Conditions for inflation in an initially inhomogeneous universe

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Using a long wavelength iteration scheme to solve Einstein's equations near the big-bang singularity of a universe driven by a massive scalar field, we find how big initial quasi-isotropic inhomogeneities can be before they can prevent inflation to set in.

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I. INTRODUCTION

The scale factor $a(t)$ of a spatially flat isotropic and homogeneous Robertson-Walker (fRW) universe driven by a massive scalar field φ evolves as follows: Near the big bang (chosen to be $t = 0$) the scalar field goes to ∞ as $-\ln t$ and behaves like a perfect stiff fluid (whose pressure equals the energy density, that is, whose adiabatic index Γ is 2), so that a grows as $t^{1/3}$ [1]. Then a grows quasiexponentially during the inflationary regime during which φ slowly rolls down its potential well [2]. Finally at the end of inflation when φ oscillates in the bottom of the well a behaves as if the Universe was driven by dust and grows on average as $t^{2/3}$ [3].

A question is, how stable is this evolution against departures from spatial flatness, isotropy, and homogeneity? The effect of curvature was studied in [4]; since its role after the big bang can become predominant, it can prevent inflation from ever starting if strong enough. Inflation in homogeneous albeit anisotropic Bianchi models was thoroughly analyzed, with similar conclusions: See, e.g., [5] for a review. Finally, the role of inhomogeneities was studied analytically under simplifying assumptions in [6]. They were studied numerically in the case of planar symmetry in [7] and in the case of spherical symmetry in [8]. In [8], setting the initial conditions in such a way that inflation would occur only in the central region if the Universe evolved like a fRW spacetime, the conclusion was that inflation would indeed occur only if the central region was larger than a few times the local Hubble radius.

Here we shall tackle the problem semianalytically in the long wavelength approximation.

The long wavelength iteration scheme, the history of which goes back to [9], is a way to build, out of "seed" spatial metrics, approximate solutions of Einstein's equations which describe inhomogeneous but quasi-isotropic universes on scales larger than the local Hubble radius (see [10] and references therein for a detailed description of the scheme). When matter satisfies the strong energy condition (that is, does not inflate) this approximation is well suited to describe the early Universe since most scales are then larger than the Hubble radius on average (this is the well-known horizon problem). These approximate solutions, however, are not generic in the sense that they are built out of a seed spatial metric, that is, on three physically distinct arbitrary functions, instead of six (four for the gravitational field, plus two for the scalar field). The three missing functions can be seen as describing departures from isotropy. These anisotropies cannot be neglected near the big bang (see, e.g., [9,10]) but they decay with time much faster than all other contributions to inhomogeneity and hence will be ignored (see [11] for an analysis of the generic solution near the big bang).

Within that scheme the growth or decay of inhomogeneities according to their equation of state can be easily inferred, at least qualitatively (see [10,12]): They decay when matter violates the strong energy condition, that is, inflates, and grow otherwise, that is, when the effective adiabatic index Γ_{eff} of matter is $> 2/3$. Therefore when matter is a scalar field the inhomogeneities first grow ($\Gamma_{\text{eff}} = 2$) and then decay during the inflationary period, to grow again at the end of inflation ($\Gamma_{\text{eff}} = 1$), *at the condition* that they do not grow so large during the first phase as to prevent inflation from starting.

The purpose of this paper is to give quantitative estimates on when inflation may be halted by the presence of initial quasi-isotropic inhomogeneities. This will be done by integrating numerically the ordinary second-order linear differential equations that govern their evolution in the long wavelength approximation scheme.

II. EQUATIONS

The long wavelength approximation scheme [10] consists in looking for solutions of Einstein's equations for gravity coupled to a scalar field whose three-metric (in the $t =$ const slices of a synchronous reference frame) can be expanded as a sum of spatial tensors of increasing order in the gradients of a "seed" metric with time-dependent coefficients. The line element is thus of the form

