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Closed-form expression for the momentum radiated from cosmic string loops

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We modify the recent analytic formula given by Allen and Casper for the rate at which piecewise linear cosmic string loops lose energy to gravitational radiation to yield the analogous analytic formula for the rate at which loops radiate momentum. The resulting formula (which is exact when the effects of gravitational back reaction are neglected) is a sum of $O(N^4)$ polynomial and log terms where N is the total number of segments on the piecewise linear string loop. As an illustration, we write the formula explicitly for a simple one-parameter family of loops with $N = 5$. For most loops the large number of terms makes evaluation "by hand" impractical, but a computer or symbolic manipulator may by used to yield accurate results. The formula has been used to correct numerical results given in the existing literature. To assist future work in this area, a small catalog of results for a number of simple string loops is provided.

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I. INTRODUCTION

Cosmic strings are one-dimensional topological defects that may have formed at phase transitions as the Universe expanded and cooled [1–4]. A cosmic string loop is formed when two sections of a long string (a string with length greater than the horizon length) meet and intercommute. After a loop is formed, it begins to oscillate under its own tension. As cosmic string loops oscillate, they lose energy in the form of gravitational radiation. The formation and subsequent decay of cosmic string loops is of fundamental importance to the evolution of the cosmic string network. In addition, most of the observational limits on cosmic strings are obtained by considering the effects of the gravitational radiation emitted as the loops decay ([4,5] and references therein).

The power emitted in gravitational radiation by a cosmic string loop depends upon its shape and velocity. If the loop configuration is asymmetric, then the energy may be radiated in an asymmetric way. In that case, the loop will radiate and lose momentum as well as energy. In this paper, we obtain an analytic formula for the momentum radiated for any piecewise linear cosmic string loop.

In the center-of-mass frame, a cosmic string loop is specified by the position $\mathbf{x}(t,\sigma)$ of the string as a function of two variables: time t and a spacelike parameter σ that runs from 0 to L. The total energy of the loop is μL where μ is the mass per unit length of the string. L is referred to as the "invariant length" of the loop. When gravitational back reaction is neglected, the string loop satisfies equations of motion whose most general solution in the center-of-mass frame is

$$\mathbf{x}(t,\sigma) = \frac{1}{2}\left[\mathbf{a}(t+\sigma) + \mathbf{b}(t-\sigma)\right]. \tag{1.1}$$

Here $\mathbf{a}(u) \equiv \mathbf{a}(u+L)$ and $\mathbf{b}(v) \equiv \mathbf{b}(v+L)$ are a pair of periodic functions, satisfying the "gauge condition" $|\mathbf{a}'(u)| = |\mathbf{b}'(v)| = 1$, where a prime denotes differentiation with respect to the function's argument. Because the functions $\mathbf{a}$ and $\mathbf{b}$ are periodic, each can be described by a closed loop. These loops are referred to, respectively, as the $\mathbf{a}$ loop and the $\mathbf{b}$ loop. Together, the $\mathbf{a}$ and $\mathbf{b}$ loops define the trajectory of the string loop.

If we define the four-momentum of the gravity waves emitted by a string loop to be $P^\alpha = (E, P^i)$ where $i = x,y,z$, then the average rate of energy and momentum loss by an oscillating string loop is given by the four-vector $-\dot{P}^\alpha$, where

$$\dot{P}^\alpha = (\dot{E}, \dot{P}^i) = \gamma^\alpha G\mu^2. \tag{1.2}$$

Here G is Newton's constant and we use units with $c = 1$ and metric signature $(-,+,+,+)$. Throughout this paper, a dot appearing over a symbol denotes the time derivative of that quantity. In Eq. (1.2), $\dot{E}$ is the energy radiated (i.e., the power) and $\dot{P}^i$ are the three spatial components of the momentum radiated, averaged over a single oscillation of the loop. With our definition of γ^α and metric signature, the string loop is losing energy in the form of gravitational radiation when γ^0 is positive (which is always the case). Note that in Ref. [6] the quantity γ^0 is denoted simply by γ. When one of the components of γ^i is positive, the loop is radiating a net amount of energy and momentum in that direction, and the loop itself will recoil and begin to accelerate in the opposite direction. Thus, if $\gamma^x > 0$, then the loop will be-

