Quantum-chromodynamic potential model for light-heavy quarkonia and the heavy quark effective theory

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(Received 29 July 1994)

We have investigated the spectra of light-heavy quarkonia with the use of a quantum-chromodynamic potential model, which is similar to that used earlier for heavy quarkonia. An essential feature of our treatment is the inclusion of the one-loop radiative corrections to the quark-antiquark potential, which contribute significantly to the spin splitting among the quarkonium energy levels. Unlike $c\bar{c}$ and $b\bar{b}$, the potential for a light-heavy system has a complicated dependence on the light and heavy quark masses m and M, and it contains a spin-orbit mixing term. We have obtained excellent results for the observed energy levels of D^0, D_s, B^0, and B_s, and we are able to provide predicted results for many unobserved energy levels. Our potential parameters for different quarkonia satisfy the constraints of quantum chromodynamics. We have also used our investigation to test the accuracy of the heavy quark effective theory. We find that the heavy quark expansion yields generally good results for the B^0 and B_s energy levels provided that M^{-1} and $M^{-1}\ln M$ corrections are taken into account in the quark-antiquark interactions. It does not, however, provide equally good results for the energy levels of D^0 and D_s, which indicates that the effective theory cannot be applied accurately to the c quark.

PACS number(s): 12.39.Pn, 12.39.Hg, 14.40.Lb, 14.40.Nd

I. INTRODUCTION

The light-heavy quarkonia D, D_s, B, and B_s are of much experimental and theoretical interest, and their exploration is necessary for our understanding of the strong as well as the electroweak interactions [1,2]. We shall here investigate the spectra of light-heavy quarkonia with the use of a quantum chromodynamic model similar to the highly successful model used earlier for the heavy quarkonia $c\bar{c}$ and $b\bar{b}$ [3]. The complexity of the model is necessarily enhanced for a light-heavy system because the potential has a complicated dependence on the light and heavy quark masses m and M, and it contains a spin-orbit mixing term. We shall obtain results for the observed and the unobserved energy levels of $D^0(c\bar{u})$, $D_s(c\bar{s})$, $B^0(b\bar{d})$, and $B_s(b\bar{s})$, compare them with the available experimental data, and examine their scaling behavior. The u-d mass difference and the electromagnetic interaction will be ignored in the present investigation [4].

We shall also use our results to test the accuracy of the heavy quark effective theory [5,6] both in the limit of $M \to \infty$ as well as with the inclusion of the M^{-1} and $M^{-1}\ln M$ corrections.

The approximate heavy quark symmetry, like the approximate chiral symmetry, points to an underlying difference between the (u, d, s) and the (c, b, t) quarks. It is interesting that this fundamental difference was recognized in our mass-matrix approach to quark mixing and CP violation [7], which predicted the value $M_t \leq$ 170 GeV for the top quark mass in excellent agreement with the recently reported experimental value of 174 ± 17 GeV [8].

II. LIGHT-HEAVY QUARKONIUM SPECTRA

Our treatment for the light-heavy quarkonia is similar to that for $c\bar{c}$ and $b\bar{b}$ [3] except for the complications arising from the difference in the quark and antiquark masses. Thus, our model is based on the Hamiltonian

$$H = H_0 + V_p + V_c, \tag{1}$$

where

$$H_0 = (m^2 + \mathbf{p}^2)^{1/2} + (M^2 + \mathbf{p}^2)^{1/2} \tag{2}$$

is the relativistic kinetic energy term, and V_p and V_c are nonsingular quasistatic perturbative and confining potentials, which are given in Appendix A. Our trial wave function for obtaining the quarkonium energy levels and wave functions is of the same form as in the earlier investigations. Since our potentials are nonsingular, we are able to avoid the use of an illegitimate perturbative treatment.

The experimental and theoretical results for the energy levels of the light-heavy quarkonia D^0, B^0, D_s, and B_s, together with the ${}^3P_1'$-${}^1P_1'$ mixing angles arising from the spin-orbit mixing terms, are given in Tables I–IV. For experimental data we have relied on the Particle Data Group [9] except that we have used the more recent results from the CLEO collaboration [10,11] for D_1^0, D_2^{*0}, and D_{s2} and from the Collider Detector at Fermilab (CDF) Collaboration [12] for B_s. In these tables, one set of theoretical results corresponds to the direct use of our model, while the other two sets are obtained

TABLE I. D^0 energy levels in MeV. Effective theory results are given with the M^{-1} and $M^{-1}\ln M$ corrections as well as in the limit of $M\to\infty$. Experimental results are from Refs. [9] and [10].

	Expt.	Theory	Effective theory	$M\to\infty$
$1\,^1S_0$ (D^0)	1864.5±0.5	1864.5	1864.5	1864.5
$1\,^3S_1$ (D^{*0})	2007±1.4	2007.0	2010.9	1864.5
$2\,^1S_0$		2547.7	2566.5	2431.9
$2\,^3S_1$		2647.0	2662.1	2431.9
$1\,^3P_0$		2278.6	2310.2	2244.8
$1\,^3P_1'$		2407.3	2414.6	2244.8
$1\,^3P_2$ (D_2^0)	2465±4.2	2465.0	2474.0	2287.2
$1\,^1P_1'$ (D_1^0)	2421±2.8	2421.0	2438.2	2287.2
θ		29.0°	30.9°	35.6°

TABLE II. D_s energy levels in MeV. Experimental results are from Refs. [9] and [11].

