

Eleventh-order calculation of Ising-limit Green's functions for scalar quantum field theory in arbitrary space-time dimension D

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This paper extends an earlier high-temperature lattice calculation of the renormalized Green's functions of a D -dimensional Euclidean scalar quantum field theory in the Ising limit. The previous calculation included all graphs through sixth order. Here, we present the results of an eleventh-order calculation. The extrapolation to the continuum limit in the previous calculation was rather clumsy and did not appear to converge when $D > 2$. Here, we present an improved extrapolation which gives uniformly good results for all real values of the dimension between $D = 0$ and $D = 4$. We find that the four-point Green's function has the value 0.620 ± 0.007 when $D = 2$ and 0.98 ± 0.01 when $D = 3$ and that the six-point Green's function has the value 0.96 ± 0.03 when $D = 2$ and 1.2 ± 0.2 when $D = 3$.

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There have been many attempts to calculate the coefficients in the effective potential of a Euclidean scalar quantum field theory in the Ising limit. These coefficients are just the dimensionless renormalized $2n$ -point Green's functions evaluated at zero momentum. Techniques that have been used to determine these Green's functions include high-temperature lattice expansions, Monte Carlo methods, and ϵ expansions. For the case $D = 3$, a complete list of references is given in a recent Monte Carlo study by Tsypin [1].

In a series of papers [2,3], high-temperature lattice techniques were used to obtain the dependence of the Green's functions upon the Euclidean space-time dimension D for D ranging continuously between 0 and 4. In this work a strong-coupling calculation to sixth order was performed analytically on a hypercubic lattice in D dimensions and Padé extrapolation techniques were

invented to obtain the continuum limit [4]. Here we extend the strong-coupling calculation in Refs. [2,3] to 11th order. Furthermore, we use an improved Padé extrapolation method that relies on information taken from the results of a large- N calculation and our recent studies of dimensional expansions for quantum field theory [5-7].

Our strong-coupling calculations are identical to those described in Ref. [2] except that the graphs were generated using a FORTRAN program and evaluated analytically using MACSYMA. The 11th-order calculation involves several hundred times as many graphs as the sixth-order calculation. We have verified the accuracy of our expansions for the specific cases of $D = 2, 3$, and 4 dimensions by comparing them with previous calculations [8].

The lattice series for the renormalized four- and six-point Green's functions are

$$\begin{aligned} \gamma^4 = & \frac{y^{-D/2}}{12} [1 + 4Dy + (4D^2 - 10D)y^2 + 16Dy^3 + (30D - 80D^2)y^4 + (256D^3 + 104D^2 - 192D)y^5 \\ & + (-704D^4 - 1736D^3 + 2508D^2 - 656D)y^6 + (1792D^5 + 10432D^4 - 11232D^3 - 3872D^2 + 4992D)y^7 \\ & + (-4352D^6 - 45600D^5 + 11456D^4 + 123672D^3 - 128440D^2 + 35542D)y^8 \\ & + (10240D^7 + 168320D^6 + 181248D^5 - 1052576D^4 + \frac{2615584}{3}D^3 + 76664D^2 - \frac{681472}{3}D)y^9 \\ & + (-23552D^8 - 558208D^7 - 1630272D^6 + 5391904D^5 - \frac{1011536}{3}D^4 - 10102936D^3 + \frac{29622092}{3}D^2 - 2720752D)y^{10} \\ & + (53248D^9 + 1718272D^8 + 9081856D^7 - 18274816D^6 - \frac{113682176}{3}D^5 + \frac{367432576}{3}D^4 \\ & - \frac{292976128}{3}D^3 + \frac{18425408}{3}D^2 + 14757984D)y^{11} + \dots] \end{aligned} \quad (1)$$