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gin to accelerate in the $-x$ direction. The dimensionless quantities $\gamma^\alpha = (\gamma^0, \gamma^i)$ depend only upon the "shape" of the cosmic string loop. That is, the energy and momentum radiated in gravitational radiation from a loop are invariant under a rescaling (magnification or shrinking) of the loop, provided that the velocity at each point on the rescaled loop is unchanged [3,7]. Thus, without loss of generality, we consider only loops with invariant length $L = 1$.

In a recent paper, we presented a new formula for γ^0. The formula is an exact analytic closed form for any piecewise linear cosmic string loop [6]. A piecewise linear loop is one which, at any time, is composed of straight segments, each of which has constant velocity. Equivalently, a piecewise linear loop is any loop for which the corresponding **a** and **b** loops are piecewise linear. As shown in Ref. [6], the piecewise linear requirement is not very restrictive since in practice a smooth cosmic string loop may be well approximated by a piecewise linear loop with a moderately small number of segments, N.

In the present paper, we show how the formula given in [6] may be modified to give an exact analytic closed form for the spatial momentum γ^i as well as for γ^0. The formula for the components of the momentum radiated is very similar to the formula for the radiated power. In each case, the formula is a sum of $O(N^4)$ terms, each of which involves nothing more complicated than logarithmic or arctangent functions.[1]

The remainder of the paper is organized as follows. Section II explains how the formulas of Ref. [6] may be modified to yield analytic, closed forms for the three spatial components of the radiated momentum. This section generalizes the work done in Sec. III of Ref. [6]. The final steps of the solution which are described in Secs. IV–VI of Ref. [6] are unchanged. Thus, those sections are not repeated in this paper. In Sec. III the resulting formula for both the radiated power and momentum is written explicitly for the case of a simple one-parameter family of string loops. The values of γ^α for this family of loops are compared to those given by an independent numerical method as well as to the results given by our C-code implementation of the general formulas. Excellent agreement is found in all cases. The formula is then used to correct the small number of numerical values for the momentum radiated which appear in the existing literature. These values are typically off by a factor of 2, though in some cases they are off by as much as a factor of 10. Section III is followed by a short conclusion. Finally, the Appendix contains a catalog of γ^α values for some simple loop trajectories.

II. MOMENTUM RADIATED BY COSMIC STRING LOOPS

In this section we derive a general formula for both the radiated energy ($\dot{E}$) and the radiated momentum ($\dot{P}^i$) for an arbitrary cosmic string loop. As in Ref. [6], we work in the weak-field limit. In this limit the gravitational back reaction on the loop is neglected so that the loop oscillates periodically in time. This derivation is almost identical to the derivation of the formula for the radiated energy given in Sec. III of Ref. [6]. The difference is that the present derivation yields the four-vector $\dot{P}^\alpha = (\dot{E}, \dot{P}^i)$. The total radiated momentum is simply given by $\dot{P} = (\dot{P}^i \dot{P}_i)^{1/2}$. We find that the three components of the radiated momentum may be obtained exactly for any piecewise linear loop by the same method used previously for the radiated energy.

The rate at which a loop loses four-momentum (averaged over a single oscillation) is [8,9]

$$\dot{P}^\alpha = \gamma^\alpha G\mu^2 = \sum_{n=0}^{\infty} \int d\Omega k^\alpha \frac{d\dot{E}_n}{d\Omega}, \tag{2.1}$$

where the constant four-vector $\gamma^\alpha = (\gamma^0, \gamma^i)$ depends only upon the shape of the string loop's trajectory. In Eq. (2.1), the four-vector k^α is defined to be $k^\alpha \equiv (1, \mathbf{\Omega})$ where

$$\mathbf{\Omega} \equiv (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.2}$$

is a unit spatial vector in the direction of the outgoing wave. The integral $\int d\Omega$ appearing in (2.1) denotes integration over angles on the two-sphere:

$$\int d\Omega \equiv \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi. \tag{2.3}$$

