Observables for two-dimensional black holes

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We consider the most general dilaton gravity theory 1+1 dimensions. By suitably parametrizing the metric and scalar field we find a simple expression that relates the energy of a generic solution to the magnitude of the corresponding Killing vector. In theories that admit black hole solutions, this relationship leads directly to an expression for the entropy $S = 2\pi\tau_0/G$, where τ_0 is the value of the scalar field (in this parametrization) at the event horizon. This result agrees with the one obtained using the more general method of Wald. Finally, we point out an intriguing connection between the black hole entropy and the imaginary part of the "phase" of the exact Dirac quantum wave functionals for the theory.

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I. INTRODUCTION

Two-dimensional theories of gravity have been the subject of much interest for a number of years because of their connection to string theory and their interesting mathematical properties. There has been an explosion of work in this area in the last few years due to the discovery by Callan *et al.* [1] that such string theories may provide models for black hole evaporation in which fundamental questions concerning the end point of collapse could in principle be addressed rigorously, if not exactly.

Most theories that have been considered contain one scalar field plus the graviton field. The most such general coordinate invariant theory (with at most two derivatives) has been examined by Banks and O'Laughlin [2] and subsequently by others [3]. Special cases of current interest include the string-derived model, the Jackiw-Teitelboim model [4], and spherically symmetric gravity [5].

Here we consider the classical observables in the most general two-dimensional (2D) dilaton gravity theory. Our main interest is in theories that have black hole solutions. By choosing a convenient parametrization for the scalar field and metric tensor, it is possible to write a very simple coordinate invariant expression for the Killing vector in the general theory. This can then be used to shed considerable light on the remaining observables in the theory. For example we prove that the coordinate invariant constant parametrizing the solutions is the conserved quantity associated with translations along the Killing direction (i.e., the energy). We also show that the momentum conjugate to the energy is the (Killing) time separation at infinity. This result has been previously shown for spherically symmetric gravity [6,7]. In addition, knowledge of the Killing vector enables us to calculate the surface gravity for a generic 2D black hole, and derive a simple expression relating the energy to the value of the scalar field at the horizon. This leads to a very simple derivation of the entropy for a generic 2D black hole. From this expression we are able to show a deep connection between the entropy, and the imaginary part of the phase of the physical quantum wave functional, which was derived for the general theory in [8].

II. ACTION AND KILLING VECTOR

The most general action functional depending on the metric tensor $g_{\mu\nu}$ and a scalar field ϕ in two spacetime dimensions, such that it contains at most second derivatives of the fields can be written [2]

$$S[g,\phi] = \frac{1}{2G}\int d^2x\sqrt{-g}\Big(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{\ell^2}\tilde{V}(\phi) + D(\phi)R\Big). \tag{1}$$

The metric, scalar field, and (1+1)-dimensional gravitation constant G are assumed to be dimensionless. This requires the introduction of a dimensionful parameter into the potential. We have chosen to make this parameter explicit in the Lagrangian, since it plays an important role in determining the dimensionally correct physical observables in the generic theory. For spherically symmetric gravity, $\ell = \ell_P$ is the Planck length. As first discussed in [2] and shown explicitly in [8], by reparametrizing the fields

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$$g_{\mu\nu} \to h_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu}, \quad (2)$$

$$\phi \to \tau = D(\phi), \quad (3)$$

with $\Omega^2 = \exp(\frac{1}{2}\int \frac{d\phi}{(dD/d\phi)})$ one can eliminate the kinetic term for the scalar and put the action in the form

$$I[h,\tau] = \frac{1}{2G}\int_{M^2} d^2x\sqrt{-h}\left(\tau R(h) - \frac{1}{\ell^2}V(\tau)\right), \quad (4)$$

where $V(\tau)$ is an arbitrary function of the scalar field τ.

