String cosmology with a time-dependent antisymmetric tensor potential

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(Received 7 October 1994)

We present a class of exact solutions for homogeneous, anisotropic cosmologies in four dimensions derived from the low-energy string effective action including a homogeneous dilaton ϕ and antisymmetric tensor potential $B_{\mu\nu}$. Making this potential time dependent produces an anisotropic energy-momentum tensor, and leads us to consider a Bianchi type I cosmology. The solution for the axion field must then only be a linear function of one spatial coordinate. This in turn places an upper bound on the product of the two scale factors evolving perpendicular to the gradient of the axion field. The only late-time isotropic solution is then a *contracting* universe.

I. INTRODUCTION

The massless excitations of a string consist not only of the graviton field $g_{\mu\nu}$ of general relativity, but also a dilaton field ϕ which determines the strength of the gravitational coupling, and an antisymmetric tensor potential $B_{\mu\nu}$. While the cosmological consequences of the dilaton have been extensively discussed [1,2], the role of the antisymmetric tensor field strength

$$H_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]} \tag{1.1}$$

is often less clear. This is partly due to the difficulty of handling the many new degrees of freedom this introduces in higher dimensions. Here we will consider the field restricted to a four-dimensional cosmology where we have only one degree of freedom which can be represented by a pseudoscalar "axion" field. The omission of $H_{\mu\nu\lambda}$ is often justified due to the existence of duality transforms of the string action which relate the dilaton-only solutions to nontrivial H field solutions, but the complete equivalence of the solutions is only true if this duality extends to the full action. In a previous paper [3] we gave exact solutions of the lowest-order string β function equations for four-dimensional cosmologies with a time-dependent axion field (see also [4]) which are related to the homogeneous dilaton-vacuum cosmologies by an SL(2, R) transform [5].

Another commonly invoked symmetry is the O(d, d) duality [6–8] which requires (in a cosmological setting) both the antisymmetric potential and the metric to be functions only of time. Here we will give explicit solutions for cosmologies including a time dependent $B_{\mu\nu}$, which can be seen to preclude any time dependence of $H_{\mu\nu\lambda}$ from its definition. We shall show that, as it is the field H that appears in the metric equations of motion, this is a highly restrictive prescription. In particular it introduces an anisotropic energy-momentum tensor which we shall show inevitably leads to an anisotropic cosmology.

We will solve the string β function equations only to lowest order, which can be derived from the low-energy effective action of the bosonic sector of a string theory

reduced to four dimensions [9]:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, e^{-\phi} \left[R + (\nabla\phi)^2 - n\,(\nabla\beta)^2 \right.$$
$$\left. - V - \frac{1}{12}H^2 \right], \tag{1.2}$$

where $H^2 = H_{\mu\nu\lambda}H^{\mu\nu\lambda}$, $\kappa^2 = 8\pi G$, and the modulus field β represents the evolution of n compact dimensions. For simplicity we assume that these dimensions are described by a spatially flat (Bianchi type I) metric with scale factors b_i, and we define $n\dot\beta^2 \equiv \sum_i(\dot b_i/b_i)^2$. We have adopted the sign conventions denoted (+++) by Misner, Thorne, and Wheeler [10]. The constant V is proportional to the central charge of the string theory.

The effect of certain types of "stringy matter" has been considered elsewhere in the literature [1], where an equation of state for matter was assumed. The symmetries of the vacuum, as well as any additional gauge symmetries that may be present, will affect the matter Lagrangian as well, and also the value of the central charge V. We shall assume that the original string theory, from which the effective action Eq. (1.2) is derived, contrives to set the central charge $V = 0$, by adding appropriate bosonic or fermionic conformal matter. Initially we shall restrict ourselves to vacuum solutions as regards these matter fields in order to examine the dynamical effect of the bosonic fields. Later we will briefly discuss the possible effect of other matter, in particular radiation.

The field equations are derived by varying this action (with $V = 0$) with respect to $g_{\mu\nu}$, $B_{\mu\nu}$, and ϕ, respectively:

$$R_\mu^\nu - \frac{1}{2}g_\mu^\nu R = \frac{1}{12}\left(3H_{\mu\lambda\kappa}H^{\nu\lambda\kappa} - \frac{1}{2}g_\mu^\nu H^2 \right)$$
$$+ n\left(g_\mu^\lambda g^{\nu\kappa} - \frac{1}{2}g_\mu^\nu g^{\lambda\kappa} \right) \nabla_\lambda\beta\nabla_\kappa\beta$$
$$- \frac{1}{2}g_\mu^\nu (\nabla\phi)^2 + \left(g_\mu^\nu g^{\lambda\kappa} - g_\mu^\lambda g^{\nu\kappa} \right) \nabla_\lambda\nabla_\kappa\phi \tag{1.3}$$

51   1569

$$\nabla_\mu \left(e^{-\phi} H^{\mu\nu\lambda} \right) = 0 \,, \tag{1.4}$$

$$\nabla_\mu \left(e^{-\phi} \nabla^\mu \beta \right) = 0, \tag{1.5}$$

$$2 \,\Box\, \phi = -R + (\nabla\phi)^2 + n\,(\nabla\beta)^2 + \frac{1}{12} H^2 \,. \tag{1.6}$$

These equations can be rewritten in a more familiar general relativistic form in terms of the conformally transformed Einstein metric, defined as

$$\tilde{g}_{\mu\nu} = e^{-\phi} g_{\mu\nu} \,. \tag{1.7}$$

In terms of this metric, the action (with $V = 0$) appears as the Einstein-Hilbert action of general relativity while the dilaton appears simply as a matter field, albeit one interacting with the other matter fields:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \Big[\tilde{R} - \frac{1}{2} \left(\tilde{\nabla}\phi \right)^2 - n \left(\tilde{\nabla}\beta \right)^2 - \frac{1}{12} e^{-2\phi} \tilde{H}^2 \Big]. \tag{1.8}$$

In this expression, raising of the indices was done with the inverse $\tilde{g}^{\mu\nu}$ of the transformed metric (1.7). Note that $\tilde{H}_{\mu\lambda\kappa} \equiv H_{\mu\lambda\kappa}$, the definition being metric independent.