Because the loops oscillate with period $L/2 = 1/2$, they radiate only at discrete angular frequencies given by

$$\omega_n = 4\pi n \quad \text{for } n = 1, 2, 3, \dots . \tag{2.4}$$

The energy radiated per unit solid angle into the nth mode is given by

$$\frac{d\dot{E}_n}{d\Omega} = \frac{G}{\pi}\omega_n^2[\tau^*_{\mu\nu}(\omega_n\mathbf{\Omega})\tau^{\mu\nu}(\omega_n\mathbf{\Omega}) - \tfrac{1}{2}|\tau^\mu{}_\mu(\omega_n\mathbf{\Omega})|^2]. \tag{2.5}$$

The Fourier transform of the stress tensor for a string loop is

$$\tau_{\mu\nu}(\omega_n\mathbf{\Omega}) = 2\mu \int_0^1 du \int_0^1 dv\, G_{\mu\nu}(u,v) \times e^{i\omega_n\{u+v-\mathbf{\Omega}\cdot[\mathbf{a}(u)+\mathbf{b}(v)]\}/2}, \tag{2.6}$$

where

$$G^{\mu\nu}(u,v) = \partial_u x^\mu \partial_v x^\nu + \partial_v x^\mu \partial_u x^\nu \tag{2.7}$$

[1] Our C code, which provides one implementation of the formulas, is publicly available via anonymous FTP from the directory pub/pcasper at the internet site alpha1.csd.uwm.edu.

and $x^\mu = (t, \mathbf{x}(t,\sigma))$. Combining Eqs. (2.1), (2.5), and (2.6), we find that

$$\dot{P}^\alpha = \frac{2G\mu^2}{\pi} \sum_{n=-\infty}^{\infty} \omega_n^2 \int d\Omega \int_0^1 du \int_0^1 dv \int_0^1 d\tilde{u} \int_0^1 d\tilde{v}\, \psi(u,v,\tilde{u},\tilde{v}) k^\alpha e^{i\omega_n[\Delta t(u,v,\tilde{u},\tilde{v}) - \mathbf{\Omega}\cdot\Delta\mathbf{x}(u,v,\tilde{u},\tilde{v})]} , \tag{2.8}$$

where we have defined

$$\begin{aligned}\psi(u,v,\tilde{u},\tilde{v}) &= G_{\mu\nu}(u,v)G^{\mu\nu}(\tilde{u},\tilde{v}) - \frac{1}{2}G^\mu{}_\mu(u,v)G^\nu{}_\nu(\tilde{u},\tilde{v}) \\ &= \frac{1}{8}\{[\mathbf{a}'(u)\cdot\mathbf{a}'(\tilde{u}) - 1][\mathbf{b}'(v)\cdot\mathbf{b}'(\tilde{v}) - 1] + [\mathbf{a}'(u)\cdot\mathbf{b}'(\tilde{v}) - 1][\mathbf{b}'(v)\cdot\mathbf{a}'(\tilde{u}) - 1] \\ &\qquad - [\mathbf{a}'(u)\cdot\mathbf{b}'(v) - 1][\mathbf{a}'(\tilde{u})\cdot\mathbf{b}'(\tilde{v}) - 1]\}. \end{aligned} \tag{2.9}$$