$$\begin{aligned} ds^2 &= -dt^2 + \gamma_{ij}(t,x^k)dx^i dx^j, \quad \gamma_{ij} = {}^{(1)}\gamma_{ij} + {}^{(3)}\gamma_{ij} + {}^{(5)}\gamma_{ij} + \cdots, \\ {}^{(1)}\gamma_{ij} &= a^2(t)h_{ij}(x^k), \quad {}^{(3)}\gamma_{ij} = a^2\left[a_2(t)Rh_{ij} + b_2(t)R_{ij}\right], \\ {}^{(5)}\gamma_{ij} &= a^2\left[a_4(t)R^2 h_{ij} + b_4(t)RR_{ij} + c_4(t)R_{lm}R^{lm}h_{ij} + d_4(t)R_{im}R^m_j\right] \\ &\quad + a^2\left[e_4(t)\nabla_m\nabla^m R h_{ij} + f_4(t)\nabla_i\nabla_j R + g_4(t)\nabla_m\nabla^m R_{ij}\right], \end{aligned} \tag{1}$$

and the scalar field of the form

$$\varphi = \phi(t) + \phi_2(t)R + \phi_4(t)R^2 + \psi_4(t)R_{lm}R^{lm} + \xi_4(t)\nabla_m\nabla^m R + \cdots \ . \tag{2}$$

This solution is built on a "seed" $h_{ij}(x^k)$ of six functions of space which can be reduced to three by a suitable choice of spatial coordinates. As shown in [10], it is not a generic solution (it should then depend on six physically distinct functions), but it is an attractor of a class of generic solutions. At lowest order the metric reduces to $\gamma_{ij} = a^2(t)h_{ij}(x^k)$ and $\varphi = \phi(t)$, hence the name "quasi-isotropic" given to the solution. ∇_i is the covariant derivative with respect to h_{ij}, R_{ij} its Ricci tensor, and $R \equiv R^{ij}h_{ij}$ the curvature scalar. The coefficients $a(t)$, $a_2(t)$, etc., are functions of time which are determined by Einstein's equations. We shall denote by L the characteristic comoving length on which the spatial metric varies: $\partial_i\gamma_{jk} \simeq L^{-1}\gamma_{jk}$.

At zeroth order Einstein's equations reduce to the Friedmann equations for a spatially flat Robertson-Walker (fRW) universe and determine $a(t)$ and $\phi(t)$. Introducing the dimensionless variables

$$T = mt, \quad F = 2\sqrt{3\pi G}\phi, \quad H = \frac{3}{m}\frac{1}{a}\frac{da}{dt}, \quad S = am^{1/3}, \tag{3}$$

where m is the mass of the scalar field, they read

$$\frac{dS}{dT} = \frac{HS}{3}, \quad \frac{dH}{dT} = -(H^2 - F^2), \quad \frac{dF}{dT} = -\sqrt{H^2 - F^2}. \tag{4}$$

The solutions of Eqs. (4) depend on three integration "constants": the time $\tilde{T}(x^k)$ of the big bang that we shall restrict to be $\tilde{T} = 0$ (see [11] for an analysis of delayed big bang solutions); the size $\tilde{S}(x^k)$ of the scale factor at some given initial time: a "constant" which can be absorbed without loss of generality in a redefinition of the seed metric h_{ij}; and finally the initial value $\tilde{F}$ for the scalar field. *A priori* $\tilde{F}$ depends on space but the first-order Einstein equations (see [10] for details): $\partial_i H = -4(dF/dT)\partial_i F$ imposes that it does not. Its value, the same for all x^k, tells us which curve of the $(F, dF/dT)$ phase diagram the solution follows, and hence determines the total amount of inflation. We shall therefore integrate these equations up to the beginning of inflation with, as initial conditions at time $T = \epsilon$,

$$S_{\rm in} = \epsilon^{1/3}, \quad H_{\rm in} = \frac{1}{\epsilon}, \quad F_{\rm in} = \tilde{F}, \tag{5}$$

ϵ being chosen such that $(\ln\epsilon)^2$ is numerically negligible compared to ϵ^{-2}. If matter were a perfect fluid with adiabatic index Γ, the scale factor S would grow as $T^{2/3\Gamma}$. In the initial regime (5) the scalar field hence behaves like a stiff fluid ($\Gamma = 2$) [1]. When $\tilde{F}$ is large enough the solution then enters an inflationary phase characterized by a slow linear decrease of F and a quasiexponential growth of S. The comoving Hubble radius $L_H = 3/SH$ hence first increases as $T^{2/3}$ during the stiff fluid regime and then decreases exponentially. The moment it reaches its maximum value can be taken as the beginning of inflation. We shall denote $L_H^{\rm inf}$ and $F_{\rm inf}$ the values of the Hubble radius and the scalar field at that moment.