The corresponding field equations are then those for interacting fields in general relativity:

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = \kappa^2 \left({}^{(H)}\tilde{T}_{\mu\nu} + {}^{(\beta)}\tilde{T}_{\mu\nu} + {}^{(\phi)}\tilde{T}_{\mu\nu} \right), \tag{1.9}$$

$$\tilde{\nabla}_\mu (e^{-2\phi} \tilde{H}^{\mu\nu\lambda}) = 0 \,, \tag{1.10}$$

$$\tilde{\Box}\, \beta = 0, \tag{1.11}$$

$$\tilde{\Box}\, \phi + \frac{1}{6} e^{-2\phi} \tilde{H}^2 = 0. \tag{1.12}$$

The energy-momentum tensors appearing on the right-hand side of the Einstein equations correspond to the energy-momentum tensors for the dilaton, moduli, and H fields, respectively:

$$\kappa^2 \, {}^{(\phi)}\tilde{T}_\mu^\nu = \frac{1}{2} \left(\tilde{g}_\mu^\lambda \tilde{g}^{\nu\kappa} - \frac{1}{2}\tilde{g}_\mu^\nu \tilde{g}^{\lambda\kappa} \right) \tilde{\nabla}_\lambda\phi \tilde{\nabla}_\kappa\phi \,, \tag{1.13}$$

$$\kappa^2 \, {}^{(\beta)}\tilde{T}_\mu^\nu = n \left(\tilde{g}_\mu^\lambda \tilde{g}^{\nu\kappa} - \frac{1}{2}\tilde{g}_\mu^\nu \tilde{g}^{\lambda\kappa} \right) \tilde{\nabla}_\lambda\beta \tilde{\nabla}_\kappa\beta \,, \tag{1.14}$$

$$\kappa^2 \, {}^{(H)}\tilde{T}_\mu^\nu = \frac{1}{12} e^{-2\phi} \left(3\tilde{H}_{\mu\lambda\kappa}\tilde{H}^{\nu\lambda\kappa} - \frac{1}{2}\tilde{g}_\mu^\nu \tilde{H}^2 \right) . \tag{1.15}$$

While the total energy momentum must be conserved (as guaranteed by the Ricci identity) there are interactions between the components.

Because we assume that all fields are independent of the compact dimensions we can immediately solve the equation of motion for the antisymmetric tensor field, Eq. (1.10), in four-dimensional spacetime by the ansatz

$$\tilde{H}^{\mu\nu\lambda} = e^{2\phi}\tilde{\epsilon}^{\mu\nu\lambda\kappa}\tilde{\nabla}_\kappa h, \tag{1.16}$$

where $\tilde{\epsilon}^{\mu\nu\lambda\kappa}$ is the antisymmetric volume form in four dimensions (obeying $\tilde{\nabla}_\rho\tilde{\epsilon}^{\mu\nu\lambda\kappa} = 0$). The field h obeys a new equation of motion, derived from the integrability

condition, $\partial_{[\mu}\tilde{H}_{\nu\lambda\kappa]} = 0$, which becomes

$$\tilde{\Box}\, h + 2\tilde{\nabla}^\mu\phi\tilde{\nabla}_\mu h = 0 \,. \tag{1.17}$$

We shall follow the usual string nomenclature and refer to h as the axion, even though its axionlike properties are not relevant to our analysis. The effective energy-momentum tensor for the antisymmetric tensor field in the Einstein frame can then be written as

$$\kappa^2 \, {}^{(H)}\tilde{T}_\mu^\nu = \frac{1}{2} \left(\tilde{g}_\mu^\lambda \tilde{g}^{\nu\kappa} - \frac{1}{2}\tilde{g}_\mu^\nu \tilde{g}^{\lambda\kappa} \right) e^{2\phi}\tilde{\nabla}_\lambda h \tilde{\nabla}_\kappa h. \tag{1.18}$$

Similarly the dilaton equation of motion, Eq. (1.12), can be rewritten in terms of h rather than of $H_{\mu\nu\lambda}$:

$$\tilde{\Box}\, \phi = e^{2\phi}(\tilde{\nabla}h)^2 \,. \tag{1.19}$$

II. SOLUTIONS

There are several possible homogeneous four-dimensional cosmologies one may have in this system of dilaton and axion coupled to gravity. The case where the axion is time dependent was discussed in a previous paper [3]. Here we consider the case where the components of the antisymmetric tensor potentials $B_{\mu\nu}$ depend only on time, $B_{0i} = 0$ and $B_{ij} \equiv B_{ij}(t)$. Note that B_{0i} can be always set to zero by utilizing the symmetry of the action under the vector gauge transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\Lambda_{\nu]}$. This is the case commonly discussed in the literature in the context of the $O(d, d)$ symmetry of the low-energy action, Eq. (1.2), when the metric and antisymmetric tensor potentials are independent of $d = D - 1$ of the spacetime coordinates. As can be easily checked, for our ansatz (1.16), a homogeneous $B_{\mu\nu}$ corresponds to the situation where $\partial_t h = 0$.

The modulus and dilaton fields are taken to be homogeneous as well: $\beta \equiv \beta(t)$ and $\phi \equiv \phi(t)$. They then act like stiff fluids in the Einstein frame with an isotropic pressure equal to their density, and so the energy-momentum tensors are

$$\tilde{T}_\mu^\nu = \mathrm{diag}(-\tilde{\rho}, \tilde{\rho}, \tilde{\rho}, \tilde{\rho}) \,, \tag{2.1}$$

where the energy densities are

$$\tilde{\rho}_\beta = \frac{n}{2\kappa^2}\dot{\beta}^2 \,, \qquad \tilde{\rho}_\phi = \frac{1}{4\kappa^2}\dot{\phi}^2 \,, \tag{2.2}$$

and an overdot denotes $d/d\tilde{t}$.

