{d\bar r}{\bar r^3}\left\{ \left(\operatorname{csch}\bar r - \frac{1}{\bar r} + \frac{\bar r}{6} - \frac{7\bar r^3}{360} \right)\left(1 - \frac{\bar r}{2}\frac{d}{d\bar r}\right)(\coth\bar r - 1) + \left(\operatorname{csch}\bar r - \frac{1}{\bar r} + \frac{\bar r}{6} \right) \right\} + O(\epsilon)$$

$$= \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2 \left[-\frac{29}{10}\frac{\zeta'(-3)}{\zeta(-3)} + 5\frac{\zeta'(-1)}{\zeta(-1)} - \frac{21}{10}\gamma_{\rm E} + \frac{43}{30} - \frac{27}{10}\ln 2 \right] + O(\epsilon)\,, \tag{D6}$$

$$I_{\rm b}^{\rm f} = \frac{T^4}{(4\pi)^2}\frac{1}{\epsilon}\left\{\left(J_{-1}^{\rm f} - 2J_{-1}^{\rm b}\right)J_{-1}^{\rm f} - A\left(\frac{4\pi\mu^2}{T^2}\right)^\epsilon S_0(\epsilon) - \frac{8J_1^{\rm f}}{d-1}\left[S_0(2) - d\,S_1(3)\right]\right\}$$
$$= \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\left[-\frac{29}{10}\frac{\zeta'(-3)}{\zeta(-3)} + 5\frac{\zeta'(-1)}{\zeta(-1)} - \frac{21}{10}\gamma_{\rm E} + \frac{43}{30} - \frac{27}{10}\ln 2\right] + O(\epsilon)\,, \tag{D7}$$

$$I_{\rm c}^{\rm f} = 2AT^4\left(\frac{4\pi\mu^2}{T^2}\right)^\epsilon\left\{J_{-1}^{\rm f}S_0(1+\epsilon) + \frac{4J_1^{\rm f}}{d-1}\left[S_0(2+\epsilon) - d\,S_1(3+\epsilon)\right]\right\}$$
$$= \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\left[\frac{1}{20\epsilon} + \frac{3}{10}\ln\frac{\bar\mu}{4\pi T} + \frac{21}{10}\frac{\zeta'(-3)}{\zeta(-3)} - 6\frac{\zeta'(-1)}{\zeta(-1)} + \frac{21}{5}\gamma_{\rm E} - \frac{43}{8} + \frac{17}{10}\ln 2\right] + O(\epsilon)\,, \tag{D8}$$

where A and $S_n(\alpha)$ are defined in Ref. [1]. Putting together (D6), (D7), and (D8),

$$\sum\!\!\!\!\!\!\int \Pi^{f(T)}\Pi^{(0)} = \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\left[\frac{1}{20\epsilon} + \frac{3}{10}\ln\frac{\bar\mu}{4\pi T} - \frac{37}{10}\frac{\zeta'(-3)}{\zeta(-3)} + 4\frac{\zeta'(-1)}{\zeta(-1)} - \frac{301}{120} - \frac{37}{10}\ln 2\right] + O(\epsilon)\,. \tag{D9}$$

The result for $\sum\!\!\!\!\!\!\int \left[\Pi^{(0)}\right]^2$ is given in Ref. [1], and adding it to (4.6) and (D9) yields the final result (4.7) for the fermionic basketball.