At third order Einstein equations determine the time-dependent coefficients $a_2(t)$, $b_2(t)$, and $\phi_2(t)$ in (1) and (2). In terms of the variables defined in (3) and introducing

$$B_2 = b_2 m^{4/3}, \qquad A_2 = a_2 m^{4/3},$$

$$F_2 = 4\sqrt{\frac{\pi G}{3}}m^{4/3}\phi_2, \tag{6}$$

they can be written as (see [10])

$$\frac{du}{dT} = -2S, \qquad \frac{dB_2}{dT} = \frac{u}{S^3}, \tag{7}$$

$$\frac{dv}{dT} = -\frac{u}{4S}\frac{d}{dT}\left[\left(S\frac{dF}{dT}\right)^{-2}\right], \quad \frac{dA_2}{dT} = v\left(\frac{dF}{dT}\right)^2, \tag{8}$$

$$F_2 = -\frac{1}{dF/dT}\left(\frac{1}{4}\frac{dB_2}{dT} + \frac{dA_2}{dT}\right). \tag{9}$$

It is easy to see that when the inflationary regime has set in, B_2 tends to a constant (which we shall call I_2) as well as F_2. As for A_2 it decreases linearly in time. Now, as shown in [10], a gauge transformation modifies the coefficient of the scalar curvature R in ${}^{(3)}\gamma_{ij}$ but leaves untouched the Ricci term. During inflation it amounts to adding an arbitrary linear function of time to A_2. Therefore the linear decrease of A_2 can be gauged away by a suitable choice of initial conditions for A_2, v_2, leaving B_2

as the only relevant quantity to be studied. Physically the geometry evolves from one configuration, h_{ij}, to another, $h_{ij} + (I_2/m^{4/3})R_{ij}$, where I_2 is the "imprint" on the geometry left by the initial inhomogeneity. We shall therefore integrate (7) with the initial conditions (5) at $T = \epsilon$ together with $dF/dT|_{\rm in} = -1/\epsilon$ and

$$u_{\rm in} = -\frac{3}{2}\epsilon^{4/3}, \quad B_2|_{\rm in} = -\frac{9}{8}\epsilon^{4/3}. \tag{10}$$

III. SOLUTIONS

Equations (4) with initial conditions (5) and Eqs. (7) with initial conditions (10) are integrated numerically. The results can be encapsulated in a plot of $\tau_2 \equiv \sqrt{|I_2|}/L_H^{\rm inf}$ as a function of $F_{\rm inf}$. See the curve $k = 0$ of Fig. 1. We see that as $F_{\rm inf}$ increases, τ_2 tends to a constant close to 1 [when a brutal extrapolation of the analytical behaviors (5) and (10) would have given $1/2\sqrt{2}$]. Going back to the expressions (1) and (2) for the metric and the scalar field, a sufficient condition for inflation then appears clearly: Inflation will set in if the corrective terms ${}^{(3)}\gamma_{ij}$ remain small compared to ${}^{(1)}\gamma_{ij}$, that is, if $I_2/L^2 < 1$, which is equivalent, because τ_2 tends to 1, to $L > L_H^{\rm inf}$. This condition says that the size L of the inhomogeneity must be larger than the (comoving) Hubble radius at the onset of inflation, when the brutal analytic extrapolation gives $L > (1/2\sqrt{2})L_H^{\rm inf}$. Moreover, when the zeroth order barely inflates, that is, when $F_{\rm inf}$ is small, $\tau_2 = \alpha > 1$, so that a sufficient condition for inflation is $L > \alpha L_H^{\rm inf}$. We thus recover in this semianalytical approach the result of Ref. [8].

IV. CONVERGENCE OF THE ITERATION SCHEME

A rigorous mathematical analysis of the convergence of the series (1) and (2) is certainly beyond the scope of this paper. Using the results of the Appendix of Ref. [10] we can, however, compute the next order, that is, the coefficients a_4, b_4, etc., and see if including them spoils the conclusion of the preceding paragraph, drawn from the first iteration.