The functions $\Delta t = (u + v - \tilde{u} - \tilde{v})/2$ and $\Delta\mathbf{x} = [\mathbf{a}(u) + \mathbf{b}(v) - \mathbf{a}(\tilde{u}) - \mathbf{b}(\tilde{v})]/2$ in (2.8) describe the temporal and spatial separation of the two points on the string world sheet with coordinates (u,v) and $(\tilde{u},\tilde{v})$, respectively. To save space, in some of the formulas that follow, the arguments of Δt and $\Delta\mathbf{x}$ are not shown. Since each term in the sum over n equals its complex conjugate, as may be shown by redefining $(u,v) \rightleftharpoons (\tilde{u},\tilde{v})$, $\dot{P}^\alpha$ is explicitly real. For this reason we have changed the sum over n to a sum from $-\infty$ to ∞ at the expense of introducing an overall factor of 1/2 into (2.8). From here on, this sum will simply be denoted by $\sum_n$. Note that the timelike $\alpha = 0$ component of (2.8) is identical to the formula for the radiated energy given in Eq. (3.8) of Ref. [6].

It is possible to carry out both the sum over n and the integral over the solid angle in (2.8) in closed form. This is done by making use of the identity

$$\sum_n \omega_n \int d\Omega e^{i\omega_n(\Delta t - \Omega\cdot\Delta\mathbf{x})} = 2\pi i \sum_{k=-\infty}^{\infty} \epsilon(\Delta t + k/2)\delta((\Delta t + k/2)^2 - |\Delta\mathbf{x}|^2), \tag{2.10}$$

where $\epsilon(x) = 2\theta(x) - 1$ is $+1$ for $x > 0$ and -1 for $x < 0$ and δ is the Dirac delta function. We define a four-vector linear differential operator

$$D^\alpha(u,v,\tilde{u},\tilde{v}) \equiv U^\alpha(u,v,\tilde{u},\tilde{v})\partial_u + V^\alpha(u,v,\tilde{u},\tilde{v})\partial_v - \tilde{U}^\alpha(u,v,\tilde{u},\tilde{v})\partial_{\tilde{u}} - \tilde{V}^\alpha(u,v,\tilde{u},\tilde{v})\partial_{\tilde{v}}, \tag{2.11}$$

where the vector functions $U^\alpha, V^\alpha, \tilde{U}^\alpha$, and $\tilde{V}^\alpha$ are defined by the effect of D^α on the exponential:

$$D^\alpha \exp(i\omega_n[\Delta t - \mathbf{\Omega}\cdot\Delta\mathbf{x}]) = i\omega_n k^\alpha \exp(i\omega_n[\Delta t - \mathbf{\Omega}\cdot\Delta\mathbf{x}]). \tag{2.12}$$

Because D^α is chosen to be a linear differential operator, (2.12) is equivalent to the 16 equations

$$\begin{aligned} D^0\Delta t(u,v,\tilde{u},\tilde{v}) &= 1, \qquad & D^0\Delta x_j(u,v,\tilde{u},\tilde{v}) &= 0, \\ D^i\Delta t(u,v,\tilde{u},\tilde{v}) &= \mathbf{0}, \qquad & D^i\Delta x_j(u,v,\tilde{u},\tilde{v}) &= -\delta^i{}_j, \end{aligned} \tag{2.13}$$

where $\delta^i{}_j$ is the Kronecker delta function. With this definition of the operator D^α, the identity (2.10) can be used to express the radiated energy and momentum (2.8) as

$$\dot{P}^\alpha = 4G\mu^2 \sum_{k=-\infty}^{\infty} \int_0^1 du \int_0^1 dv \int_0^1 d\tilde{u} \int_0^1 d\tilde{v}\, \psi(u,v,\tilde{u},\tilde{v}) D^\alpha(u,v,\tilde{u},\tilde{v})[\epsilon(\Delta t + k/2)\delta((\Delta t + k/2)^2 - |\Delta\mathbf{x}|^2)]. \tag{2.14}$$

This equation has the same functional form as Eq. (3.15) of Ref. [6]. By the exact same arguments as those given following (3.15) in [6], we find that Eq. (2.14) may be written

$$\dot{P}^\alpha = 8G\mu^2 \int_0^1 du \int_0^1 dv \int_0^1 d\tilde{u} \int_{-2}^2 d\tilde{v}\, \psi(u,v,\tilde{u},\tilde{v}) D^\alpha[\theta(\Delta t)\delta((\Delta t)^2 - |\Delta\mathbf{x}|^2)]. \tag{2.15}$$

All that remains is to write down the explicit expression for the differential operator D^α.