We will consider a Bianchi type I cosmology $g_{\mu\nu} = \mathrm{diag}(-1, a_1^2(t), a_2^2(t), a_3^2(t))$, the simplest form for an anisotropic metric, where the homogeneous hypersurfaces of constant time have zero spatial curvature. We shall see that anisotropic expansion is a necessary consequence of our choice of a homogeneous tensor potential. We solve first the evolution in the Einstein frame, (1.7). This metric is given by

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}_1^2(dx^1)^2 + \tilde{a}_2^2(dx^2)^2 + \tilde{a}_3^2(dx^3)^2, \tag{2.3}$$

where we have defined $\widetilde{a}_i = e^{-\phi/2}a_i$ and $d\widetilde{t} = e^{-\phi/2}dt$. Since our metric is diagonal, the equation of motion for h, Eq. (1.17), takes the form

$$\frac{1}{\widetilde{a}_1^2}\partial_1^2 h + \frac{1}{\widetilde{a}_2^2}\partial_2^2 h + \frac{1}{\widetilde{a}_3^2}\partial_3^2 h = 0. \qquad (2.4)$$

Also, from the off-diagonal components of Einstein equations, Eq. (1.9), we have (since $\partial_i\phi = 0$)

$$0 = \frac{1}{2}e^{2\phi}\partial_i h \partial_j h \quad (i,j = 1,2,3, i \neq j, \text{no sum}). \qquad (2.5)$$

In addition, by our assumption of homogeneity the stress-energy tensor depends only on the time coordinate, which implies that the only solution of Eq. (2.4) is

$$\partial_i h = L_i, \qquad (2.6)$$

where $L_i(i = 1,2,3)$, are constants, since $\partial_t h = 0$. Then we have $L_i L_j = 0$, for all $i \neq j$, which tells us that only one of L_i can be nonzero. For definiteness, we choose $L_1 = L_2 = 0$. The resulting energy-momentum tensor for the axion field is then

$$^{(H)}\widetilde{T}^\nu_\mu = \text{diag}\left(-\widetilde{\rho}_H, -\widetilde{\rho}_H, -\widetilde{\rho}_H, \widetilde{\rho}_H\right), \qquad (2.7)$$

where

$$\widetilde{\rho}_H = \frac{e^{2\phi}L_3^2}{4\kappa^2\widetilde{a}_3^2}. \qquad (2.8)$$

The axion thus exerts an anisotropic pressure — positive in the x_3 direction but negative pressure along x_1 and x_2.

The Einstein equations, Eqs. (1.9), then lead to the equations of motion for the scale factors, $\widetilde{\alpha}_i = \ln\widetilde{a}_i$:

$$\widetilde{\alpha}_1 = C_1(\lambda - \lambda_1),$$
$$\widetilde{\alpha}_2 = C_2(\lambda - \lambda_2),$$
$$\widetilde{\alpha}_3 + \tfrac{1}{2}\phi = C_3(\lambda - \lambda_3),$$

and

$$\beta = C_n(\lambda - \lambda_n), \qquad (2.21)$$

where C_i, λ_i $(i = 1,2,3,n)$ are constants of integration. The "free" fields $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, and β are monotonic functions of time, while ϕ and $\widetilde{\alpha}_3$, both driven by the axion field, are linked. The conformal transform back to the string metric cancels out this dependence of the third scale factor on the dilaton, leaving a_3 a "free" field, while it is the evolution of a_1 and a_2 that becomes tied to the dilaton.

These expressions can be substituted into the constraint equation (2.12) to give

$$\left(\frac{d\phi}{d\lambda}\right)^2 + 2\left(\frac{d\phi}{d\lambda}\right)(C_1 + C_2) - 4C_1 C_2 - 4C_3(C_1 + C_2)$$
$$+ 2nC_n^2 + e^{2\phi}L_3^2\widetilde{a}_1^2\widetilde{a}_2^2 = 0. \qquad (2.22)$$

$$\ddot{\widetilde{\alpha}}_1 + \dot{\widetilde{\alpha}}_1(\dot{\widetilde{\alpha}}_1 + \dot{\widetilde{\alpha}}_2 + \dot{\widetilde{\alpha}}_3) = 0, \qquad (2.9)$$

$$\ddot{\widetilde{\alpha}}_2 + \dot{\widetilde{\alpha}}_2(\dot{\widetilde{\alpha}}_1 + \dot{\widetilde{\alpha}}_2 + \dot{\widetilde{\alpha}}_3) = 0, \qquad (2.10)$$

$$\ddot{\widetilde{\alpha}}_3 + \dot{\widetilde{\alpha}}_3(\dot{\widetilde{\alpha}}_1 + \dot{\widetilde{\alpha}}_2 + \dot{\widetilde{\alpha}}_3) = \frac{e^{2\phi}L_3^2}{2\widetilde{a}_3^2}, \qquad (2.11)$$

together with the constraint equation

$$\dot{\widetilde{\alpha}}_1\dot{\widetilde{\alpha}}_2 + \dot{\widetilde{\alpha}}_2\dot{\widetilde{\alpha}}_3 + \dot{\widetilde{\alpha}}_3\dot{\widetilde{\alpha}}_1 = \frac{1}{4}\dot{\phi}^2 + \frac{n}{2}\dot{\beta}^2 + \frac{1}{4}\frac{e^{2\phi}L_3^2}{\widetilde{a}_3^2}, \qquad (2.12)$$

where overdots denote differentiation with respect to time in the Einstein frame, $\widetilde{t}$. The modulus and dilaton equations can be written as

$$\ddot{\beta} + \dot{\beta}(\dot{\widetilde{\alpha}}_1 + \dot{\widetilde{\alpha}}_2 + \dot{\widetilde{\alpha}}_3) = 0, \qquad (2.13)$$

$$\ddot{\phi} + \dot{\phi}(\dot{\widetilde{\alpha}}_1 + \dot{\widetilde{\alpha}}_2 + \dot{\widetilde{\alpha}}_3) = -\frac{e^{2\phi}L_3^2}{\widetilde{a}_3^2}. \qquad (2.14)$$

The axion field drives the evolution of ϕ and $\widetilde{a}_3$ but leaves $\widetilde{a}_1$, $\widetilde{a}_2$, and β to evolve as "free" fields, subject only to damping by the spatial expansion.