The relevant coefficients, which are not affected by a gauge transformation, are c_4, d_4, and g_4. We shall concentrate on g_4 which satisfies the linear second-order differential equations

$$\frac{dw}{dT} = B_2 S, \quad \frac{dG_4}{dT} = -\frac{w}{S^3}, \tag{11}$$

where we have introduced $G_4 \equiv m^{8/3} g_4$ with the initial conditions

$$w_{\rm in} = -\frac{27}{64}\epsilon^{8/3}, \quad G_4|_{\rm in} = -\frac{81}{8^3}\epsilon^{8/3}. \tag{12}$$

(Knowing G_4 and B_2, the coefficient $D_4 \equiv m^{8/3} d_4$ is readily obtained: $D_4 = -4G_4 + \frac{1}{2}B_2^2$; see [10].)

The results of the integration of (11) and (12) are summarized in the diagram $\tau_4 \equiv |G_4|^{1/4}/L_H^{\rm inf}$ as a function of $F_{\rm inf}$ (see the curve $k = 0$ of Fig. 2). We see that τ_4 follows the same pattern as τ_2. As $F_{\rm inf}$ increases it tends to a constant value of the order of 0.63 when an extrapolation of the analytical behavior (16) would have given $8^{-3/4} = 0.21$. As for the ratio $|D_4|^{1/4}/L_H^{\rm inf}$ it tends to 0.60.

The inclusion of these fourth-order terms therefore does not spoil the conclusion of the previous paragraph

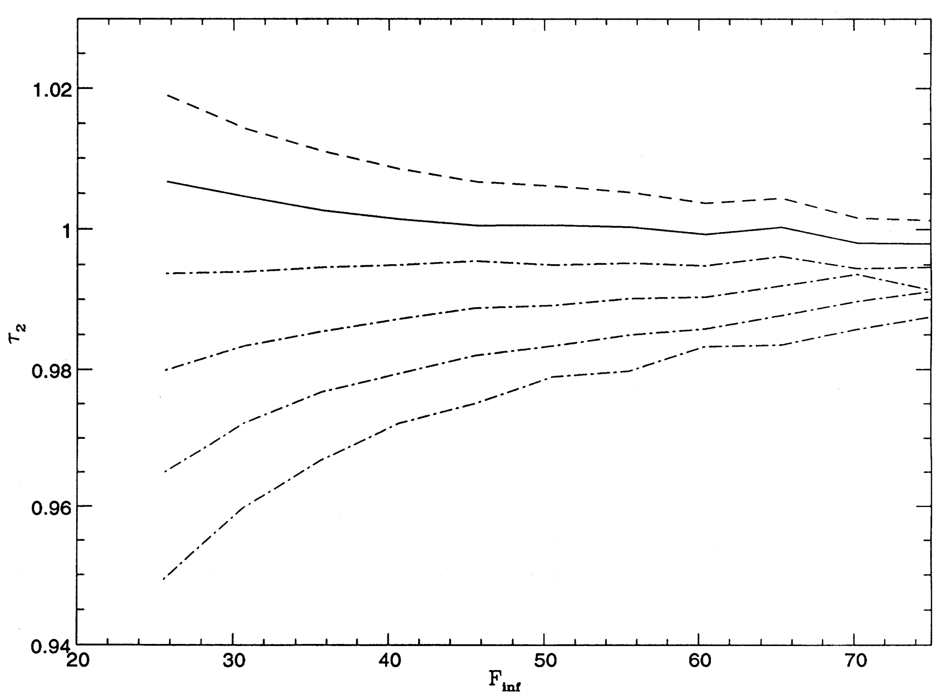

FIG. 1. Plot of τ_2 as a function of $F_{\rm inf}$. The solid line corresponds to $k = 0$, the dashed line above to positive k, and the dot-dashed lines below to negative k.