The operator D^α is determined by the equations given in (2.13). These equations may be written in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -\mathbf{a}'(u) & -\mathbf{b}'(v) & -\mathbf{a}'(\tilde{u}) & -\mathbf{b}'(\tilde{v}) \end{pmatrix} \begin{pmatrix} U^\alpha \\ V^\alpha \\ \tilde{U}^\alpha \\ \tilde{V}^\alpha \end{pmatrix} = 2I , \tag{2.16}$$

where I denotes the identity matrix. Equation (2.16) may be solved to find U^α, V^α, $\tilde{U}^\alpha$, and $\tilde{V}^\alpha$ (and therefore D^α) explicitly. If we denote minus twice the inverse determinant of the prefactor matrix by

$$Q(u,v,\tilde{u},\tilde{v}) = 2[\mathbf{b}'(v)\cdot\mathbf{a}'(\tilde{u})\times\mathbf{b}'(\tilde{v}) - \mathbf{a}'(u)\cdot\mathbf{a}'(\tilde{u})\times\mathbf{b}'(\tilde{v}) + \mathbf{a}'(u)\cdot\mathbf{b}'(v)\times\mathbf{b}'(\tilde{v}) - \mathbf{a}'(u)\cdot\mathbf{b}'(v)\times\mathbf{a}'(\tilde{u})]^{-1}, \quad (2.17)$$

then we find that the coefficients of the partial derivatives in D^0 are

$$\begin{aligned} U^0 &= \mathbf{b}'(v)\cdot\mathbf{a}'(\tilde{u})\times\mathbf{b}'(\tilde{v})\,Q, & V^0 &= -\mathbf{a}'(u)\cdot\mathbf{a}'(\tilde{u})\times\mathbf{b}'(\tilde{v})\,Q,\\ \tilde{U}^0 &= \mathbf{a}'(u)\cdot\mathbf{b}'(v)\times\mathbf{b}'(\tilde{v})\,Q, & \tilde{V}^0 &= -\mathbf{a}'(u)\cdot\mathbf{b}'(v)\times\mathbf{a}'(\tilde{u})\,Q, \end{aligned} \quad (2.18)$$

which is exactly the result found in Ref. [6]. In addition, the coefficients which define the operators D^i are given by

$$\begin{aligned} U^i &= [\mathbf{b}'(v)\times\mathbf{a}'(\tilde{u}) + \mathbf{b}'(\tilde{v})\times\mathbf{b}'(v) + \mathbf{a}'(\tilde{u})\times\mathbf{b}'(\tilde{v})]^i\,Q,\\ V^i &= [\mathbf{b}'(v)\times\mathbf{a}'(u) + \mathbf{a}'(u)\times\mathbf{b}'(\tilde{v}) + \mathbf{b}'(\tilde{v})\times\mathbf{a}'(\tilde{u})]^i\,Q,\\ \tilde{U}^i &= [\mathbf{a}'(u)\times\mathbf{b}'(v) + \mathbf{b}'(v)\times\mathbf{b}'(\tilde{v}) + \mathbf{b}'(\tilde{v})\times\mathbf{a}'(u)]^i\,Q,\\ \tilde{V}^i &= [\mathbf{b}'(v)\times\mathbf{a}'(u) + \mathbf{a}'(u)\times\mathbf{a}'(\tilde{u}) + \mathbf{a}'(\tilde{u})\times\mathbf{b}'(v)]^i\,Q. \end{aligned} \quad (2.19)$$

Equation (2.15) may now be expressed in closed form for any piecewise linear loop by exactly the same method as given in Secs. IV–VI of [6]. The *only* difference between the calculation of the radiated energy and the calculation of a component of the momentum radiated is in the choice of the coefficients of the differential operator D^α.