The equations of motion take the simple form

$$R = \frac{1}{\ell^2}\frac{dV}{d\tau}, \quad (5)$$

and

$$\nabla_\mu \nabla_\nu \tau + \frac{1}{2\ell^2} g_{\mu\nu} V = 0 \ . \quad (6)$$

The most general solution to these equations has been found [9]. In the convenient gauge

$$\tau = x/\ell, g_{t,x} = 0 \ , \quad (7)$$

the solution is

$$ds^2 = -[-J(x/\ell) - C]dt^2 + [-J(x/\ell) - C]^{-1}dx^2 \ , \quad (8)$$

where $J'(\tau) = V(\tau)$ and C is a coordinate-invariant constant of integration that characterizes the physically distinct solutions in the theory. It can be expressed in covariant form:

$$C = -|\nabla\tau|^2\ell^2 - J(\tau) \ . \quad (9)$$

We will show later that $C/2\ell$ is the energy of the solution.

Since the solutions given above depend only on the spatial coordinate, clearly they each have a Killing vector, so that the generalization to Birkhoff's theorem holds for 2D dilaton gravity, as shown in [9]. The Killing vector can in fact be written in any coordinate system as

$$k^\mu = \ell\eta^{\mu\nu}\tau_{,\nu} \ . \quad (10)$$

In the above $\eta^{\mu\nu} = -\eta^{\nu\mu} = \frac{1}{\sqrt{-g}}\epsilon^{\mu\nu}$ is the antisymmetric *tensor*. The constant ℓ is required to make the Killing vector components dimensionless. It can easily be verified that Eq. (6) implies that k^μ satisfies the Killing equation $\nabla_{(\mu}k_{\nu)} = 0$ on shell. Moreover, it is clear that $\tau_{,\mu}k^\mu = 0$ identically, so that the scalar field is also invariant along the Killing directions. Note that

$$|k|^2 = -\ell^2|\nabla\tau|^2 = C + J(\tau) \ . \quad (11)$$

The question of which of the generic dilaton theories admit black hole solutions can be addressed at this point [3]. A necessary condition that a model admit a black hole configuration is that there exists at least one curve in spacetime given by $\tau(x,t) = \tau_0$ =const, such that $J(\tau_0) = -C$. In addition, $J(\tau)$ must be monotonic (in τ) in a neighborhood of τ_0.

Before closing this section, we will display the various quantities defined here in the special case of spherically symmetric gravity, for which $V(\tau) = -1/\sqrt{2\tau}$. The static solution for the metric in our parametrization is related to the usual Schwarzschild solution by the conformal reparametrization $ds^2 = \sqrt{2\tau}ds^2_{\text{Schwarz}}$. In terms of the coordinate $r = \ell\sqrt{2\tau}$, the metric Eq. (8) takes the form

$$h_{\mu\nu}dx^\mu dx^\nu = \frac{r}{\ell}\{-(1-2m/r)dt^2 + (1-2m/r)^{-1}\}dr^2 \ , \quad (12)$$

where the mass $m = \ell C/2$. Finally, $(k^\mu) = (1,0)$ and $|k|^2 = (2m - r)/\ell$.

III. HAMILTONIAN ANALYSIS

We now review a Hamiltonian analysis of the general (1+1)-dimensional theory [8]. Spacetime is split into a product of space and time: $M_2 \simeq \Sigma \times R$ and the metric $h_{\mu\nu}$ is given an Arnowitt-Deser-Misner- (ADM-)like parametrization [10]:

$$ds^2 = e^\alpha[-(M^2+N^2)dt^2 + (dx + Mdt)^2] \ , \quad (13)$$

where α, M, and N are functions on spacetime M_2. We define the quantity σ by $\sigma^2 := M^2 + N^2$. Also, in the following, we denote by the overdot and prime, respectively, derivatives with respect to the time coordinate t and spatial coordinate x.