and

$$\begin{aligned} \gamma_6 = & \frac{y^{-D}}{30} [1 + 6Dy + (12D^2 - 6D)y^2 + (8D^3 - 12D^2 - 20D)y^3 + (48D^2 + 48D)y^4 + (-96D^3 - 816D^2 + 528D)y^5 \\ & + (192D^4 + 4640D^3 - 2736D^2 - 560D)y^6 + (-384D^5 - 18432D^4 - 10800D^3 + 46512D^2 - 23040D)y^7 \\ & + (768D^6 + 61440D^5 + 188352D^4 - 510816D^3 + 357324D^2 - 72492D)y^8 \\ & + (-1536D^7 - 184576D^6 - 1274880D^5 + 2653440D^4 - 77496D^3 - 2911496D^2 + 1698240D)y^9 \\ & + (3072D^8 + 517632D^7 + 6280704D^6 - 6584832D^5 - 27745840D^4 \\ & + 65401176D^3 - 49332608D^2 + 11853912D)y^{10} \\ & + (-6144D^9 - 1382400D^8 - 25928448D^7 - 13343232D^6 + 286690784D^5 \\ & - 516057392D^4 + 211594432D^3 + 210150872D^2 - 153291366D)y^{11} + \dots], \end{aligned} \quad (2)$$

where $y = (Ma)^{-2}$, a is the lattice spacing, and M is the renormalized mass, which is obtained from the two-point function as explained in Ref. [2].

The quantity \sqrt{y} is the dimensionless correlation length. The continuum limit $a \rightarrow 0$ corresponds to infinite correlation length. To obtain the continuum Green's functions, it is necessary to extrapolate the formulas in (1) and (2) to their values at $y = \infty$. Direct extrapolation to the continuum limit of either series in (1) or (2) leads to a sequence of extrapolants that becomes badly behaved when D increases beyond 2; we find that extrapolations as functions of D do not converge to a limiting curve (see Figs. 1 and 2). However, for D near 0 these extrapolants are well behaved and converge rapidly to the known exact values [2] $\gamma_4 = \frac{1}{4}$ and $\gamma_6 = \frac{1}{4}$ at $D = 1$.

To improve our extrapolation, we make the following observation. We consider a scalar field theory having an $O(N)$ symmetry. The model we have studied above corresponds to the case $N = 1$. In the limit $N \rightarrow \infty$, one can solve for the Green's functions exactly. We obtain the lattice results

$$N\gamma_4^{(N=\infty)} = \frac{y^{-D/2}}{4 \int_0^\infty dt t e^{-t} [e^{-2ty} I_0(2ty)]^D} \quad (3)$$

and

$$N^2\gamma_6^{(N=\infty)} = \frac{y^{-D} \int_0^\infty dt t^2 e^{-t} [e^{-2ty} I_0(2ty)]^D}{12 \left\{ \int_0^\infty dt t e^{-t} [e^{-2ty} I_0(2ty)]^D \right\}^3}, \quad (4)$$

where we have summed over the external indices. In the continuum limit $y \rightarrow \infty$, we have

$$N\gamma_4^{(N=\infty)} = \frac{(4\pi)^{D/2}}{4\Gamma(2-D/2)} \quad (5)$$

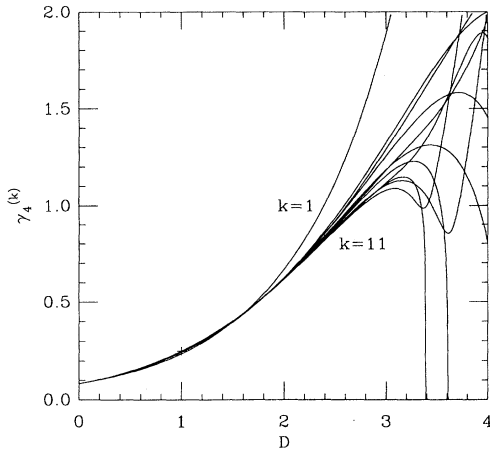


FIG. 1. First k Padé approximants, $k = 1, 2, \dots, 11$, to the continuum limit of the four-point Green's function. These approximants are constructed from the lattice series in (1) using the Padé procedure explained in Ref. [2]. Observe that the approximants are well behaved when D is near 0; in particular, they converge nicely to the exact value $\frac{1}{4}$ at $D = 1$ (indicated by a plus sign). However, when D increases beyond 2, the approximants become irregular and do not seem to converge to a limiting function of D . (Some of the approximants reach zero and terminate as D increases because they become complex.)

and

$$N^2\gamma_6^{(N=\infty)} = \frac{(4\pi)^D \Gamma(3-D/2)}{12[\Gamma(2-D/2)]^3}, \quad (6)$$

where D lies in the range $0 \leq D \leq 4$. Each of these functions rises from its value at $D = 0$, attains a maximum, and falls to 0 at $D = 4$.