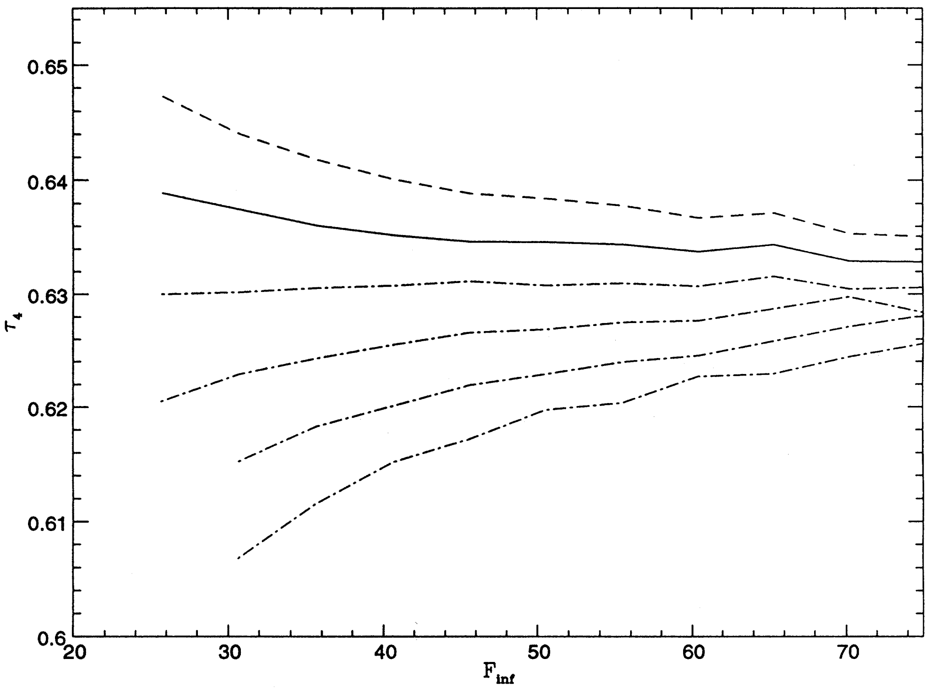

FIG. 2. Plot of τ_4 as a function of $F_{\rm inf}$. The conventions are the same as in Fig. 1.

and, since $\tau_4 < \tau_2$, is an indication (although certainly not a proof) that the exact solution does not differ drastically from the second-order solution. Assuming then that the series converges we conclude that, if the size of the inhomogeneity is larger than the Hubble radius, then the solution will inflate and the inhomogeneity be washed away.

V. CURVATURE VERSUS GRADIENT EFFECTS AND AN IMPROVED SCHEME IN THE CASE OF SPHERICAL SYMMETRY

In the case when the inhomogeneity is imposed to be spherically symmetric, the seed line element $d\sigma^2 = h_{ij}dx^i dx^j$ can be written in the form

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega, \tag{13}$$

where $d\Omega$ is line element on the sphere and where $k(r)$ is an arbitrary function of r.

Expressing the Ricci tensor and its derivatives as a function of $k(r)$ we can write the long wavelength metric (1) and scalar field (2) in the form

$$ds^2 = -dt^2 + R^2(C^2 dr^2 + r^2 d\Omega), \tag{14}$$

with

$$R = R_{\rm cRW}^{\rm app}\left[1 + \left(a_2 + \frac{1}{4}b_2\right)k'r + \cdots\right], \quad R_{\rm cRW}^{\rm app} = a\left[1 + (3a_2 + b_2)\,k + \cdots\right], \tag{15}$$

$$C = C_{\rm cRW}\left[1 + \left(a_2 + \frac{1}{2}b_2\right)k'r + \cdots\right], \quad C_{\rm cRW} = \frac{1}{\sqrt{1-kr^2}}, \tag{16}$$

and

$$\varphi = \phi_{\rm cRW}^{\rm app} + 2k'r\phi_2 + \cdots, \quad \phi_{\rm cRW}^{\rm app} = \phi + 6k\phi_2 + \cdots \ , \tag{17}$$

where a prime denotes differentiation with respect to r and where the time-dependent coefficients (a, ϕ), (a_2, b_2, ϕ_2), etc., are the same as before and satisfy Eqs. (3)–(5), (6)–(10), (11), and (12).

The rewriting of Eqs. (1) and (2) under the form (14)–(17) shows clearly the two ways in which a spherically symmetric inhomogeneity makes the solution depart from the fRW solution. The first (trivial) effect is that of curvature: If $k \neq 0$ but all its derivative are taken to be zero, the long wavelength solution (14)–(17) can be checked to be nothing but the Taylor expansion in t of the exact curved Robertson-Walker (cRW) solution. In this case then the exact metric and scalar field are given by (14) with $C = (1-kr^2)^{-1/2}$, and $R = a$ and ϕ satisfying the Friedmann equation