The canonical momenta for the fields $\{\alpha, \tau\}$ are, respectively,

$$\Pi_\alpha = \frac{1}{2G\sigma}(M\tau' - \dot{\tau}), \quad (14)$$

$$\Pi_\tau = \frac{1}{2G_0}(-\dot{\alpha} + M\alpha' + 2M'). \quad (15)$$

The vanishing of the momenta canonically conjugate to M and σ yield the primary constraints for the system. Following the standard Dirac prescription [11], we obtain the canonical Hamiltonian (up to spatial divergences)

$$H_0 = \int dx\left(M\mathcal{F} + \frac{1}{2G}\sigma\mathcal{G}\right). \quad (16)$$

where we have defined

$$\mathcal{F} := \alpha'\Pi_\alpha + \tau'\Pi_\tau - 2\Pi'_\alpha, \quad (17)$$

$$\mathcal{G} := 2\tau'' - \alpha'\tau' - (2G)^2\Pi_\alpha\Pi_\tau + \frac{1}{\ell^2}e^\alpha V(\tau) \ . \quad (18)$$

Clearly $\frac{1}{2G}\sigma$ and M play the role of Lagrange multipliers that enforce the secondary constraints $\mathcal{F} \approx 0$ and $\mathcal{G} \approx 0$.

The energy can be constructed by noting that the following linear combination of the constraints is a total spatial derivative:

$$\tilde{\mathcal{G}} := \frac{\ell}{2}e^{-\alpha}[(2G)^2\Pi_\alpha\mathcal{F} + \tau'\mathcal{G}]$$
$$= (q[\alpha,\tau,\Pi_\alpha,\Pi_\tau])' \approx 0, \tag{19}$$

where we have defined the variable q as

$$q := \frac{\ell}{2}[e^{-\alpha}((2G\Pi_\alpha)^2 - (\tau')^2) - \ell^{-2}J(\tau)] \ . \tag{20}$$

The expression on the right-hand side above is nominally an implicit function of the spatial coordinate, but is constant on the constraint surface. Moreover, it is straightforward to show that q commutes with both constraints $\mathcal{F},\mathcal{G}$. Thus, the constant mode of q is a physical observable in the Dirac sense.

In terms of the canonical momenta the magnitude of the Killing vector can be written as

$$|k|^2 = \ell^2 e^{-\alpha}[(2G\Pi_\alpha)^2 - (\tau')^2] \ . \tag{21}$$

Thus the observable q is

$$q = \frac{1}{2\ell}[|k|^2 - J(\tau)]$$
$$= \frac{C}{2\ell} \ . \tag{22}$$

It is worth noting that the constancy of q in *spacetime* follows by contracting the field equations Eq. (6) by $\nabla^\mu\tau$.

We now prove that the generator $\tilde{\mathcal{G}} = q'$ generates diffeomorphisms along the direction of the Killing vector k^μ. We consider an infinitesimal translation $x^\mu \to x^\mu + f^\mu$, where $f^\lambda := -vk^\lambda$. Using the ADM parametrization Eq. (13), we find on the constraint surface that

$$\delta\tau = 0 \ , \tag{23}$$
$$\delta\alpha = 4G\ell e^{-\alpha}\Pi_\alpha v' \ . \tag{24}$$

On the other hand, the transformations generated by $\tilde{\mathcal{G}}(v)$ are

$$\bar{\delta}\tau := \{\tilde{\mathcal{G}}(v),\tau(x)\} = 0, \tag{25}$$
$$\bar{\delta}\alpha = \{\tilde{\mathcal{G}}(v),\alpha(x)\} \tag{26}$$
$$= (2G)^2\ell e^{-\alpha}\Pi_\alpha v'. \tag{27}$$

Comparing these two transformations one finds that the observable q/G is the conserved quantity associated with translations along the Killing vector, i.e., it is the energy. This can also be verified by writing the canonical Hamiltonian in terms of $\tilde{\mathcal{G}}$. One finds

$$H_0 = -\left(\frac{\dot{\tau}}{\tau'}\right)\mathcal{F} + \left(\frac{\sigma e^\alpha}{\ell\tau'}\right)\frac{q'}{G}. \tag{28}$$

In order to obtain Hamilton's equations, it is necessary to add the following surface term to the canonical Hamiltonian:

$$H_{\rm ADM} = \int dx \left[\left(\frac{\sigma e^\alpha}{\ell\tau'}\right)\frac{q}{G}\right]' \ . \tag{29}$$

It is easy to verify that for solutions of the form Eq. (8), $\sigma e^\alpha/\ell\tau' = 1$. Hence, $H_{\rm ADM} = q/G$ is the ADM energy, as expected.