	Expt.	Theory	Effective theory	$M\to\infty$
$1\,^1S_0$ (D_s)	1968.8±0.7	1968.8	1968.8	1968.8
$1\,^3S_1$ (D_s^*)	2110.3±2.0	2110.5	2113.1	1968.8
$2\,^1S_0$		2656.5	2678.8	2536.5
$2\,^3S_1$		2757.8	2774.3	2536.5
$1\,^3P_0$		2387.8	2422.2	2382.2
$1\,^3P_1'$		2521.2	2528.8	2382.2
$1\,^3P_2$ (D_{s2})	2573.2±1.9	2573.1	2582.8	2402.8
$1\,^1P_1'$ (D_{s1})	2536.5±0.8	2536.5	2552.1	2402.8
θ		26.0°	31.8°	35.6°

by means of heavy quark expansions of our potentials to test the accuracy of the heavy quark effective theory with the inclusion of the M^{-1} and $M^{-1}\ln M$ corrections as well as without these corrections. The approximate potentials corresponding to the effective theory are given in Appendix B.

We expect the dynamics of a light-heavy system to be primarily dependent on the light quark. Therefore, our potential parameters for D^0 and B^0 are the same except for the difference in the c and b quark masses, and they are given by

$$\begin{aligned} m_{u,d} &= 0.350 \text{ GeV},\\ M_c &= 1.690 \text{ GeV},\\ M_b &= 5.400 \text{ GeV},\\ \mu &= 0.932 \text{ GeV},\\ \alpha_s &= 0.3965,\\ A &= 0.185 \text{ GeV}^2,\\ B &= 0.152. \end{aligned} \tag{3}$$

Similarly, the parameters for D_s and B_s are

$$\begin{aligned} m_s &= 0.514 \text{ GeV},\\ M_c &= 1.578 \text{ GeV},\\ M_b &= 5.040 \text{ GeV},\\ \mu &= 1.250 \text{ GeV},\\ \alpha_s &= 0.340,\\ A &= 0.198 \text{ GeV}^2,\\ B &= 0.131. \end{aligned} \tag{4}$$

We have ensured that the values of α_s, M_c, and M_b in (3) and (4) are related through the quantum chromodynamic transformation relations

$$\alpha_s' = \frac{\alpha_s}{1+\beta_0(\alpha_s/4\pi)\ln(\mu'^2/\mu^2)} \tag{5}$$

and

$$M' = M\left(\frac{\alpha_s'}{\alpha_s}\right)^{2\gamma_0/\beta_0}, \tag{6}$$

where $\beta_0 = 11 - \frac{2}{3}n_f$, $n_f = 3$, and $\gamma_0 = 2$. The use of the one-loop transformation relations is consistent with the inclusion of the one-loop radiative corrections in the quarkonium potentials. Moreover, since u, d, and s are the dynamical quarks in the light-heavy systems, a higher value of μ for quarkonia with the s quark is to be expected.

A precise determination of the potential parameters for the light-heavy quarkonia is difficult because of the availability of only limited experimental data. This difficulty, however, has been mitigated in our treatment by requiring that the parameters for the four systems satisfy reasonable physical and quantum-chromodynamic constraints.

We have also looked at the correlation of our parameters for the light-heavy quarkonia with those of other quarkonia. When applied to $u\bar{d}$ and $u\bar{s}$, our parameters yield good results for the π-ρ and K-K^* splittings. We are, however, unable to correlate our parameters for the light-heavy quarkonia with those for the heavy quarkonia through the transformation relations (5) and (6), and keeping in mind the past success of our quarkonium model, we can only offer the following possible explanation.

Strictly speaking, the QCD transformation relations are applicable only to the current quarks. According to our experience, the transformation relations seem to hold reasonably well for the heavy quarkonia $c\bar{c}$ and $b\bar{b}$ as well

TABLE III. B^0 energy levels in MeV. Experimental results are from Ref. [9].

	Expt.	Theory	Effective theory	$M\to\infty$
$1\,^1S_0$ (B^0)	5278.7±2.1	5278.7	5278.7	5278.7
$1\,^3S_1$ (B^{*0})	5324.6±2.1	5324.0	5325.8	5278.7
$2\,^1S_0$		5892.1	5893.9	5846.3
$2\,^3S_1$		5924.3	5927.1	5846.3
$1\,^3P_0$		5689.5	5692.5	5659.1
$1\,^3P_1'$		5730.8	5734.1	5659.1
$1\,^3P_2$		5759.1	5761.4	5701.5
$1\,^1P_1'$		5743.6	5745.4	5701.5
θ		31.7°	31.3°	35.6°

TABLE IV. B_s energy levels in MeV. Experimental results are from Refs. [9] and [12].