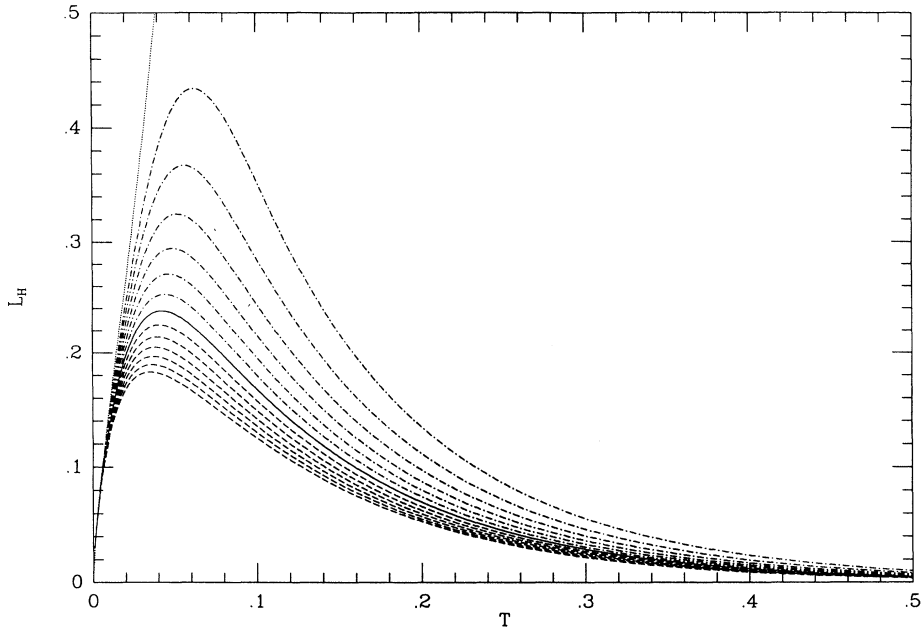

FIG. 3. Plot of the comoving Hubble radius as a function of time for various values of the curvature term k and a given initial value for the scalar field. The solid line corresponds to $k = 0$, the dashed lines below to negative k, and the dot-dashed ones to positive k. Finally the dotted line is an example of a curvature strong enough to prevent inflation from setting in.

$$\frac{dS}{dT} = \frac{HS}{3}, \quad \frac{dH}{dT} = -(H^2 - F^2) - \frac{6K}{S^2},$$

$$\frac{dF}{dT} = -\sqrt{H^2 - F^2 + 9K/S^2}, \tag{18}$$

where T, F, H, S are defined by (3) and where $K = km^{-2/3}$.

Integrating (18) numerically with the same initial conditions as before [Eq. (5)], we recover the results of Ref. [4]; that is, that if the curvature term K/S^2 is too large, then inflation is halted. This can be seen in Fig. 3 where L_H is plotted as a function of T for different values of K and a given initial value $\tilde{F}$ for the scalar field. We see that a negative K "favors" inflation whereas a positive value delays its setting in and, if large enough, can even prevent it (dotted line).

The second effect is that of the gradients, that is, the derivatives of $k(r)$, which reflects the point-to-point correlation due to the variation of the curvature. It can be enhanced by choosing a seed $k(r)$ such that it is small everywhere but has a steep gradient $k'r$ around, say, $r = R$. From (16) and the result of Sec. III, that $\tau_2 \simeq 1$, we know that these gradient effects will not prevent inflation to set in if the function $k(r)$ is everywhere such that $k'r < 2m^{4/3}/(L_H^{\text{inf}})^2$.

One can also improve the long wavelength scheme by replacing in (14)–(17) the approximate cRW values by their exact values as given by (18) and taking for A_2, B_2, F_2 the solutions of Eqs. (7)–(9) where S, H, and F are taken to satisfy (18) instead of (4). The results are summarized in Figs. 1 and 2 and show that the improved and the standard schemes coalesce for large F_{inf}, that is, for strongly inflating solutions. This confirms what the previous section already indicated, that is, that the iteration scheme seems to converge nicely.

VI. CONCLUSIONS

An important question in inflationary cosmology is, how generic is it? As already shown by Goldwirth and Piran and confirmed here, inflation by itself requires a certain level of homogeneity: It can start if initial inhomogeneities are larger than the local Hubble radius. While the numerical calculations of [8] can explore in a detailed way a specific (spherically symmetric) case of strong initial inhomogeneity, the semianalytical approach presented here is limited to rather small perturbations, but it gives a better global picture on the factors that control the behavior of the system. It gives only sufficient conditions for the onset of inflation, not as strong as the necessary conditions obtained in [8], but they are general and do not assume any spatial symmetry. We plan to extend the comparison between these analytical and numerical approaches by solving an identical initial value problem. This requires further investigation of the various coordinates systems used.

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