Under the assumption that the Green's functions for $N = 1$ vanish at $D = 4$ like those in (5) and (6), we can extract such a behavior from the series (1) and (2) by performing a Borel summation as follows. Consider the lattice series in (1) for γ_4 . This series has the general form

$$\gamma_4 = \frac{y^{-D/2}}{12} \sum_{k=0}^{\infty} P_k(D) y^k, \quad (7)$$

where $P_k(D)$ are polynomials of maximum degree k . One can read off the first 11 polynomials $P_k(D)$ from (1). We can rewrite (7) as

$$\frac{1}{12\gamma_4} = y^{D/2} \sum_{k=0}^{\infty} Q_k(D) y^k, \quad (8)$$

where $Q_k(D)$ is another polynomial in D . Next, we insert the identity

$$1 = \frac{1}{(k+1)!} \int_0^\infty dt t^{k+1} e^{-t} \quad (9)$$

for each term in the sum. This converts (8) to the form

$$\frac{1}{12\gamma_4} = \int_0^\infty dt t^{1-D/2} e^{-t} f(yt)^{D/2}, \quad (10)$$

where we define

$$f(x) = x \left(\sum_{k=0}^{\infty} \frac{Q_k(D) x^k}{(k+1)!} \right)^{2/D} = x \sum_{k=0}^{\infty} R_k(D) x^k, \quad (11)$$

where again $R_k(D)$ are polynomials in D .

We now take the continuum limit of the expression (10). Assuming that $f(\infty)$ exists in the limit $y \rightarrow \infty$ so that (10) separates into a product of two terms, we have

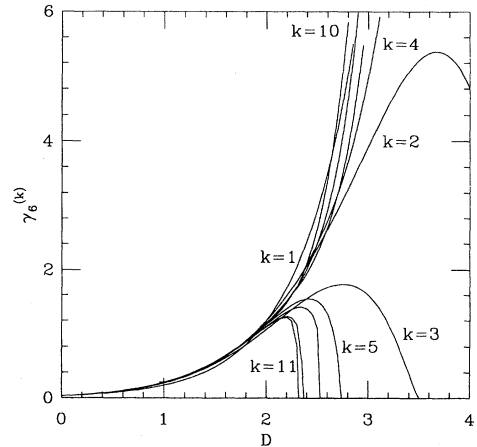


FIG. 2. Same as in Fig. 1, except that we have plotted the first k approximants to the six-point Green's function for $k = 1, 2, \dots, 11$. Again, we find that the approximants converge to the exact value $\frac{1}{4}$ at $D = 1$, but that they are irregular for $D > 2$.

$$\gamma_4 = \frac{f(\infty)^{-D/2}}{12\Gamma(2-D/2)}. \quad (12)$$

Thus we have forced the continuum limit of the four-point function to take the form of the large- N result in (5) apart from $f(\infty)$, which is a smoothly varying function of D .

There is an immediate indication that the Borel summation leading to (12) has a significant impact. We find that the polynomials $R_k(D)$ in (11) are significantly simpler than the original polynomials $P_k(D)$ in (7); the polynomials $R_k(D)$ have maximum degree $[k/2] - 1$, about half the degree of the polynomials $P_k(D)$ [9]. The polynomials $R_k(D)$, $k = 0, 1, 2, \dots, 11$, are

$$\begin{aligned} R_0(D) &= 1, \\ R_1(D) &= -2, \\ R_2(D) &= \frac{11}{3}, \\ R_3(D) &= -\frac{16}{3}, \\ R_4(D) &= -\frac{7}{45}D + \frac{233}{36}, \\ R_5(D) &= \frac{2}{3}D - \frac{125}{18}, \\ R_6(D) &= \frac{73}{2835}D^2 - \frac{1346}{945}D + \frac{76691}{11340}, \\ R_7(D) &= -\frac{221}{1134}D^2 + \frac{2713}{1260}D - \frac{16939}{2835}, \\ R_8(D) &= -\frac{13}{8100}D^3 + \frac{98323}{170100}D^2 - \frac{8837}{3240}D + \frac{2627137}{544320}, \\ R_9(D) &= \frac{163}{3150}D^3 - \frac{14699}{14175}D^2 + \frac{17111}{5670}D - \frac{235681}{64800}, \\ R_{10}(D) &= -\frac{457}{267300}D^4 - \frac{1338257}{5613300}D^3 + \frac{2015579}{1403325}D^2 \\ &\quad - \frac{9257497}{3207600}D + \frac{230357209}{89812800}, \\ R_{11}(D) &= -\frac{611}{69300}D^4 + \frac{6066953}{11226600}D^3 - \frac{3923681}{2245320}D^2 \\ &\quad + \frac{110092603}{44906400}D - \frac{75132389}{44906400}. \end{aligned} \quad (13)$$