Let us introduce a new time coordinate λ via the relation

$$d\lambda \equiv \frac{d\widetilde{t}}{\widetilde{a}_1\widetilde{a}_2\widetilde{a}_3} = \frac{e^\phi dt}{a_1 a_2 a_3}. \qquad (2.15)$$

In terms of this variable the above equations simplify considerably:

$$\frac{d^2}{d\lambda^2}\widetilde{\alpha}_1 = \frac{d^2}{d\lambda^2}\widetilde{\alpha}_2 = \frac{d^2}{d\lambda^2}\beta = \frac{d^2}{d\lambda^2}\left(\widetilde{\alpha}_3 + \frac{1}{2}\phi\right) = 0, \qquad (2.16)$$

$$\frac{d^2\phi}{d\lambda^2} = -e^{2\phi}L_3^2\widetilde{a}_1^2\widetilde{a}_2^2. \qquad (2.17)$$

The equations for the scale factors and modulus can be readily solved:

$$a_1 = \exp[\tfrac{1}{2}\phi + C_1(\lambda - \lambda_1)], \qquad (2.18)$$
$$a_2 = \exp[\tfrac{1}{2}\phi + C_2(\lambda - \lambda_2)], \qquad (2.19)$$
$$a_3 = \exp[C_3(\lambda - \lambda_3)], \qquad (2.20)$$

Since the last term is necessarily non-negative, the requirement that the dilaton ϕ be real translates to a constraint on the constants C_i:

$$C_0^2 \equiv 2(C_1 + C_2 + C_3)^2 - (C_1 - C_2)^2 - 2C_3^2 - 2nC_n^2$$
$$\geq 0. \qquad (2.23)$$

We will choose C_0 to be non-negative. (It can be zero only when $L_3 = 0$, corresponding to a vacuum solution.)

Note that from the definition of λ in Eq. (2.15), we have

$$t - t_0 = \frac{1}{C_1 + C_2 + C_3}\exp[C_1(\lambda - \lambda_1) + C_2(\lambda - \lambda_2) + C_3(\lambda - \lambda_3)], \qquad (2.24)$$

where t_0 is a constant of integration which corresponds to an arbitrarily chosen origin of proper time in the

string frame. Note that from Eq. (2.23) it follows that $C_1 + C_2 + C_3 = 0$ only when $C_0 = 0$ and all the $C_i = 0$ ($i = 1, 2, 3, n$), corresponding to the isotropic general relativistic vacuum solution (Minkowski spacetime) which we are not interested in here. The variable λ runs from $-\infty$ to $+\infty$, which means that $t - t_0$ is on the positive or negative half-line, depending on the sign of $C_1 + C_2 + C_3$:

$$-\infty < t < t_0 \quad \text{for} \quad C_1 + C_2 + C_3 < 0, \qquad (2.25)$$

$$t_0 < t < \infty \quad \text{for} \quad C_1 + C_2 + C_3 > 0. \qquad (2.26)$$

Henceforth we will consider solutions only for $C_1 + C_2 + C_3 > 0$ without loss of generality. When $C_1 + C_2 + C_3 < 0$ we obtain the time-reversed solutions for $t < t_0$. For simplicity we shall set $t_0 = 0$ below; it can be reintroduced by substituting $t - t_0$ for t in the appropriate expressions.

A. Dilaton-vacuum solutions

For purposes of comparison, let us first give the solutions for $L_3 = 0$, when the axion field h and tensor potential $B_{\mu\nu}$ remain constant and so do not affect the dynamics. This corresponds to the well-known dilaton-vacuum cosmology [1]. We have $d\phi/d\lambda = C_1 + C_2 \pm C_0$ and thus

$$e^\phi = \exp\left[\pm C_0(\lambda - \lambda_0) - C_1(\lambda - \lambda_1) - C_2(\lambda - \lambda_2)\right], \qquad (2.27)$$

Thus we have, from Eqs. (2.18)–(2.20),

$$a_1 = \exp\left(\frac{1}{2}C_1(\lambda - \lambda_1) - \frac{1}{2}C_2(\lambda - \lambda_2)\right.$$
$$\left. \pm \frac{1}{2}C_0(\lambda - \lambda_0)\right), \qquad (2.28)$$

$$a_2 = \exp\left(\frac{1}{2}C_2(\lambda - \lambda_2) - \frac{1}{2}C_1(\lambda - \lambda_1)\right.$$
$$\left. \pm \frac{1}{2}C_0(\lambda - \lambda_0)\right), \qquad (2.29)$$

$$a_3 = \exp[C_3(\lambda - \lambda_3)]. \qquad (2.30)$$

The constants C_0, C_i are constrained by Eq. (2.23), while the constants λ_0, λ_i are free. All of these constants, of course, are fixed by the initial conditions on the cosmology.

Using Eq. (2.24) to rewrite these in terms of the proper time in the string frame we have simply power-law solutions

$$e^\phi = e^{\phi_*}\left(\frac{t}{t_*}\right)^{\mp p + q - 1}, \qquad (2.31)$$

$$a_1 = a_{1*}\left(\frac{t}{t_*}\right)^{-\frac{1}{2}(\pm p + r)}, \qquad (2.32)$$

$$a_2 = a_{2*}\left(\frac{t}{t_*}\right)^{-\frac{1}{2}(\pm p - r)}, \qquad (2.33)$$

$$a_3 = a_{3*}\left(\frac{t}{t_*}\right)^q. \qquad (2.34)$$

Here we have renamed various combinations of constants; in particular, we have

$$p = \frac{C_0}{C_1 + C_2 + C_3}, \quad q = \frac{C_3}{C_1 + C_2 + C_3},$$
$$r = \frac{C_2 - C_1}{C_1 + C_2 + C_3}, \quad s = \frac{C_n}{C_1 + C_2 + C_3}. \qquad (2.35)$$