The momentum conjugate to q is found by inspection to be [8]

$$p := -\int_\Sigma dx \frac{2\Pi_\alpha e^\alpha}{(2G\Pi_\alpha)^2 - (\tau')^2}. \tag{30}$$

It can easily be verified that the Poisson algebra for the fields and the momenta leads directly to $\{q,p\} = 1$. Under a general gauge transformation

$$\delta p := \{\mathcal{G}(v) + \mathcal{F}(w), p\}$$
$$= -\int dx \left(\frac{e^\alpha(w\Pi_\alpha - v\tau')}{(2G\Pi_\alpha)^2 - (\tau')^2}\right)' \ . \tag{31}$$

Thus p is gauge invariant only if the test functions v and w vanish sufficiently rapidly at infinity. The value of p depends on the global properties of the spacetime slicing. This is consistent with the generalized Birkhoff theorem [8] which states that there is only one independent diffeomorphism invariant parameter characterizing the space of solutions.

It is instructive to write the observable p in covariant form:

$$p = \int_\Sigma dx e^{\alpha/2} n^\mu \frac{\nabla_\mu\tau}{|k|^2} \tag{32}$$
$$= -2\int_\Sigma dx^\mu \frac{k_\mu}{|k|^2}. \tag{33}$$

Note that $dxe^{\alpha/2}$ is the measure induced on Σ by $h_{\mu\nu}$. In the expression for p the vector field n^μ is the unit (timelike) normal to Σ.

Using Eq. (33) it is straightforward to show that the global variable p is the time separation at infinity of neighboring spacelike surfaces which are asymptotically normal to the Killing vector field k^μ. We suppose that $V(\tau)$ is such that in the region exterior to the event horizon, k^μ is timelike. Let U be the "triangular region" of spacetime bounded by spacelike surfaces Σ_1, Σ_2, and by a timelike surface T at infinity tangent to k^μ. It is straightforward to show that $\nabla_\mu(\nabla^\mu\tau/|k|^2) \equiv 0$ in U for any solution of the equations of motion. Hence by Gauss' theorem

$$0 = \int_U d^2x \nabla_\mu\left(\frac{\nabla^\mu\tau}{|k|^2}\right)$$
$$= p_2 - p_1 + \int_T d\mu[T] t^\mu \frac{\nabla_\mu\tau}{|k|^2} \ , \tag{34}$$

where p_1, p_2 are the values of p on Σ_1, Σ_2, respectively, $d\mu[T]$ is the measure on T and t^μ is the outward unit normal to T. Now at infinity, the integral over T above is just the time separation of the spacelike surfaces. Indeed, by definition, $t^\mu = \tau^{,\mu}/|k|^2$. Choose the measure $d\mu[T] = \sqrt{h_{\theta\theta}}d\theta$, where θ is the parametrization of the timelike line T such that the induced metric $h_{\theta\theta} := h_{\mu\nu}\frac{\partial x^\mu}{\partial\theta}\frac{\partial x^\nu}{\partial\theta} = |k|^2$. From this we immediately get the desired result.

Finally, we observe that the integrand of the observable p has a pole at the location of any event horizon in the model. Thus, analytic continuation is, in general, required to make the expression well defined, and may introduce an imaginary part to the observable p. For example, in spherically symmetric gravity, one can show that in Kruskal coordinates the observable p integrated along a slice of constant Kruskal time T takes the simple form

$$p = \frac{2m}{G}\int dX\left[\frac{1}{X-T}-\frac{1}{X+T}\right] = \frac{2m}{G}\ln\left(\frac{X-T}{X+T}\right)\Bigg|_{X_i}^{X_f} . \tag{35}$$

p is therefore precisely the difference in Schwarzchild times at the end points of the spatial slice. In this case there are simple poles at $X = \pm T$ (i.e., at $r = 2m$), so that for an eternal black hole, with suitable analytic continuation, $\mathrm{Im}p = 2\pi m/G$. Although this potential imaginary piece is irrelevant classically for the Schwarzschild time, it may have some significance in the quantum theory in which p is a physical phase space observable. This will be discussed below.