APPENDIX E: THE PIECES OF $I_{\rm sun}^{\rm f}$

As mentioned earlier, the fermionic setting-sun integral $I_{\rm sun}^{\rm f}$ vanishes when particle masses are ignored. However, it is useful to also know the piece corresponding to

$$\sum\!\!\!\!\!\!\int \frac{\delta_{p_0}}{P^2}\Pi^{\rm f}(P) = -\sum\!\!\!\!\!\!\int \frac{1-\delta_{p_0}}{P^2}\Pi^{\rm f}(P)\,. \tag{E1}$$

This can be easily evaluated by relating it to the comparable bosonic piece using (B4) and then scaling momenta by 2:

$$\sum\!\!\!\!\!\!\int \frac{\delta_{p_0}}{P^2}\Pi^{\rm f}(P) = \sum\!\!\!\!\!\!\int \frac{\delta_{p_0}}{P^2}\left[2^{5-d}\Pi^{\rm b}(2P) - \Pi^{\rm b}(P)\right]$$
$$= \left(2^{2(4-d)} - 1\right)\sum\!\!\!\!\!\!\int \frac{\delta_{p_0}}{P^2}\Pi^{\rm b}(P)$$
$$= \left(2^{4\epsilon} - 1\right)\sum\!\!\!\!\!\!\int_{PQ} \frac{\delta_{p_0}(1-\delta_{q_0})}{P^2Q^2(P+Q)^2}$$
$$= -\frac{T^2}{(4\pi)^2}\ln 2 + O(\epsilon)\,, \tag{E2}$$

where the last equality follows from the bosonic result of Ref. [1] that

$$\sum\!\!\!\!\!\!\int \frac{\delta_{p_0}(1-\delta_{q_0})}{P^2Q^2(P+Q)^2} = \frac{T^2}{(4\pi)^2}\left[-\frac{1}{4\epsilon} + \ln\frac{2T}{\bar\mu} - \frac{1}{2}\right] + O(m,\epsilon)\,. \tag{E3}$$

APPENDIX F: DERIVATIONS OF $I_{\rm QCD}^{\rm ff}$ AND $I_{\rm QCD}^{\rm bf}$

The calculations of $I_{\rm QCD}^{\rm ff}$ and $I_{\rm QCD}^{\rm bf}$ also closely parallel those of the purely bosonic case in Appendix H of Ref. [1]. In that previous work, we found it algebraically convenient to rewrite the bosonic contribution $\Pi_{\mu\nu}^{\rm b}$ of Fig. 1(a) to the vector self-energy in terms of

$$\bar\Pi_{\mu\nu}^{\rm b} \equiv 2\delta_{\mu\nu}\sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2} - \sum\!\!\!\!\!\!\int_Q \frac{(2Q+P)_\mu(2Q+P)_\nu}{Q^2(P+Q)^2} \tag{F1}$$

via

$$\left(\Pi_{\mu\nu}^{\rm b}\right)^{ab}(P) = C_{\rm A}g^2\delta^{ab}\left[\frac{d-2}{2}\bar\Pi_{\mu\nu}^{\rm b}(P) - 2(P^2\delta_{\mu\nu} - P_\mu P_\nu)\sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2(P+Q)^2}\right]. \tag{F2}$$

In order to make our presentation of the fermionic case as similar as possible to the bosonic one, we shall do the same for the fermionic contribution $\Pi_{\mu\nu}^{\rm f}$ of Fig. 1(b):[3]

$$\left(\Pi_{\mu\nu}^{\rm f}\right)^{ab}(P) = -S_{\rm F}g^2\delta^{ab}\left[2\bar\Pi_{\mu\nu}^{\rm f}(P) - 2(P^2\delta_{\mu\nu} - P_\mu P_\nu)\sum\!\!\!\!\!\!\int_{\{Q\}} \frac{1}{Q^2(P+Q)^2}\right], \tag{F3}$$

where

$$\bar\Pi_{\mu\nu}^{\rm f} \equiv 2\delta_{\mu\nu}\sum\!\!\!\!\!\!\int_{\{Q\}} \frac{1}{Q^2} - \sum\!\!\!\!\!\!\int_{\{Q\}} \frac{(2Q+P)_\mu(2Q+P)_\nu}{Q^2(P+Q)^2}\,. \tag{F4}$$

[3]The main advantage to this is simplicity of presentation; one could just as easily do the calculation with $\Pi_{\mu\nu}^{\rm f}$ directly.