The various prefactors e^{ϕ_*}, a_{i*} ($i = 1, 2, 3$) are appropriate combinations of the constants that appear in λ-time solutions. The characteristic time t_* corresponds to the value of t when $\lambda = \lambda_0$:

$$\left(\frac{t}{t_*}\right)^{\frac{1}{C_1 + C_2 + C_3}} = \exp(\lambda - \lambda_0). \qquad (2.36)$$

In terms of these new constants the constraint, Eq. (2.23), becomes

$$\frac{1}{2}p^2 + q^2 + \frac{1}{2}r^2 + ns^2 = 1. \qquad (2.37)$$

Solutions with $p < 0$ correspond to solutions for t (and thus t_*) negative. Considering only solutions for $t > 0$ (i.e., $C_1 + C_2 + C_3 > 0$) implies that $p \geq 0$. In either case we have two possible vacuum branches corresponding to the choice of $\pm p$ in the solutions (unless C_0 and thus p are zero).

While these solutions are in general anisotropic, this is simply a consequence of having allowed ourselves the freedom to choose anisotropic initial conditions. If we pick isotropic initial conditions, the metric remains an isotropic (Friedmann-Robertson-Walker) metric.

B. Axion-dilaton solutions

When $L_3 \neq 0$ we define $v = d\phi/d\lambda + (C_1 + C_2)$ so that the equation of motion for the dilaton, Eq. (2.17), can be written as

$$\frac{dv}{d\lambda} = -e^{2\phi}L_3^2\tilde{a}_1^2\tilde{a}_2^2 = v^2 - C_0^2, \qquad (2.38)$$

which can be solved to give (note that $v^2 < C_0^2$)

$$v = -C_0 \tanh C_0(\lambda - \lambda_0), \qquad (2.39)$$

where λ_0 is a constant of integration. The solution for the dilaton then follows from Eq. (2.38):

$$e^\phi = \left|\frac{C_0}{L_3}\right|\frac{1}{\cosh C_0(\lambda - \lambda_0)}$$
$$\times \exp[-C_1(\lambda - \lambda_1) - C_2(\lambda - \lambda_2)] \quad \text{for} \quad L_3 \neq 0. \qquad (2.40)$$

We can now collect our solutions for the scale factors in the string frame:

$$a_1 = \sqrt{\left|\frac{C_0}{L_3}\right|} \frac{\exp\left[\frac{1}{2}C_1(\lambda - \lambda_1) - \frac{1}{2}C_2(\lambda - \lambda_2)\right]}{\sqrt{\cosh C_0(\lambda - \lambda_0)}}, \quad (2.41)$$

$$a_2 = \sqrt{\left|\frac{C_0}{L_3}\right|} \frac{\exp\left[\frac{1}{2}C_2(\lambda - \lambda_2) - \frac{1}{2}C_1(\lambda - \lambda_1)\right]}{\sqrt{\cosh C_0(\lambda - \lambda_0)}}, \quad (2.42)$$

$$a_3 = \exp[C_3(\lambda - \lambda_3)]. \quad (2.43)$$

Note how the dilaton-vacuum solutions contain two distinct branches according to whether we choose $\pm C_0$, whereas the axion-dilaton results above are independent of the choice of sign, smoothly evolving from the $-C_0$ vacuum branch, when $(\lambda - \lambda_0)$ is large and negative, to the $+C_0$ branch, when $(\lambda - \lambda_0)$ becomes large and positive.

In terms of the string frame time coordinate t, the solutions for the scale factors and the dilaton take the forms [with p, q, and r as defined in Eq. (2.35)]

$$e^\phi = e^{\phi_*}\left[\left(\frac{t}{t_*}\right)^{p-q+1} + \left(\frac{t}{t_*}\right)^{-p-q+1}\right]^{-1}, \quad (2.44)$$

$$a_1 = a_{1*}\left[\left(\frac{t}{t_*}\right)^{p+r} + \left(\frac{t}{t_*}\right)^{-p+r}\right]^{-\frac{1}{2}}, \quad (2.45)$$

$$a_2 = a_{2*}\left[\left(\frac{t}{t_*}\right)^{p-r} + \left(\frac{t}{t_*}\right)^{-p-r}\right]^{-\frac{1}{2}}, \quad (2.46)$$

$$a_3 = a_{3*}\left(\frac{t}{t_*}\right)^q. \quad (2.47)$$

Again we see the evolution from one vacuum branch [with the lower signs in Eqs. (2.31)–(2.34)] for $t \ll t_0$ to the other vacuum branch (with upper signs) for $t \gg t_*$.

Note that the effect of the axion field is to decelerate the scale factors a_1 and a_2, placing an upper bound on the product:

$$a_1 a_2 = \frac{a_{1*}a_{2*}}{\left(\frac{t}{t_*}\right)^p + \left(\frac{t}{t_*}\right)^{-p}} \leq \left|\frac{C_0}{L_3}\right|. \quad (2.48)$$

In contrast with the dilaton-vacuum solutions, even an initially isotropic metric ($r = 0$, $q = p/2$) becomes anisotropic in the presence of the axion resulting from a time-dependent tensor potential. The only solution that can approach isotropy at late times is a *contracting* metric.