III. RESULTS

The formula (2.15) for $\dot{P}^\alpha$ may be evaluated exactly for any piecewise linear cosmic string loop. The general solution is given in Secs. IV–VI of Ref. [6]. In most cases the large number of terms involved makes it impractical to write out the solution explicitly; however, there are cases where the solution may be written in a manageable form. As an example, in Sec. III A we give the closed-form solution for a one-parameter family of cosmic string loops obtained with the aid of MATHEMATICA. In the cases where it is impractical to write out the closed-form solution, accurate values of γ^α may still be obtained using a computer implementation of the general formula. In Sec. III B one such implementation is used to correct the small number of numerical values for the radiated momentum which appear in the existing literature.

A. Analytic results

We will now give closed forms for $\dot{P}^\alpha$ for a one-parameter family of string loops. These are defined by **a** and **b** loops consisting of two and three segments, respectively. The **a** loop is taken to lie along the z axis. One kink on the **a** loop is positioned at the origin; the parameter $u = 0$ at this kink. The other kink (at $u = 1/2$) is positioned above the first kink and has coordinates $(0, 0, 1/2)$. The three-segment **b** loop has the shape of an equilateral triangle. For the **b** loop, we again position one kink at the origin and set the parameter $v = 0$ at that kink. The position of the other two kinks depends on a parameter ϕ. The kink at $v = 1/3$ has coordinates $-\frac{1}{6}(\cos\phi, \sqrt{3}, \sin\phi)$ and the kink at $v = 2/3$ has coordinates $\frac{1}{6}(\cos\phi, -\sqrt{3}, \sin\phi)$. If we make the definition $s \equiv \sin\phi$, then the rate of energy loss by this set of string loops may be written as

$$\begin{aligned} \dot{E}(s) = \frac{16G\mu^2}{(1-s^2)(4-s^2)^2}[&12(1-s^2)(4+s^2)\ln(2) + 18(4-3s^2)\ln(3) - (1-s^2)(2-s)^2(1+2s)\ln(1-s)\\ &-(1-s^2)(2+s)^2(1-2s)\ln(1+s) - (1-s)(4-s^2)^2\ln(2-s)\\ &-(1+s)(4-s^2)^2\ln(2+s) - (1-s^2)(2-s)^2(4-s)\ln(4-s) - (1-s^2)(2+s)^2(4+s)\ln(4+s)]. \end{aligned} \quad (3.1)$$

The x and y components of the momentum radiated by these loops both vanish. The z component is given by

$$\begin{aligned} \dot{P}^z(s) = \frac{16G\mu^2}{(1-s^2)(4-s^2)^2}[&-48s(1-s^2)\ln(2) + 18s^3\ln(3) - (1-s^2)(2-s)^2(1+2s)\ln(1-s)\\ &+(1-s^2)(2+s)^2(1-2s)\ln(1+s) + (1-s)(4-s^2)^2\ln(2-s)\\ &-(1+s)(4-s^2)^2\ln(2+s) - (1-s^2)(2-s)^2(4-s)\ln(4-s) + (1-s^2)(2+s)^2(4+s)\ln(4+s)]. \end{aligned} \quad (3.2)$$

Note that $\dot{E}(s)$ and $\dot{P}(s)$ are even and odd functions of s, respectively, as they must be. The exact results given by (3.1) and (3.2) have been compared to those given by our C-code implementation of the general formula as well as to the independent numerical results given by the fast Fourier transform method of Allen and Shellard [7]. The different sets of results are compared in Fig. 1. We find excellent agreement between all three sets of results.