	Expt.	Theory	Effective theory	$M \to \infty$
$1\,^1S_0$ (B_s)	5383.3±6.7	5383.3	5383.3	5383.3
$1\,^3S_1$ (B_s^*)	5430.5±2.6	5431.9	5434.1	5383.3
$2\,^1S_0$		6000.9	6003.1	5950.9
$2\,^3S_1$		6035.8	6039.1	5950.9
$1\,^3P_0$		5810.1	5814.2	5796.7
$1\,^3P_1'$		5855.0	5857.9	5796.7
$1\,^3P_2$		5875.2	5878.1	5817.1
$1\,^1P_1'$		5860.2	5863.2	5817.1
θ		27.3°	27.1°	35.6°

as for the quarkonia containing one or two light quarks, but they are unsuitable for correlating the parameters for these two classes of quarkonia. This seems to be another manifestation of the difference between light and heavy quarks. We believe a full explanation would require an understanding of the origin of the constituent quark masses, which remains unclear at this time.

Our phenomenological confining potential for the light-heavy quarkonia is of the same form as that for the heavy quarkonia. We find that the parameter A for the spin-independent term in the confining potential is approximately the same for all quarkonia, while the spin-dependent terms vary such that the vector-exchange component is smaller for $c\bar{c}$ than for $b\bar{b}$, and still smaller for the light-heavy quarkonia.

III. CONCLUSION

We have obtained excellent results for the observed energy levels of D^0, B^0, D_s, and B_s with the use of our quantum-chromodynamic potential model, and provided predicted results for many unobserved energy levels in Tables I–IV. We have included in these tables the mixing angles for the $1\,^3P_1'$ and $1\,^1P_1'$ levels, which are needed for an understanding of their decay properties. Although the use of a semirelativistic model may seem questionable for a system containing a light quark, ultimately such an approach should be judged on the basis of its predictions [13]. Additional experimental data on the light-heavy quarkonia should be available in the near future.

We have also used our results to test the accuracy of the heavy quark effective theory [5,6]. By comparing the theoretical results without and with the effective theory

TABLE V. Scaling of energy level splittings in D^0 and B^0. Splittings are given in MeV.

	ΔD_0	$(m_c/m_b)\Delta D^0$	ΔB^0
$1\,^3S_1 - 1\,^1S_0$	142.5	44.6	45.3
$2\,^3S_1 - 2\,^1S_0$	99.3	31.1	32.2
$1\,^3P_1' - 1\,^3P_0$	128.7	40.3	41.3
$1\,^3P_2 - 1\,^1P_1'$	44.0	13.8	15.5

TABLE VI. Scaling of energy level splittings in D_s and B_s. Splittings are given in MeV.

	ΔD_s	$(m_c/m_b)\Delta D_s$	ΔB_s
$1\,^3S_1 - 1\,^1S_0$	141.7	44.4	48.6
$2\,^3S_1 - 2\,^1S_0$	101.3	31.7	35.0
$1\,^3P_1' - 1\,^3P_0$	133.4	41.8	44.9
$1\,^3P_2 - 1\,^1P_1'$	36.6	11.5	15.0

in Tables I–IV, we find that the heavy quark expansion with the inclusion of the M^{-1} and $M^{-1}\ln M$ corrections yields generally good results for the B^0 and B_s energy levels. It does not, however, provide equally good results for the energy levels of D^0 and D_s, which indicates that the effective theory can be applied more accurately to the b quark than the c quark [14].

We further find that the results for the energy levels in the limit $M \to \infty$ are unacceptable. As is well known, in this limit the energy level pairs $(^3S_1, {}^1S_0)$, $(^3P_1', {}^3P_0)$, and $(^3P_2, {}^1P_1')$ become degenerate.

Finally, we have examined the scaling behavior of energy level splittings in the light-heavy quarkonia by looking at the results obtained by the direct use of our model in Tables I–IV. As shown in Tables V and VI, the splittings between levels which become degenerate in the limit $M \to \infty$ exhibit an approximate M^{-1} scaling. This scaling behavior does not apply to splittings between other pairs of energy levels.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-85ER40209.

APPENDIX A: NONSINGULAR QUARKONIUM POTENTIALS

The nonsingular potentials for a light-heavy quarkonium are similar to those used recently for $c\bar{c}$ [3] except for the complications due to the difference in the quark and antiquark masses. The complications are further enhanced by the conversion of the singular potentials [15] into the nonsingular ones [16], which are necessary to avoid the use of an illegitimate perturbative treatment. The corresponding denominators in the singular and nonsingular potentials for a quark and an antiquark of masses m_1 and m_2 are related as

$$\frac{1}{m_1^2} \to \frac{1}{m_1^2 + \frac{1}{4}\mathbf{k}^2}\,, \qquad \frac{1}{m_2^2} \to \frac{1}{m_2^2 + \frac{1}{4}\mathbf{k}^2}\,, \tag{A1}$$

and

$$\frac{1}{m_1 m_2} \to \frac{1}{(m_1^2 + \frac{1}{4}\mathbf{k}^2)^{1/2}(m_2^2 + \frac{1}{4}\mathbf{k}^2)^{1/2}} \approx \frac{1}{m_1 m_2 + \frac{1}{8}\left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right)\mathbf{k}^2}$$

or

$$\frac{1}{m_1 m_2} \to \frac{\kappa}{\omega^2 + \frac{1}{4}\mathbf{k}^2}\,, \tag{A2}$$

where

$$\kappa = \frac{2m_1 m_2}{m_1^2 + m_2^2}\,, \qquad \omega = (\kappa m_1 m_2)^{1/2}\,. \tag{A3}$$

The potentials for a quark and an antiquark of different flavors are given below. They reduce to those for $c\bar{c}$ for $m_1 = m_2 = m$ except that, unlike $c\bar{c}$, they do not contain the annihilation terms.