The resummation of the lattice series as performed above reduces the problem of extracting the continuum limit to finding the value of $f(\infty)$. This is done using the same Padé techniques as were used in Ref. [2]. If we perform this numerical calculation, we obtain a sequence of approximants, one for each new order in perturbation theory. The first 11 such approximants for γ_4 in (12) are plotted in Fig. 3. Each approximant is a continuous function of D for $0 \leq D \leq 4$. Note that the approximants are smooth and well behaved; the sequence is monotonically increasing and appears to converge uniformly to a limiting curve. The dotted line on Fig. 3 is an extrapolation of these approximants to this limiting curve obtained using Richardson extrapolation [10].

There are a number of ways to assess the accuracy of the limiting curve. First, one can Taylor expand this limiting curve about $D = 0$ as a series in powers of D . This Taylor series has the form

$$\begin{aligned} \gamma_4^{\text{limiting curve}}(D) &= \frac{1}{12}(1 + 1.177D + 0.640D^2 + 0.195D^3 \\ &\quad + 0.028D^4 - 0.004D^5 \\ &\quad - 0.003D^6 - \dots). \end{aligned} \quad (14)$$

We may then compare this Taylor series with that recently obtained [7] using dimensional expansion methods:

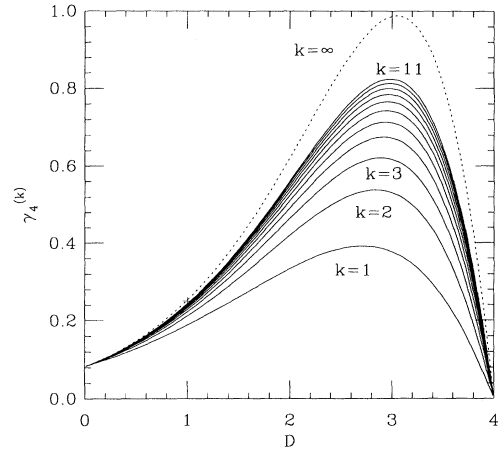


FIG. 3. First 11 approximants to γ_4 plotted as functions of D for $0 \leq D \leq 4$ [see (12)]. The approximants form a monotonically increasing sequence of curves; the labeling indicates the order. Note that the approximants are smooth curves that seem to be tending uniformly to a limiting curve. We have obtained this limiting curve (dotted curve) by means of fifth-order Richardson extrapolation [10]. The exact result $\gamma_4 = \frac{1}{4}$ at $D = 1$ is indicated by a plus sign; the limiting curve passes within 1% of this point.

$$\begin{aligned} \gamma_4^{\text{dimensional expansion}}(D) &= \frac{1}{12}(1 + 1.18D + 0.62D^2 \\ &\quad + 0.18D^3 + 0.03D^4 + \dots). \end{aligned} \quad (15)$$

Note that the coefficients of these two series are almost identical [11]. Second, we can examine the limiting curve at $D = 1$, for which the exact value $\gamma_4 = \frac{1}{4}$ is known. At this value of D , the limiting curve has the value 0.2526 , and so it is slightly higher by about 1%.

In Figs. 4 and 5, we demonstrate how we obtain the limiting curve for the cases $D = 2$ and 3 . We have plotted the n th-order Richardson extrapolants for the approximants in Fig. 3 for $n = 1, 2, \dots, 5$ versus the inverse order of the approximants. We then determine where each of these extrapolants crosses the vertical axis (each intersection is indicated by a horizontal bar). Finally, we extrapolate to the limiting value of these intersection points. This procedure gives the value $\gamma_4 = 0.620 \pm 0.007$ at $D = 2$, indicated in Fig. 4 by a fancy square. This result is to be compared with $\gamma_4 = 0.6108 \pm 0.0025$ obtained by Baker and Kincaid [12,13]. Similarly, in Fig. 5 we find that the limiting curve gives $\gamma_4 = 0.986 \pm 0.010$ at $D = 3$. This value compares reasonably well with previous results, as tabulated in Ref. [1]. For example, Baker and Kincaid obtain 0.98, Monte Carlo studies give results between 0.9 and 1, and renormalization group studies give results around 0.98. Note that the limiting curve in Fig. 3 has a maximum extremely close to $D = 3$; numerically, the maximum occurs at $D = 3.03$.