To understand how this occurs it is useful to return to the Einstein frame solutions. We can always write the Einstein constraint, Eq. (2.12), in a Bianchi type I metric as

$$\bar\theta^2 = 3\kappa^2\left(\tilde\rho_\phi + \tilde\rho_\beta + \tilde\rho_H\right) + 3\tilde\sigma^2, \quad (2.49)$$

where

$$\tilde\theta^2 \equiv \left(\dot{\tilde\alpha}_1 + \dot{\tilde\alpha}_2 + \dot{\tilde\alpha}_3\right)^2 \quad (2.50)$$

$$= \frac{1}{4\tilde a_1^2 \tilde a_2^2 \tilde a_3^2}\left\{3(C_1 + C_2) + 2C_3\right.$$

$$\left. + C_0 \tanh C_0(\lambda - \lambda_0)\right\}^2 \quad (2.51)$$

is the expansion rate and the anisotropy (or "shear") is given by

$$\tilde\sigma^2 \equiv \frac{1}{3}\left(\dot{\tilde\alpha}_1^2 + \dot{\tilde\alpha}_2^2 + \dot{\tilde\alpha}_3^2 - \dot{\tilde\alpha}_1\dot{\tilde\alpha}_2 - \dot{\tilde\alpha}_2\dot{\tilde\alpha}_3 - \dot{\tilde\alpha}_3\dot{\tilde\alpha}_1\right) \quad (2.52)$$

$$= \frac{1}{4\tilde a_1^2 \tilde a_2^2 \tilde a_3^2}\left\{C_1^2 + C_2^2 + 2(C_1 - C_2)^2\right.$$

$$\left. + [2C_3 + C_0 \tanh C_0(\lambda - \lambda_0)]^2\right\} \quad (2.53)$$

for the axion-dilaton solutions given in Eqs. (2.41)–(2.43).

Each term in the constraint equation (2.49) is nonnegative and so the relative importance of each term on the right-hand side in determining the expansion rate is simply given by their magnitude. Thus at early or late times we recover the dilaton-vacuum solutions where the expansion rate is proportional to square of the volume in the Einstein frame ($\tilde\theta^2 \propto \tilde\rho_\phi \propto \tilde\rho_\beta \propto \tilde\sigma^2$ as $\lambda \to \pm\infty$), while the axion energy density evolves as

$$\tilde\rho_H = \frac{L_3^2}{4\kappa^2}\frac{a_1^2 a_2^2}{\tilde a_1^2 \tilde a_2^2 \tilde a_3^2} \quad (2.54)$$

$$= \frac{C_0^2}{4\kappa^2 \tilde a_1^2 \tilde a_2^2 \tilde a_3^2}\left\{1 - \tanh^2 C_0(\lambda - \lambda_0)\right\} \quad (2.55)$$

and vanishes relative to the other terms as $\lambda \to \pm\infty$.

The axion field only plays a dynamical role for a brief period around $\lambda \simeq \lambda_0$ ($t \simeq t_*$). It is the only anisotropic fluid in the system, and so it delivers an "anisotropic impulse" to the metric. As can be seen from Eq. (2.53) this causes a change in the shear, around $\lambda = \lambda_0$ ($t = t_*$), $\Delta\tilde\sigma^2 \propto pq$. The only stable late-time vacuum solutions have the area perpendicular to the gradient of the axion field ($a_1 a_2$) decreasing.

In order to solve for $B_{\mu\nu}$, we first note that we have already chosen $B_{0i} = 0$, and setting $B_{ij} \equiv B_{ij}(t)$ implies $H_{ijk} = 0$. Further, the choice $L_1 = L_2 = 0$, made because of Eq. (2.5), implies that B_{23} and B_{31} are constants, and it follows that $H_{012} = \partial_t B_{12}$. Then, combining the two expressions

$$\tilde H^{012} = -\frac{e^\phi}{\tilde a_1^2 \tilde a_2^2}\partial_t B_{12}, \quad (2.56)$$

$$\tilde H^{012} = \frac{e^{2\phi}L_3}{\sqrt{\det \tilde g}}, \quad (2.57)$$

and using the definition of λ, Eq. (2.15), we obtain

$$\partial_\lambda B_{12} = -L_3 e^{2\phi}\tilde a_1^2 \tilde a_2^2 = -L_3 a_1^2 a_2^2. \quad (2.58)$$

Of course, if $L_3 = 0$, then B_{12} is just a constant. Otherwise, use of Eqs. (2.18), (2.19), and (2.40) allows us to rewrite the above equation and solve for B_{12}:

$$\partial_\lambda B_{12} = -\frac{C_0^2}{L_3}\frac{1}{\cosh^2 C_0(\lambda - \lambda_0)} \qquad \Rightarrow \qquad B_{12} = -\frac{C_0}{L_3}\left[\tanh C_0(\lambda - \lambda_0) + \widehat{B}_{12}\right], \tag{2.59}$$

where $\widehat{B}_{12}$ is a constant fixed solely by the initial conditions on the antisymmetric tensor and is independent of the choice of all the other constants. For the sake of completeness, the other components of $B_{\mu\nu}$ are given by $B_{0i} = 0, B_{23} = \widehat{B}_{23}, B_{31} = \widehat{B}_{31}$, both constants.

Thus we see that, except for $\lambda \sim \lambda_0$ (or equivalently $t \sim t_*$), the tensor potential remains very nearly constant and we recover the vacuum solutions. Only in the vicinity of $\lambda = \lambda_0$ does the potential change, resulting in a nonzero axion field, which delivers an anisotropic impulse to the metric, before the potential becomes roughly constant, again returning to the vacuum branch.

III. DUALITY

The class of homogeneous solutions of the metric, dilaton, and antisymmetric tensor potential in four dimensions has been shown to have a global $O(3,3)$ invariance [in general, a global $O(D-1, D-1)$ invariance in D dimensions] [6–8] under which

$$M \to M' = \Omega^T M \Omega, \qquad \bar{\phi} \equiv \phi - \ln\sqrt{\det G} \to \bar{\phi}, \tag{3.1}$$

where Ω is a 6×6 constant matrix satisfying

$$\Omega^T \eta \Omega = \eta, \qquad \eta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \tag{3.2}$$

($\mathbf{1}$ is the 3×3 identity matrix), and

$$M \equiv \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \tag{3.3}$$

where G and B are, respectively, g_{ij} and B_{ij} written as 3×3 matrices. Any 6×6 constant matrix Ω obeying Eq. (3.2) generates new solutions for the metric, antisymmetric tensor, and dilaton, corresponding to M', from the original set of solutions.