1. Perturbative quantum-chromodynamic potential

The potential in the momentum space is

$$\begin{aligned}
V_p(\mathbf{k}) = &-\frac{16\pi\alpha_s}{3\mathbf{k}^2}\left[1 - \frac{3\alpha_s}{2\pi} - \frac{\alpha_s}{12\pi}(33 - 2n_f)\ln\left(\frac{\mathbf{k}^2}{\mu^2}\right)\right] \\
&+\frac{8\pi\alpha_s}{3}\left(\frac{1}{\mathbf{k}^2 + 4m_1^2} + \frac{1}{\mathbf{k}^2 + 4m_2^2}\right)\left[\delta_{l0}\left(1 - \frac{3\alpha_s}{2\pi}\right) - \frac{\alpha_s}{12\pi}(33 - 2n_f)\ln\left(\frac{\mathbf{k}^2}{\mu^2}\right)\right] \\
&-\frac{8\pi^2\alpha_s^2}{9|\mathbf{k}|}\frac{\kappa}{\mathbf{k}^2 + 4\omega^2}\left[9(m_1 + m_2) - \frac{8m_1 m_2}{m_1 + m_2}\right] \\
&+\frac{64\pi\alpha_s}{3}\mathbf{S}_1\cdot\mathbf{S}_2\frac{\kappa}{\mathbf{k}^2 + 4\omega^2}\left\{\delta_{l0}\left[\frac{2}{3} - \frac{19\alpha_s}{9\pi} - \frac{\alpha_s}{12\pi}\left(8\frac{m_1 - m_2}{m_1 + m_2} + \frac{m_1 + m_2}{m_1 - m_2}\right)\ln\left(\frac{m_2}{m_1}\right)\right]\right. \\
&\left.-\frac{\alpha_s}{18\pi}(33 - 2n_f)\ln\left(\frac{\mathbf{k}^2}{\mu^2}\right) + \frac{7\alpha_s}{4\pi}\ln\left(\frac{\mathbf{k}^2}{m_1 m_2}\right)\right\} \\
&-\frac{64\pi\alpha_s}{3}\frac{\mathbf{S}_1\cdot\mathbf{k}\,\mathbf{S}_2\cdot\mathbf{k} - \frac{1}{3}\mathbf{k}^2\mathbf{S}_1\cdot\mathbf{S}_2}{\mathbf{k}^2}\frac{\kappa}{\mathbf{k}^2 + 4\omega^2}\left[1 + \frac{4\alpha_s}{3\pi} - \frac{\alpha_s}{12\pi}(33 - 2n_f)\ln\left(\frac{\mathbf{k}^2}{\mu^2}\right) + \frac{3\alpha_s}{2\pi}\ln\left(\frac{\mathbf{k}^2}{m_1 m_2}\right)\right] \\
&-\frac{16\pi\alpha_s}{3}\frac{i\mathbf{S}\cdot(\mathbf{k}\times\mathbf{p})}{\mathbf{k}^2}\left\{\left(\frac{1}{\mathbf{k}^2 + 4m_1^2} + \frac{1}{\mathbf{k}^2 + 4m_2^2}\right)\left[1 - \frac{\alpha_s}{6\pi} - \frac{\alpha_s}{12\pi}(33 - 2n_f)\ln\left(\frac{\mathbf{k}^2}{\mu^2}\right)\right]\right. \\
&+\frac{3\alpha_s}{2\pi}\frac{1}{\mathbf{k}^2 + 4m_1^2}\ln\left(\frac{\mathbf{k}^2}{m_1^2}\right) + \frac{3\alpha_s}{2\pi}\frac{1}{\mathbf{k}^2 + 4m_2^2}\ln\left(\frac{\mathbf{k}^2}{m_2^2}\right) \\
&\left.+\frac{4\kappa}{\mathbf{k}^2 + 4\omega^2}\left[1 - \frac{5\alpha_s}{6\pi} - \frac{\alpha_s}{12\pi}(33 - 2n_f)\ln\left(\frac{\mathbf{k}^2}{\mu^2}\right) + \frac{3\alpha_s}{4\pi}\ln\left(\frac{\mathbf{k}^2}{m_1 m_2}\right)\right]\right\} \\
&-\frac{16\pi\alpha_s}{3}\frac{i(\mathbf{S}_1 - \mathbf{S}_2)\cdot(\mathbf{k}\times\mathbf{p})}{\mathbf{k}^2}\left\{\left(\frac{1}{\mathbf{k}^2 + 4m_1^2} - \frac{1}{\mathbf{k}^2 + 4m_2^2}\right)\left[1 - \frac{\alpha_s}{6\pi} - \frac{\alpha_s}{12\pi}(33 - 2n_f)\ln\left(\frac{\mathbf{k}^2}{\mu^2}\right)\right]\right. \\
&\left.+\frac{3\alpha_s}{2\pi}\frac{1}{\mathbf{k}^2 + 4m_1^2}\ln\left(\frac{\mathbf{k}^2}{m_1^2}\right) - \frac{3\alpha_s}{2\pi}\frac{1}{\mathbf{k}^2 + 4m_2^2}\ln\left(\frac{\mathbf{k}^2}{m_2^2}\right) + \frac{3\alpha_s}{2\pi}\frac{2\kappa}{\mathbf{k}^2 + 4\omega^2}\ln\left(\frac{m_2}{m_1}\right)\right\}.
\end{aligned} \tag{A4}$$