The same procedure that was used to extrapolate (1) to the continuum and thereby to obtain a plot of γ_4 as a function of D can be applied to (2). We perform a Borel summation of the series in (2) as follows. The lattice series in (2) for γ_6 has the general form

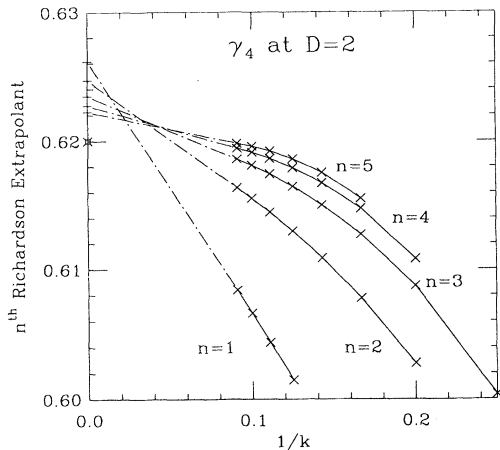


FIG. 4. Plot of the n th-order Richardson extrapolants for the approximants in Fig. 3 for $n = 1, 2, \dots, 5$ versus the inverse order of the approximants for the case $D = 2$. Each Richardson extrapolant is linearly extended (dot-dashed line) until it intersects the vertical axis. Each intersection is indicated by a horizontal bar. We then extrapolate to the limiting value of these intersection points, indicated by a fancy square. This procedure predicts that $\gamma_4 = 0.620$ at $D = 2$.

$$\gamma_6 = \frac{y^{-D}}{30} \sum_{k=0}^{\infty} S_k(D) y^k, \quad (16)$$

where $S_k(D)$ are polynomials of maximum degree k . One can read off the first 11 polynomials $S_k(D)$ from (2). From the structure of $\gamma_6^{(N=\infty)}$ in (4), we are motivated to rewrite (16) in the form

$$\frac{y^{D/2} \int_0^{\infty} dt t^2 e^{-t} [e^{-2ty} I_0(2ty)]^D}{30\gamma_6} = \left[y^{D/2} \sum_{k=0}^{\infty} T_k(D) y^k \right]^3, \quad (17)$$

where $T_k(D)$ is another polynomial in D . Again, we insert the identity (9) for each term in the sum in (17). This converts (17) to the form

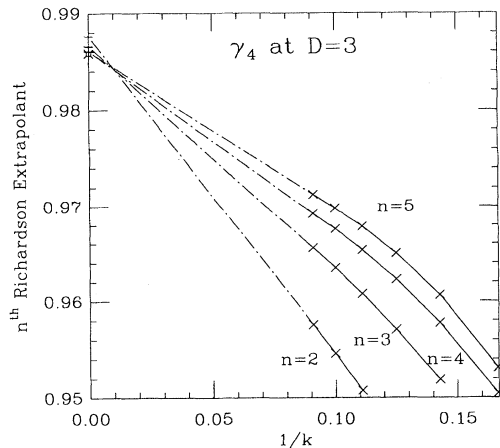


FIG. 5. Same as in Fig. 4, except that $D = 3$. This extrapolation procedure predicts that $\gamma_4 = 0.986$ at $D = 3$.

$$\frac{y^{D/2} \int_0^{\infty} dt t^2 e^{-t} [e^{-2ty} I_0(2ty)]^D}{30\gamma_6} = \left[\int_0^{\infty} dt t^{1-D/2} e^{-t} g(ty)^{D/2} \right]^3, \quad (18)$$

where we define

$$g(x) = x \left(\sum_{k=0}^{\infty} \frac{T_k(D) x^k}{(k+1)!} \right)^{2/D} = x \sum_{k=0}^{\infty} U_k(D) x^k, \quad (19)$$

where $U_k(D)$ are polynomials in D of degree $[k/2] - 1$ similar in structure to those in (13) [9].