B. Corrected results

In this section we compare the results for the radiated momentum given by other authors in the previous literature to the values given by our formula. The only family of loop trajectories for which numerical values of the radiated momentum have been previously published is a two-parameter family of loops first studied by Vachaspati and Vilenkin [8]. The **a** and **b** loops which define these trajectories are given by

$$\mathbf{a}(u) = \frac{1}{2\pi}[\sin(2\pi u)\hat{\mathbf{x}} - \cos(2\pi u)(\cos\phi\hat{\mathbf{y}} + \sin\phi\hat{\mathbf{z}})],$$

$$\mathbf{b}(v) = \frac{1}{2\pi}\Bigg[\left(\frac{\alpha}{3}\sin(6\pi v) - (1-\alpha)\sin(2\pi v)\right)\hat{\mathbf{x}} - \left(\frac{\alpha}{3}\cos(6\pi v) + (1-\alpha)\cos(2\pi v)\right)\hat{\mathbf{y}} - [\alpha(1-\alpha)]^{1/2}\sin(4\pi v)\hat{\mathbf{z}}\Bigg], \quad (3.3)$$

where α and ϕ are constant parameters, $0 \le \alpha \le 1$ and $-\pi < \phi \le \pi$. These loops have also been studied by Durrer [9]. In both cases, the authors determined $\dot{P}^\alpha$ by numerically evaluating a finite number of terms of the infinite sum appearing in (2.1), and then adding an estimate of the contribution from the truncated infinite "tail." Although formally correct, it is difficult to obtain highly accurate numerical results by this method. It is shown in Ref. [6] that the errors in the numerical values of γ^0 calculated by this method are typically due to errors in the estimates of the contribution from the tail. This appears to also be the case for the values of $\dot{P}^i$ calculated by this method, which are typically off by a factor of 2. The results found by Vachaspati and Vilenkin, and Durrer, are shown in Figs. 2 and 3 along with the results of our new method for the cases $\alpha = 0.5$ and $\alpha = 0.8$. The results given by our formula are found by evaluating γ^α for a piecewise linear loop with approximately the same shape as the smooth loops given by (3.3). The approximation becomes more accurate as the number of piecewise linear segments used is increased. Although there is no general way known to calculate the exact error when approximating smooth loops by piecewise linear loops, it appears that the error typically falls off like $1/N$ (see Ref. [6]). The piecewise linear loops whose γ^α values were used in Figs. 2 and 3 had 100 segments for both the **a** and **b** loops. With this number of segments, the error due to the piecewise linear approximation is estimated to be no more than 5%.

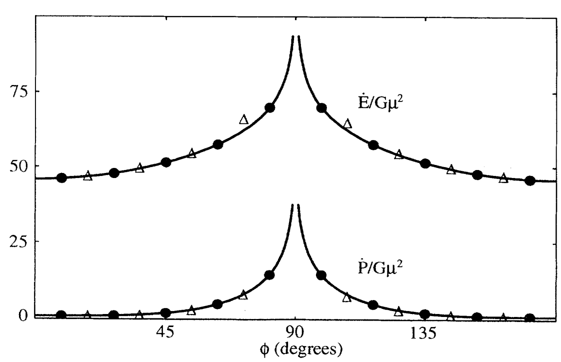

FIG. 1. The power ($\dot{E}$) and radiated momentum ($\dot{P}$) for a simple one-parameter family of loop trajectories given in Sec. III A. The solid lines are the exact results given in (3.1) and (3.2). The crosses are the results given by the C-code implementation of the general formula and the triangles are the results of the numerical fast Fourier transform method. There is excellent agreement between all three sets of results.

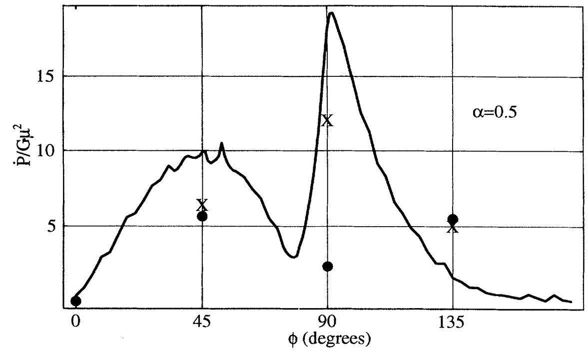

FIG. 2. Values of $\dot{P} = (\dot{P}^i\dot{P}_i)^{1/2}$ are shown (in units of $G\mu^2$) for the loop trajectories given in (3.3) with $\alpha = 0.5$. The results of our formula are given every few degrees and are connected by straight segments to form the solid line. Durrer's results are shown as dots while the results of Vachaspati and Vilenkin are shown as crosses.