IV. THERMODYNAMICAL PROPERTIES

We now calculate the surface gravity and entropy of a generic 2D black hole. The surface gravity κ is determined by the following expression, evaluated at the event horizon [12]:

$$\kappa^2 = -\frac{1}{2}\nabla^\mu k^\nu \nabla_\mu k_\nu . \tag{36}$$

Using Eq. (10) for k^μ and the field equations Eq. (6) it is straightforward to show that

$$\kappa = -\frac{1}{2\ell}V(\tau_0) , \tag{37}$$

where $V(\tau_0)$ is the potential evaluated at $\tau = \tau_0$ (i.e., on the event horizon). The sign in Eq. (37) was chosen to yield a positive surface gravity for positive energy. Note that τ_0 is given implicitly as a function of the energy q by requiring $|k|^2 = 0$ in Eq. (22).

The Hawking temperature for the generic black hole solution can easily be calculated by defining the Euclidean time $t_E = it$ in Eq. (8) and then finding the periodicity condition on t_E that makes the solution everywhere regular. This is done by defining the coordinate $R^2 := -a[J(\tau) + C]$ and choosing the constant a so that the spatial part of the metric goes to dR^2 at the event horizon τ_0. A straightforward calculation gives $a = |2\ell/V(\tau_0)|$, so that the Hawking temperature, which is the inverse of the period of t_E, is

$$T_H = \frac{1}{2\pi}\frac{V(\tau_0)}{2\ell} = \frac{\kappa}{2\pi} , \tag{38}$$

as expected. Note that this calculation does not depend on the details of the model: it merely requires the existence of a horizon at which $J(\tau_0) = -C$.

The entropy S can now easily be determined by inspection of Eq. (22). In particular, if we vary the solution, but stay on the event horizon, we find that the variation of the energy is

$$\delta\mathcal{E} = \delta(q/G) = -\frac{1}{2\ell G}V(\tau_0)\delta\tau_0 . \tag{39}$$

Identifying the Hawking temperature and surface gravity derived above, we find that the first law of thermodynamics $\delta E = T\delta S$ will be satisfied providing we identify the entropy to be

$$S = \frac{2\pi}{G}\tau_0. \tag{40}$$

Recently Wald [13] formulated a local geometric expression for the entropy of a black hole in any Lagrangian-based theory which admits black hole solutions. Following is a brief summary of this construction.

Denote any dynamical fields in the theory by ϕ. Under a diffeomorphism generated by v^μ, the Lagrangian L considered as a two form, transforms as

$$\delta L = E\cdot\mathcal{L}_v\phi + d\Theta, \tag{41}$$

where the product in the first term includes a summation over the dynamical fields and contraction over the tensor indices. The components of E are just the Euler-Lagrange expressions for the action, and hence the first term vanishes on-shell. The second term is the exterior derivative of a one-form field Θ, which depends on ϕ and $\mathcal{L}_v\phi$. From the identity $\mathcal{L}_v\gamma = v\cdot d\gamma + d(v\cdot\gamma)$ for any differential form γ, the invariance of the action under diffeomorphisms implies that on shell the expression

$$j := \Theta - v\cdot L \tag{42}$$

is closed. Furthermore it can be demonstrated [14] that on-shell j is exact, i.e., $j = dQ$, where Q is a zero-form, locally constructed from the dynamical fields and their Lie derivatives with respect to v.