In the coordinate space, it takes the form

$$
\begin{aligned}
V_p(\mathbf{r}) = -\frac{4\alpha_s}{3r}\Bigg\{ & 1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)\left[\ln(\mu r)+\gamma_E\right]-\delta_{l0}\left(1-\frac{3\alpha_s}{2\pi}\right)\left(\frac{e^{-2m_1r}+e^{-2m_2r}}{2}\right) \\
& -\frac{\alpha_s}{12\pi}(33-2n_f)\left[\ln(\mu r)\left(e^{-2m_1r}+e^{-2m_2r}\right)+E_+(2m_1r)+E_+(2m_2r)\right] \\
& +\frac{\alpha_s}{6}\left[9(m_1+m_2)-\frac{8m_1m_2}{m_1+m_2}\right]\frac{\kappa}{\omega}\left[\ln(2\omega r)e^{-2\omega r}-E_-(2\omega r)\right]\Bigg\} \\
+\kappa\frac{32\alpha_s}{9r}\mathbf{S}_1\cdot\mathbf{S}_2\Bigg\{ & \delta_{l0}\left[1-\frac{19\alpha_s}{6\pi}-\frac{\alpha_s}{8\pi}\left(8\frac{m_1-m_2}{m_1+m_2}+\frac{m_1+m_2}{m_1-m_2}\right)\ln\left(\frac{m_2}{m_1}\right)\right]e^{-2\omega r} \\
& +\frac{\alpha_s}{6\pi}(33-2n_f)\left[\ln(\mu r)e^{-2\omega r}+E_+(2\omega r)\right]-\frac{21\alpha_s}{4\pi}\left[\ln(\sqrt{m_1m_2}\,r)e^{-2\omega r}+E_+(2\omega r)\right]\Bigg\} \\
+\kappa\frac{4\alpha_s}{3r}S_T\Bigg\{ & \left(1+\frac{4\alpha_s}{3\pi}\right)f_2(2\omega r)+\frac{\alpha_s}{6\pi}(33-2n_f)\left[f_2(2\omega r)\ln(\mu r)+g_2(2\omega r)\right] \\
& -\frac{3\alpha_s}{\pi}\left[f_2(2\omega r)\ln(\sqrt{m_1m_2}\,r)+g_2(2\omega r)\right]\Bigg\} \\
+\frac{4\alpha_s}{3r}\mathbf{L}\cdot\mathbf{S}\Bigg\{ & \left(1-\frac{\alpha_s}{6\pi}\right)\left[f_1(2m_1r)+f_1(2m_2r)\right] \\
& +\frac{\alpha_s}{6\pi}(33-2n_f)\left[f_1(2m_1r)\ln(\mu r)+g_1(2m_1r)+f_1(2m_2r)\ln(\mu r)+g_1(2m_2r)\right] \\
& -\frac{3\alpha_s}{\pi}\left[f_1(2m_1r)\ln(m_1r)+g_1(2m_1r)+f_1(2m_2r)\ln(m_2r)+g_1(2m_2r)\right] \\
& +4\kappa\Bigg[\left(1-\frac{5\alpha_s}{6\pi}\right)f_1(2\omega r)+\frac{\alpha_s}{6\pi}(33-2n_f)\left[f_1(2\omega r)\ln(\mu r)+g_1(2\omega r)\right] \\
& \quad -\frac{3\alpha_s}{2\pi}\left[f_1(2\omega r)\ln(\sqrt{m_1m_2}\,r)+g_1(2\omega r)\right]\Bigg]\Bigg\} \\
+\frac{4\alpha_s}{3r}\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2)\Bigg\{ & \left(1-\frac{\alpha_s}{6\pi}\right)\left[f_1(2m_1r)-f_1(2m_2r)\right] \\
& +\frac{\alpha_s}{6\pi}(33-2n_f)\left[f_1(2m_1r)\ln(\mu r)+g_1(2m_1r)-f_1(2m_2r)\ln(\mu r)-g_1(2m_2r)\right] \\
& -\frac{3\alpha_s}{\pi}\left[f_1(2m_1r)\ln(m_1r)+g_1(2m_1r)-f_1(2m_2r)\ln(m_2r)-g_1(2m_2r)\right] \\
& +\frac{3\alpha_s}{\pi}\ln\left(\frac{m_2}{m_1}\right)\kappa f_1(2\omega r)\Bigg\}\,.
\end{aligned} \tag{A5}
$$