And now let's focus on computing

$$I^{\rm ff}_{\rm sQED} \equiv \sumint_P \frac{[\Delta\bar\Pi^{\rm f}_{\mu\nu}(P)]^2}{P^4}\,, \qquad I^{\rm bf}_{\rm sQED} \equiv \sumint_P \frac{\Delta\bar\Pi^{\rm b}_{\mu\nu}(P)\Delta\bar\Pi^{\rm f}_{\mu\nu}(P)}{P^4}\,. \tag{F5}$$

Following through the same steps as in Appendix H of Ref. [1], one gets the obvious generalizations of Eqs. (H27)–(H29) of that reference:

$$\sumint_P \frac{1}{P^4}\left[\Delta\bar\Pi^{{\rm f}(T)}_{\mu\nu}(P)\right]^2 = \sumint_P \left[\Pi^{{\rm f}(T)}(P)\right]^2 + 4(d-2)b_2 f_1^2 + O(\epsilon)\,, \tag{F6}$$

$$\sumint_P \frac{1}{P^4}\Delta\bar\Pi^{{\rm b}(T)}_{\mu\nu}(P)\Delta\bar\Pi^{{\rm f}(T)}_{\mu\nu}(P) = \sumint_P \Pi^{{\rm b}(T)}(P)\Pi^{{\rm f}(T)}(P) + 4(d-2)b_2 b_1 f_1 + O(\epsilon)\,, \tag{F7}$$

$$\sumint_P \frac{1}{P^4}\bar\Pi^{{\rm f}(T)}_{\mu\nu}(P)\bar\Pi^{(0)}_{\mu\nu}(P) = \frac{1}{d-1}\sumint_P \Pi^{{\rm f}(T)}(P)\Pi^{(0)}(P) + 2\frac{(d-2)}{(d-1)}f_1\sumint_P \frac{1}{P^2}\Pi^{(0)}(P)\,, \tag{F8}$$

$$\sumint_P \frac{1}{P^4}\bar\Pi^{{\rm b}(T)}_{\mu\nu}(P)\bar\Pi^{(0)}_{\mu\nu}(P) = \frac{1}{d-1}\sumint_P \Pi^{{\rm b}(T)}(P)\Pi^{(0)}(P) + 2\frac{(d-2)}{(d-1)}b_1\sumint_P \frac{1}{P^2}\Pi^{(0)}(P)\,, \tag{F9}$$

$$\sumint_P \frac{1}{P^4}\left[\bar\Pi^{(0)}_{\mu\nu}(P)\right]^2 = \frac{1}{d-1}\sumint_P \left[\Pi^{(0)}(P)\right]^2 . \tag{F10}$$

Summing these results, and incorporating the results for the assorted basic integrals, gives

$$I^{\rm ff}_{\rm sQED} = \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\left[\frac{11}{6\epsilon} + 11\ln\frac{\bar\mu}{4\pi T} + \frac{1}{3}\frac{\zeta'(-3)}{\zeta(-3)} + \frac{20}{3}\frac{\zeta'(-1)}{\zeta(-1)} + 4\gamma_{\rm E} + \frac{281}{60} - 13\ln 2\right] + O(\epsilon)\,, \tag{F11}$$

$$I^{\rm bf}_{\rm sQED} = \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\left[-\frac{59}{12\epsilon} - \frac{59}{2}\ln\frac{\bar\mu}{4\pi T} + \frac{11}{6}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{70}{3}\frac{\zeta'(-1)}{\zeta(-1)} - 8\gamma_{\rm E} - \frac{2063}{120} + \frac{169}{10}\ln 2\right] + O(\epsilon)\,. \tag{F12}$$