If black hole solutions exist, Wald showed that the quantity

$$S := \frac{2\pi}{\kappa}\mathcal{Q}(x_0), \tag{43}$$

behaves like the entropy of the black hole. For the generic dilaton gravity models, it can be shown that

$$\mathcal{Q} = \frac{1}{2G}\eta_{\mu\nu}(2v^\mu\nabla^\nu\tau + \tau\nabla^\mu v^\nu) , \tag{44}$$

where v is an arbitrary diffeomorphism. Now for the case that $v^\mu = k^\mu = \ell\eta^{\mu\nu}\nabla_\nu\tau$, it follows that Wald's expression for the entropy is

$$S = \frac{2\pi}{G}\tau(x_0), \tag{45}$$

in agreement with the result obtained above. It also gives the correct answer for spherically symmetrical gravity (for which $\tau_0 = 2m^2/\ell_p^2$) and agrees with the results obtained by Frolov [15] and Iyer and Wald [16] for string motivated models.

It is perhaps worth noting that the very simple expressions given above for the Killing vector, surface gravity, and entropy are only valid in the given parametrization, which was obtained from the generic form by a conformal reparametrization of the metric. It is therefore worthwhile to ask how such conformal reparametrizations affect the physical quantities described above. First of all, the Killing vector is invariant under such a transformation, since for solutions, the conformal factor $\Omega^2(\tau)$ is also invariant along the Killing directions. A straightforward calculation shows that the surface gravity for a given solution is also unchanged. Since the energy, q, which is the conserved quantity associated with translations along the Killing direction is also presumably invariant under reparametrizations of the fields (that leave the Killing vector invariant), the above arguments would lead to precisely the same value for the entropy in any parametrization (although the dependence of the entropy on the fields will, in general, be considerably more complicated in different parametrizations).

The exact quantum wave functional which solves the constraints with a particular factor ordering has been found to be [8]

$$\psi_{\text{phys}}[q;\alpha,\tau] = \exp\left(\frac{i}{G}\chi[q;\alpha,\tau]\right) , \tag{46}$$

where the "phase" is given by

$$\chi[q;\alpha,\tau] = \int dx \left[Q + \frac{\tau'}{2}\ln\left(\frac{\tau' - Q}{\tau' + Q}\right)\right] , \tag{47}$$

with

$$Q := \sqrt{(\tau')^2 + e^{\alpha}\left(\frac{2q}{\ell} + \frac{J(\tau)}{l^2}\right)}, \tag{48}$$

which is equal to $(2G\Pi_\alpha)$ on the constraint surface. If we restrict to classically allowed regions, for which $Q^2 \geq 0$ then the phase S can acquire an imaginary part from the logarithm when $(\tau')^2 - Q^2 \leq 0$. This is precisely the region where the Killing vector for the solution is spacelike (in a nonsingular coordinate system for which e^α is positive). Therefore for theories with an event horizon, the logarithm in Eq. (47) can be analytically continued so that

$$\text{Im}\chi = \frac{i\pi\tau_0}{2} = i\frac{S}{4} . \tag{49}$$

The imaginary part of the phase is therefore proportional to the entropy of the black hole. This is consistent with an earlier heuristic result obtained for spherically symmetric gravity [17].

V. CONCLUSIONS

We have shown that in a suitable parametrization the Killing vector for a generic 2D black hole takes a particularly simply form, and can be used to shed considerable light on the nature of the physical observables in the theory. In particular we were able to show that the space of physical observables consists of two conjugate variables: the energy and the Killing time-separation at infinity. The former is the conserved quantity associated with translations along the Killing direction. The latter depends explicitly on the global properties of the spacetime slicing, as required by the generalized Birkhoff's theorem valid for such theories. Moreover, in the calculation of the time-separation, it is necessary to continue analytically through the event horizon (in those models which display this feature). We also used the explicit expression for the Killing vector to calculate the surface gravity and entropy for the general theory. The latter agrees with the result obtained using Wald's more general method. Finally we showed an intriguing relationship between the entropy and imaginary part of the phase of the exact quantum wave functional in the Dirac quantized theory.

We therefore believe that the above formalism provides a powerful tool for analyzing the classical, quantum and thermodynamical properties of generic (1+1)-dimensional black holes.

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