Note that the tensor operator is defined as

$$S_T = 3\,\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\;\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2, \tag{A6}$$

the functions $E_\pm$ are expressible in terms of the exponential-integral function Ei as

$$E_\pm(x)=\frac{1}{2}\left[e^{x}\mathrm{Ei}(-x)\pm e^{-x}\mathrm{Ei}(x)\right]\mp e^{-x}\ln x, \tag{A7}$$

and

$$\begin{aligned} f_1(x) &= \frac{1-(1+x)e^{-x}}{x^2}, \\ f_2(x) &= \frac{1-\left(1+x+\frac{1}{3}x^2\right)e^{-x}}{x^2}, \\ g_1(x) &= \frac{\gamma_E - [E_+(x) - xE_-(x)]}{x^2}, \\ g_2(x) &= \frac{\gamma_E - \left[\left(1+\frac{1}{3}x^2\right)E_+(x) - xE_-(x)\right]}{x^2}. \end{aligned} \tag{A8}$$

2. Phenomenological confining potential

The scalar-vector-exchange confining potential is given by

$$V_c = (1-B)V_S + BV_V, \tag{A9}$$

where, in momentum space,

$$V_S(\mathbf{k}) = -8\pi A\left[\frac{1}{\mathbf{k}^4} - \frac{i\mathbf{S}\cdot(\mathbf{k}\times\mathbf{p})}{\mathbf{k}^4}\left(\frac{1}{\mathbf{k}^2+4m_1^2}+\frac{1}{\mathbf{k}^2+4m_2^2}\right) - \frac{i(\mathbf{S}_1-\mathbf{S}_2)\cdot(\mathbf{k}\times\mathbf{p})}{\mathbf{k}^4}\left(\frac{1}{\mathbf{k}^2+4m_1^2}-\frac{1}{\mathbf{k}^2+4m_2^2}\right)\right], \tag{A10}$$

$$\begin{aligned} V_V(\mathbf{k}) = -8\pi A \Bigg[&\frac{1}{\mathbf{k}^4} - \frac{1}{2\mathbf{k}^2}\left(\frac{1}{\mathbf{k}^2+4m_1^2}+\frac{1}{\mathbf{k}^2+4m_2^2}\right) - \frac{8\kappa\mathbf{S}_1\cdot\mathbf{S}_2}{3\mathbf{k}^2(\mathbf{k}^2+4\omega^2)} + \frac{4\kappa\left(\mathbf{S}_1\cdot\mathbf{k}\,\mathbf{S}_2\cdot\mathbf{k} - \frac{1}{3}\mathbf{k}^2\mathbf{S}_1\cdot\mathbf{S}_2\right)}{\mathbf{k}^4(\mathbf{k}^2+4\omega^2)} \\ &+\frac{i\mathbf{S}\cdot(\mathbf{k}\times\mathbf{p})}{\mathbf{k}^4}\left(\frac{1}{\mathbf{k}^2+4m_1^2}+\frac{1}{\mathbf{k}^2+4m_2^2}+\frac{4\kappa}{\mathbf{k}^2+4\omega^2}\right) + \frac{i(\mathbf{S}_1-\mathbf{S}_2)\cdot(\mathbf{k}\times\mathbf{p})}{\mathbf{k}^4}\left(\frac{1}{\mathbf{k}^2+4m_1^2}-\frac{1}{\mathbf{k}^2+4m_2^2}\right)\Bigg]. \end{aligned} \tag{A11}$$

The coordinate-space potentials are

$$V_S(\mathbf{r}) = Ar - \frac{A}{4r}\mathbf{L}\cdot\mathbf{S}\left[\frac{1-2f_1(2m_1r)}{m_1^2}+\frac{1-2f_1(2m_2r)}{m_2^2}\right] - \frac{A}{4r}\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2)\left[\frac{1-2f_1(2m_1r)}{m_1^2}-\frac{1-2f_1(2m_2r)}{m_2^2}\right], \tag{A12}$$

$$\begin{aligned} V_V(\mathbf{r}) = Ar &+ \frac{A}{4r}\left(\frac{1-e^{-2m_1r}}{m_1^2}+\frac{1-e^{-2m_2r}}{m_2^2}\right) + \frac{4A}{3r}\mathbf{S}_1\cdot\mathbf{S}_2\left(\frac{1-e^{-2\omega r}}{m_1m_2}\right) + \frac{A}{12r}S_T\left[\frac{1-6f_2(2\omega r)}{m_1m_2}\right] \\ &+\frac{A}{r}\mathbf{L}\cdot\mathbf{S}\left[\frac{1-2f_1(2m_1r)}{4m_1^2}+\frac{1-2f_1(2m_2r)}{4m_2^2}+\frac{1-2f_1(2\omega r)}{m_1m_2}\right] \\ &+\frac{A}{4r}\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2)\left[\frac{1-2f_1(2m_1r)}{m_1^2}-\frac{1-2f_1(2m_2r)}{m_2^2}\right]. \end{aligned} \tag{A13}$$

It is understood that the confining potential also contains an additive phenomenological constant C.