Using (F2) and (F3) and our standard reduction tricks yields[4]

$$\begin{aligned} I^{\rm ff}_{\rm QCD} &= 4I^{\rm ff}_{\rm sQED} + 4(d-3)I^{\rm ff}_{\rm ball} - 16(d-2)f_1\sumint{}' \frac{1}{P^2}\Pi^{\rm f}(P) \\ &= \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\left[\frac{40}{3\epsilon} + 80\ln\frac{\bar\mu}{4\pi T} - \frac{32}{3}\frac{\zeta'(-3)}{\zeta(-3)} + \frac{224}{3}\frac{\zeta'(-1)}{\zeta(-1)} + 16\gamma_{\rm E} + \frac{124}{3} + \frac{448}{5}\ln 2\right] + O(\epsilon)\,, \end{aligned} \tag{F13}$$

$$\begin{aligned} I^{\rm bf}_{\rm QCD} &= (2-d)I^{\rm bf}_{\rm sQED} - 3(d-2)I^{\rm bf}_{\rm ball} + 8(d-2)f_1\sumint_P{}' \frac{1}{P^2}\Pi^{\rm b}(P) + 2(d-2)^2 b_1\sumint_P{}' \frac{1}{P^2}\Pi^{\rm f}(P) \\ &= \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\Bigg[-\frac{29}{3\epsilon} - 58\ln\frac{\bar\mu}{4\pi T} - \frac{38}{3}\frac{\zeta'(-3)}{\zeta(-3)} + \frac{104}{3}\frac{\zeta'(-1)}{\zeta(-1)} \\ &\qquad +16\gamma_{\rm E} - \frac{251}{10} + \frac{398}{5}\ln 2 - 96\ln(2\pi)\Bigg] + O(\epsilon)\,. \end{aligned} \tag{F14}$$

Before leaving this section, we note that one can use our results to derive the basic integral

$$H_4 \equiv \sumint_{P\{QK\}} \frac{(Q\cdot K)^2}{P^4 Q^2 K^2 (P+Q)^2 (P+K)^2} \tag{F15}$$

defined by Parwani and Corianò. By applying our usual reduction techniques to $I^{\rm ff}_{\rm sQED}$, and using the fact that $I^{\rm f}_{\rm sun} = 0$, one can derive that

$$I^{\rm ff}_{\rm sQED} = 16H_4 - I^{\rm ff}_{\rm ball} + 4(d-4)b_2 f_1^2\,. \tag{F16}$$

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[4]Because of our slightly different methods of bookkeeping of infrared divergences, our reduction of $I^{\rm ff}_{\rm QCD}$ differs from a similar reduction in Ref. [2] by $M_4^2 \sumint (1-\delta_{p_0})P^{-2}\Pi^{\rm f}(P)$. There is a canceling difference in our treatment of diagram (o) of Fig. 5.

Solving for H_4 and using our results for $I^{\rm ff}_{\rm sQED}$ and $I^{\rm ff}_{\rm ball}$ then yields

$$H_4 = \frac{1}{(4\pi)^2}\left(\frac{T^2}{12}\right)^2\left[\frac{5}{24\epsilon}+\frac{5}{4}\ln\frac{\bar{\mu}}{4\pi T}-\frac{1}{6}\frac{\zeta'(-3)}{\zeta(-3)}+\frac{7}{6}\frac{\zeta'(-1)}{\zeta(-1)}+\frac{1}{4}\gamma_{\rm E}+\frac{23}{24}-\frac{8}{5}\ln 2\right]+O(\epsilon)\,, \tag{F17}$$

which agrees with the numerical result of Parwani and Corianò within errors.

[1] P. Arnold and C. Zhai, Phys. Rev. D **50**, 7603 (1994).
[2] C. Corianò and R. Parwani, Phys. Rev. Lett. **73**, 2398 (1994); R. Parwani, Phys. Lett. B **334**, 420 (1994); R. Parwani and C. Corianò, Argonne National Lab Report No. ANL-HEP-PR-94-32 (1994), hep-ph/9409269 (unpublished).