Next, we take the continuum limit of the expression (18). Assuming that $g(\infty)$ exists in the limit $y \rightarrow \infty$, we find that (18) separates into a product of several terms and we have

$$\gamma_6 = \frac{(4\pi)^{-D/2} \Gamma(3 - D/2)}{30\Gamma(2 - D/2)^3} g(\infty)^{-3D/2}. \quad (20)$$

Thus we have forced the continuum limit of the six-point function to take the form of the large- N result given in (6) apart from $g(\infty)$, which is a slowly varying function of D .

Again, the resummation of the lattice series reduces the problem of extracting the continuum limit to finding the value of $g(\infty)$. This is done using the same Padé techniques as were used in Ref. 2. We perform this numerical calculation and obtain a sequence of approximants, one for each new order in perturbation theory. The first 11 such approximants for γ_6 in (20) are plotted in Fig. 6. Each approximant is a continuous function of D for $0 \leq D \leq 4$. As in Fig. 3, the approximants are smooth and well behaved; the sequence is monotonically increasing and appears to converge uniformly to a

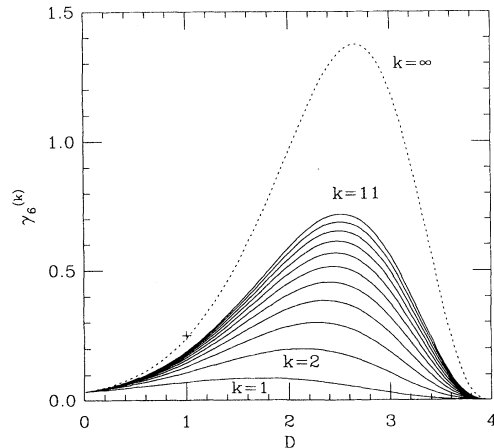


FIG. 6. First 11 approximants to γ_6 plotted as functions of D for $0 \leq D \leq 4$ [see (20)]. The approximants form a monotonically increasing sequence of curve as indicated by the labeling. As in Fig. 3 for the four-point function, the approximants are smooth curves that seem to be tending uniformly to a limiting curve. This limiting curve (dotted curve) is a fifth-order Richardson extrapolation. The exact result $\gamma_6 = \frac{1}{4}$ at $D = 1$ is indicated by a plus sign; the limiting curve passes within 4% of this point.

limiting curve indicated in Fig. 6 by a dotted line. This limiting curve is again obtained using fifth-order Richardson extrapolation. The limiting curve at $D = 1$ passes through the value $\gamma_6 = 0.240$, which differs from the exact value $\gamma_6 = \frac{1}{4}$ by about 4%.

The limiting curve predicts that $\gamma_6 = 0.96 \pm 0.04$ at $D = 2$ and $\gamma_6 = 1.2 \pm 0.1$ at $D = 3$. The value at $D = 3$ is lower than most previous results, as tabulated in Ref. [1], but it is certainly finite. An earlier conjecture [7] that γ_6 might be singular at $D = 3$ seems unjustified now in light of this result [11]. By comparison, an ϵ expansion around $D = 4$ gives [1,14]

$$\frac{\gamma_6}{(\gamma_4)^2} = 2\epsilon - \frac{20}{27}\epsilon^2 + 1.2759\epsilon^3 + \dots \quad (21)$$

This series may be divergent, but a direct optimal truncation of the series after one term with $\epsilon = 1$ gives the value $\gamma_6 = 1.9 \pm 0.7$. (Here we have substituted the value $\gamma_4 = 0.986$ given above.) However, if we perform

a (1,1)-Padé summation of this series, which seems justified because of the alternating sign pattern, we obtain $\gamma_6 = 1.66 \pm 0.28$ with a central value in better agreement with that of our predicted value. Of course, the numerical results for γ_6 at $D = 3$ that have been obtained thus far with Monte Carlo and strong-coupling lattice calculations are not yet very good; hopefully, they will be improved in the future.

Finally, we observe that the maxima of γ_{2i} as a function of D appear to follow a pattern. We observed already that γ_4 has a maximum that is close to $D = 3$. Here we find that the limiting curve for γ_6 has a maximum at $D = 2.66$, which is very close to $\frac{8}{3}$. An interesting conjecture is that in general the maximum might be located at $D_{\max} = \frac{2(i+1)}{i}$, the value of D for which a ϕ^{2i+2} theory becomes free.

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