In the case where B_{ij} vanishes the special choice $\Omega = \eta$ is called the "scale factor duality" transformation [6,7] because it takes the scale factors $a_i \to a_i^{-1}$, thus exchanging "large" directions with "small" directions. Let us consider what happens to the solutions of the previous section with this particular choice of Ω. This generates nonequivalent solutions; i.e., it is not just a spatial rotation or gauge transformation [7], while it will allow us remain within our class of Bianchi type I solutions if we set $\widehat{B}_{23} = \widehat{B}_{31} = 0$.

In this case, a bit of algebra shows (with primes denoting the duality transformed functions)

$$a'^2_1 = \frac{a_2^2}{a_1^2 a_2^2 + B_{12}^2}, \tag{3.4}$$

$$a'^2_2 = \frac{a_1^2}{a_1^2 a_2^2 + B_{12}^2}, \tag{3.5}$$

$$a'^2_3 = \frac{1}{a_3^2}, \tag{3.6}$$

$$B'_{12} = -\frac{B_{12}}{a_1^2 a_2^2 + B_{12}^2}. \tag{3.7}$$

From our solutions (2.41), (2.42), and (2.59) we have

$$a_1^2 a_2^2 + B_{12}^2 = \frac{C_0^2}{L_3^2 \cosh C_0(\lambda - \lambda_0)}\left[(1 + \widehat{B}_{12}^2)\cosh C_0(\lambda - \lambda_0) + 2\widehat{B}_{12}\sinh C_0(\lambda - \lambda_0)\right]. \tag{3.8}$$

It follows that the dual solutions can be divided into two separate classes.

(a) $\widehat{B}_{12} \neq \pm 1$. In this case, we can define a constant λ_0'' such that

$$2\widehat{B}_{12} = b\sinh C_0\lambda_0'', \tag{3.9}$$

where $b = |1 - \widehat{B}_{12}^2|$. Then $1 + \widehat{B}_{12}^2 = b\cosh C_0\lambda_0''$, and Eq. (3.8) becomes

$$a_1^2 a_2^2 + B_{12}^2 = \frac{bC_0^2}{L_3^2}\frac{\cosh C_0(\lambda - \lambda_0')}{\cosh C_0(\lambda - \lambda_0)}, \tag{3.10}$$

with $\lambda_0' = \lambda_0 - \lambda_0''$. The dual transformed solutions [(3.4)–(3.7)] can be then rewritten as

$$a_1' = \sqrt{\left|\frac{L_3}{bC_0}\right|}\frac{\exp\left[\frac{1}{2}C_2(\lambda - \lambda_2) - \frac{1}{2}C_1(\lambda - \lambda_1)\right]}{\sqrt{\cosh C_0(\lambda - \lambda_0')}}, \tag{3.11}$$

$$a_2' = \sqrt{\left|\frac{L_3}{bC_0}\right|}\frac{\exp\left[\frac{1}{2}C_1(\lambda - \lambda_1) - \frac{1}{2}C_2(\lambda - \lambda_2)\right]}{\sqrt{\cosh C_0(\lambda - \lambda_0')}}, \tag{3.12}$$

$$a_3' = \exp[-C_3(\lambda - \lambda_3)], \tag{3.13}$$

$$B_{12}' = \mp\frac{L_3}{bC_0}\left[\tanh C_0(\lambda - \lambda_0') - \widehat{B}_{12}\right], \tag{3.14}$$

where the sign for B_{12}' is positive or negative as $(1 - \widehat{B}_{12}^2)$ is negative or positive, respectively. As is obvious, these fall in the same classes of solutions as our original ones Eqs. (2.40)–(2.43), (2.59). The dilaton ϕ is shifted,

$$e^\phi \to e^{\phi'} = \left| \frac{L_3}{bC_0} \right| \frac{1}{\cosh C_0(\lambda - \lambda_0')} \exp[-C_1(\lambda - \lambda_1) - C_2(\lambda - \lambda_2) - 2C_3(\lambda - \lambda_3)], \tag{3.15}$$

while $\bar{\phi}$ as defined in Eq. (3.1) is invariant, as is the coordinate λ. Obviously, the duality transformation can be expressed as a transformation on the constants that appear in the solutions. For example, when the solutions are expressed in terms of the string time coordinate t as in Eqs. (2.44)–(2.47), the duality transformation is essentially equivalent to the following transformation on the constants:

$$p \to -p, \qquad q \to -q, \qquad r \to -r, \qquad L_3 \to \pm |1 - \hat{B}_{12}^2| \frac{C_0^2}{L_3}. \tag{3.16}$$

Note that the duality transformation changes the characteristic time defined by Eq. (2.36) (but now with λ_0 replaced by λ_0'):

$$t_* \to t_*' = t_* e^{-(C_1 + C_2 + C_3)\lambda_0''} = t_* \left| \frac{1 - \hat{B}_{12}}{1 + \hat{B}_{12}} \right|^{1/p}. \tag{3.17}$$

Thus the characteristic time tends to zero or infinity as $\hat{B}_{12}$ tends to $+1$ or -1, respectively.

(b) $\hat{B}_{12} = \pm 1$. In this case, we have

$$a_1^2 a_2^2 + B_{12}^2 = \frac{C_0^2}{L_3^2} \left[\text{sech}^2 C_0(\lambda - \lambda_0) + (\tanh C_0(\lambda - \lambda_0) \pm 1)^2 \right]$$

$$= \frac{C_0^2}{L_3^2} \frac{2 e^{\pm C_0(\lambda - \lambda_0)}}{\cosh C_0(\lambda - \lambda_0)}. \tag{3.18}$$