APPENDIX B: QUARKONIUM POTENTIALS WITH HEAVY QUARK EXPANSION

Upon replacing m_1 and m_2 by m and M, and expanding in powers of M^{-1}, the coordinate-space potentials of Appendix A take the approximate forms given below.

1. Perturbative potential

The perturbative potential is

$$V_p(\mathbf{r}) = V_{p0}(\mathbf{r}) + \left(\frac{m}{M}\right)V_{p1}(\mathbf{r}) + \mathcal{O}\left(\frac{m^2}{M^2}\right), \tag{B1}$$

with

$$V_{p0}(\mathbf{r}) = -\frac{4\alpha_s}{3r}\left\{1 - \frac{3\alpha_s}{2\pi} + \frac{\alpha_s}{6\pi}(33-2n_f)\left[\ln(\mu r)+\gamma_E\right] - \delta_{l0}\left(1-\frac{3\alpha_s}{2\pi}\right)\frac{e^{-2mr}}{2}\right.$$
$$-\frac{\alpha_s}{12\pi}(33-2n_f)\left[\ln(\mu r)e^{-2mr}+E_+(2mr)\right]$$
$$\left.+\frac{3\alpha_s}{\sqrt{2}}\left[\ln(2\sqrt{2}mr)e^{-2\sqrt{2}mr}-E_-(2\sqrt{2}mr)\right]\right\}$$
$$+\frac{4\alpha_s}{3r}\mathbf{L}\cdot\mathbf{S}\left\{\left(1-\frac{\alpha_s}{6\pi}\right)f_1(2mr)+\frac{\alpha_s}{6\pi}(33-2n_f)\left[f_1(2mr)\ln(\mu r)+g_1(2mr)\right]\right.$$
$$\left.-\frac{3\alpha_s}{\pi}\left[f_1(2mr)\ln(mr)+g_1(2mr)\right]\right\}$$
$$+\frac{4\alpha_s}{3r}\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2)\left\{\left(1-\frac{\alpha_s}{6\pi}\right)f_1(2mr)+\frac{\alpha_s}{6\pi}(33-2n_f)\left[f_1(2mr)\ln(\mu r)+g_1(2mr)\right]\right.$$
$$\left.-\frac{3\alpha_s}{\pi}\left[f_1(2mr)\ln(mr)+g_1(2mr)\right]\right\}, \tag{B2}$$

$$V_{p1}(\mathbf{r}) = -\frac{2\sqrt{2}\alpha_s^2}{9r}\left[\ln(2\sqrt{2}mr)e^{-2\sqrt{2}mr}-E_-(2\sqrt{2}mr)\right]$$
$$+\frac{64\alpha_s}{9r}\mathbf{S}_1\cdot\mathbf{S}_2\left\{\delta_{l0}\left[1-\frac{19\alpha_s}{6\pi}+\frac{9\alpha_s}{8\pi}\ln\left(\frac{M}{m}\right)\right]e^{-2\sqrt{2}mr}\right.$$
$$+\frac{\alpha_s}{6\pi}(33-2n_f)\left[\ln(\mu r)e^{-2\sqrt{2}mr}+E_+(2\sqrt{2}mr)\right]$$
$$\left.-\frac{21\alpha_s}{4\pi}\left[\ln(\sqrt{mM}\,r)e^{-2\sqrt{2}mr}+E_+(2\sqrt{2}mr)\right]\right\}$$
$$+\frac{8\alpha_s}{3r}S_T\left\{\left(1+\frac{4\alpha_s}{3\pi}\right)f_2(2\sqrt{2}mr)+\frac{\alpha_s}{6\pi}(33-2n_f)\left[f_2(2\sqrt{2}mr)\ln(\mu r)+g_2(2\sqrt{2}mr)\right]\right.$$
$$\left.-\frac{3\alpha_s}{\pi}\left[f_2(2\sqrt{2}mr)\ln(\sqrt{mM}\,r)+g_2(2\sqrt{2}mr)\right]\right\}$$
$$+\frac{32\alpha_s}{3r}\mathbf{L}\cdot\mathbf{S}\left\{\left(1-\frac{5\alpha_s}{6\pi}\right)f_1(2\sqrt{2}mr)+\frac{\alpha_s}{6\pi}(33-2n_f)\left[f_1(2\sqrt{2}mr)\ln(\mu r)+g_1(2\sqrt{2}mr)\right]\right.$$
$$\left.-\frac{3\alpha_s}{2\pi}\left[f_1(2\sqrt{2}mr)\ln(\sqrt{mM}\,r)+g_1(2\sqrt{2}mr)\right]\right\}$$
$$+\frac{8\alpha_s^2}{\pi r}\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2)\ln\left(\frac{M}{m}\right)f_1(2\sqrt{2}mr)\ . \tag{B3}$$