The dual transformed solutions turn out to be nothing more than anisotropic solutions of pure dilaton cosmology:

$$a_1' = \sqrt{\left| \frac{L_3}{2C_0} \right|} \exp \left(\frac{1}{2} C_2(\lambda - \lambda_2) - \frac{1}{2} C_1(\lambda - \lambda_1) \mp \frac{1}{2} C_0(\lambda - \lambda_0) \right), \tag{3.19}$$

$$a_2' = \sqrt{\left| \frac{L_3}{2C_0} \right|} \exp \left(\frac{1}{2} C_1(\lambda - \lambda_1) - \frac{1}{2} C_2(\lambda - \lambda_2) \mp \frac{1}{2} C_0(\lambda - \lambda_0) \right), \tag{3.20}$$

$$a_3' = \exp[-C_3(\lambda - \lambda_3)], \tag{3.21}$$

$$B_{12}' = \frac{L_3}{2C_0} \left[\sinh C_0(\lambda - \lambda_0) \pm \cosh C_0(\lambda - \lambda_0) \right] e^{\mp C_0(\lambda - \lambda_0)} = \pm \frac{L_3}{2C_0}. \tag{3.22}$$

The sign ($\pm$) in the above corresponds to the sign of $\hat{B}_{12}$. Thus we see that these "vacuum" ($H_{\mu\nu\lambda} = 0$) solutions appear as a limit of the duality transforms of type (a) above as $\lambda_0'' \to \pm\infty$ (alternatively, as $t_*' \to 0$ or ∞).

IV. CONCLUSIONS

We have considered four-dimensional cosmological solutions of low-energy effective string theory in which the metric, dilaton, modulus, and antisymmetric tensor potentials depend only on the time coordinate. This restriction inevitably leads us to a homogeneous but anisotropic universe except for the isotropic vacuum solution with the antisymmetric tensor $B_{\mu\nu} = $const. Even in a situation with $B_{ij} \neq 0$ (but with $H_{\mu\nu\lambda} = 0$) the three components of B_{ij} form a three-vector, thus specifying a chosen direction. However, it is when the components of $B_{\mu\nu}$ are allowed to vary with time that the variation (the axion field h) drives the anisotropy of the metric. In order to gain an understanding of how such a universe would evolve, we have considered Bianchi type I universes in

this paper possessing shear but no spatial curvature.

We find models which behave like dilaton-vacuum models (where the axion can be neglected) at early and late times. However, the axion field does affect the dynamics for a brief period around $t = t_*$, producing an anisotropic "impulse" at this point. The Einstein metric provides a useful frame in which to discuss the behavior of these solutions. During the effectively vacuum regimes, the shear $\tilde{\sigma}^2$ and density of the modulus, $\tilde{\rho}_\beta$, and dilaton, $\tilde{\rho}_\phi$, in this frame drive the expansion and are proportional to $(\tilde{a}_1 \tilde{a}_2 \tilde{a}_3)^{-2}$. However, the axion density $\tilde{\rho}_H \propto e^{2\phi} \tilde{a}_3^{-2}$ will grow relative to the shear, modulus, and dilaton densities at early times while $e^{2\phi} \tilde{a}_1^2 \tilde{a}_2^2 = a_1^2 a_2^2$ grows. Thus the axion's anisotropic pressure must eventually become important. It tends to decelerate the scale factors a_1 and a_2, and produces an upper bound on the product $a_1 a_2 |L_3| \leq C_0$. The dilaton-vacuum solutions ($L_3 = 0$) are thus atypical of the general axion-dilaton solutions. The stable, late-time, effectively vacuum era must have $a_1 a_2$ decreasing, while the third scale factor a_3 is free to grow (or decrease) monotonically.

Such anisotropy is not observed in our Universe to-

day. We usually expect shear will be diluted away at late times by the presence of other isotropic matter, in particular isotropic radiation in the hot big bang model. However, it is far from clear whether ordinary isotropic radiation can dominate over the axion in this model. The isotropic late-time behavior (with $a = a_1 = a_2 = a_3$) dominated by a radiation ($p = \rho/3$) is only possible if $\tilde{\rho}_H/\tilde{\rho} \propto e^{\phi}a^2 \to 0$ as $t \to 0$. Because the pure radiation plus dilaton solution has $\phi \to$const, a late-time isotropic radiation-dominated solution must be a *contracting* universe, i.e., $a \to 0$.

We cannot rule out some expanding isotropic solution with radiation at late times, but for the anisotropic axion field not to spoil this isotropy we must have $\tilde{\rho}_H/\tilde{\rho}$ remaining negligible, and thus a decreasing dilaton, or some reason (such as an inflationary era in the Einstein frame) for the axion gradient to be vanishingly small. As any variation from the standard hot big bang model (with constant dilaton) is tightly constrained by, for instance, results from primordial nucleosynthesis, radiation alone does not seem to be sufficient to erase the anisotropic influence of a time-dependent antisymmetric tensor potential. Spatial curvature (zero in the Bianchi type I metric) would in general introduce further anisotropy. Again, we would require inflation in the Einstein frame to avoid curvature dominating the evolution at late times.

The "characteristic time" t_* plays a major role in both the evolution and in the interpretation of duality transformations of the solutions. At early times, the antisymmetric tensor $B_{\mu\nu}$ is approximately constant. It changes rapidly around t_* and becomes approximately constant again, albeit at a different value. When a transformed characteristic time t'_* is defined, the duality transformation is seen to change the time dependence of the scale factors and the antisymmetric tensor potential by $(t/t_*) \to (t'_*/t)$. The duality transformation is then seen to relate a given solution at late times with another solution at early times. In particular, we find that for special choices of initial values of the antisymmetric tensor, the universe that results is duality related to a vacuum solution of pure dilaton cosmology, where $t'_* \to 0$ or ∞.

The $O(3,3)$ invariance of the low-energy action proves to be of limited use in a cosmological context. The requirement that the metric and potential $B_{\mu\nu}$ both be functions only of time is highly prescriptive. Given a homogeneous metric a more natural expectation would be that the axion field (which determines the energy-momentum tensor and thus the metric) should be time dependent [3,4], rather than the potential. The only metric (with zero vorticity) which can meet this prescription is Bianchi type I and in such a case the axion field derived from $B_{\mu\nu}(t)$ can have no time dependence and must be anisotropic. This prohibits isotropic expanding universes at late times in the string frame. In this respect the dilaton-vacuum solutions $B_{\mu\nu}$ =const are atypical of the behavior of the general axion-dilaton solutions.

ACKNOWLEDGMENTS

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