2. Confining potentials

The confining potentials are

$$V_S(\mathbf{r}) = V_{S0}(\mathbf{r}) + \mathcal{O}\left(\frac{m^2}{M^2}\right), \tag{B4}$$

with

$$V_{S0}(\mathbf{r}) = Ar - \frac{A}{4r}\mathbf{L}\cdot\mathbf{S}\frac{1-2f_1(2mr)}{m^2}$$
$$-\frac{A}{4r}\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2)\frac{1-2f_1(2mr)}{m^2} \tag{B5}$$

and

$$V_V(\mathbf{r}) = V_{V0}(\mathbf{r}) + \left(\frac{m}{M}\right)V_{V1}(\mathbf{r}) + \mathcal{O}\left(\frac{m^2}{M^2}\right), \tag{B6}$$

with

$$V_{V0}(\mathbf{r}) = Ar + \frac{A}{4r}\,\frac{1-e^{-2mr}}{m^2} + \frac{A}{4r}\mathbf{L}\cdot\mathbf{S}\frac{1-2f_1(2mr)}{m^2} + \frac{A}{4r}\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2)\frac{1-2f_1(2mr)}{m^2} \tag{B7}$$

$$V_{V1}(\mathbf{r}) = \frac{4A}{3r}\mathbf{S}_1\cdot\mathbf{S}_2\frac{1-e^{-2\sqrt{2}mr}}{m^2} + \frac{A}{12r}S_T\frac{1-6f_2(2\sqrt{2}mr)}{m^2} + \frac{A}{r}\mathbf{L}\cdot\mathbf{S}\frac{1-2f_1(2\sqrt{2}mr)}{m^2}\,. \tag{B8}$$

3. $M \to \infty$ limit

In the limit $M \to \infty$, the perturbative and confining potentials are given by

$$V_p = V_{p0}, \qquad V_S = V_{S0}, \qquad V_V = V_{V0}. \tag{B9}$$

According to (B2), (B5), and (B8), the spin-orbit and spin-orbit-mixing terms in these potentials are of the form

$$f(r)\mathbf{L}\cdot\mathbf{S} + f(r)\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2), \tag{B10}$$

which is expressible solely in terms of the light-quark spin as

$$2f(r)\mathbf{L}\cdot\mathbf{S}_1. \tag{B11}$$

We have, however, given the potentials in the form (B10) to facilitate comparison with the more accurate treatments of the light-heavy quarkonia.

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[1] For a review of the earlier work on the light-heavy quarkonia, see J. L. Rosner, Comments Nucl. Part. Phys. **16**, 109 (1986).

[2] For some of the recent theoretical investigations, see S. Godfrey and R. Kokoski, Phys. Rev. D **43**, 1679 (1991); W. Kwong and J. L. Rosner, *ibid.* **44**, 212 (1991); J. L. Rosner and M. B. Wise, *ibid.* **47**, 343 (1993); E. J. Eichten, C. T. Hill, and C. Quigg, Phys. Rev. Lett. **71**, 4116 (1993).

[3] S. N. Gupta, J. M. Johnson, W. W. Repko, and C. J. Suchyta III, Phys. Rev. D **49**, 1551 (1994); see also earlier related papers cited therein.

[4] Although our treatment is equally applicable to quarkonia containing either a u or a d quark, we do not have as much experimental data on $D^{\pm}$ and $B^{\pm}$ as we have on D^0 and B^0.

[5] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990).

[6] For a review of the heavy quark effective theory and its update, see B. Grinstein, Annu. Rev. Nucl. Part. Sci. **42**, 101 (1992); in *Proceedings of the Workshop on B Physics at Hadron Accelerators*, Snowmass, Colorado, 1993, edited by P. McBride and C. S. Mishra (Fermilab, Batavia, IL, 1994).

[7] S. N. Gupta and J. M. Johnson, Phys. Rev. D **44**, 2110 (1991).

[8] CDF Collaboration, F. Abe *et al.*, Phys. Rev. Lett. **73**, 225 (1994).

[9] Particle Data Group, K. Hikasa *et al.*, Phys. Rev. D **45**, S1 (1992).

[10] CLEO Collaboration, Y. Kubota *et al.*, Phys. Rev. Lett. **72**, 1972 (1994).

[11] CLEO Collaboration, P. Avery *et al.*, Phys. Lett. B **331**, 236 (1994).

[12] CDF Collaboration, F. Abe *et al.*, Phys. Rev. Lett. **71**, 1685 (1993).

[13] Our predicted values for the energy levels do not include the effects of the u-d mass difference and the electromagnetic interaction, which may amount to a few MeV.

[14] The accuracy with which the effective theory can be applied to c quarks has also been questioned by L. Randall and E. Sather, Phys. Rev. D **49**, 6236 (1994).

[15] S. N. Gupta and S. F. Radford, Phys. Rev. D **24**, 2309 (1981); **25**, 3430 (1982).

[16] S. N. Gupta, Phys. Rev. D **35**, 1736 (1987).