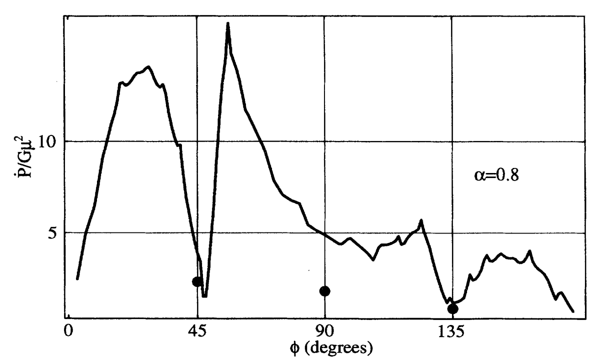

FIG. 3. Values of $\dot{P} = (\dot{P}^i\dot{P}_i)^{1/2}$ are shown (in units of $G\mu^2$) for the loop trajectories given in (3.3) with $\alpha = 0.8$. The results of our formula are given every few degrees and are connected by straight segments to form the solid line. Durrer's results are shown as dots.

IV. CONCLUSION

We have modified the method of Allen and Casper [6] to yield analytic closed-form results for the linear momentum radiated by piecewise linear cosmic string loops. Any cosmic string loop can be arbitrarily well approximated by a piecewise linear loop with the number of segments sufficiently large. An exact formula is given for a simple one-parameter family of string loops. Our computer implementation of the general formula is then used to investigate the small number of numerical results published in the previous literature. These results are found to be typically off by a factor of 2 from the correct results, though in some cases they are off by as much as a factor of 10. A small catalog of γ^α values for some simple loop trajectories has been provided in the Appendix as a set of benchmark results for future analytic or numerical work. Although the string loops studied in this paper are not physically realistic, they provide a simple set of trajectories with which to test our formula. We intend to use the method of this paper to investigate a large sample of more physically realistic loop trajectories in the near future.

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APPENDIX

This appendix gives a short catalog of γ^α values for some simple loop trajectories. This catalog expands the catalog given in Sec. VIII of Ref. [6] by including both the radiated energy and the three components of the radiated momentum. The values given in this catalog are intended (as in the original catalog) to be "benchmark" values which might prove useful in testing future analytic or numerical methods. In fact, the original catalog has already proven to be quite useful. The values of γ^α are shown as column vectors of the form

$$\gamma^\alpha = \begin{pmatrix} \gamma^0 \\ \gamma^x \\ \gamma^y \\ \gamma^z \end{pmatrix} . \tag{A1}$$

TABLE I. The γ^α values (shown as column vectors) for the first two-parameter family of string loops.

θ/ϕ	18°	36°	54°	72°
18°	$\begin{pmatrix} 59.80 \\ 0.00 \\ 0.00 \\ -1.90 \end{pmatrix}$	$\begin{pmatrix} 61.21 \\ 0.00 \\ 0.00 \\ -4.05 \end{pmatrix}$	$\begin{pmatrix} 63.51 \\ 0.00 \\ 0.00 \\ -6.62 \end{pmatrix}$	$\begin{pmatrix} 66.30 \\ 0.00 \\ 0.00 \\ -9.45 \end{pmatrix}$
36°	$\begin{pmatrix} 54.56 \\ 0.00 \\ 0.00 \\ -1.53 \end{pmatrix}$	$\begin{pmatrix} 56.86 \\ 0.00 \\ 0.00 \\ -3.47 \end{pmatrix}$	$\begin{pmatrix} 60.93 \\ 0.00 \\ 0.00 \\ -6.46 \end{pmatrix}$	$\begin{pmatrix} 67.45 \\ 0.00 \\ 0.00 \\ -11.79 \end{pmatrix}$
54°	$\begin{pmatrix} 50.15 \\ 0.00 \\ 0.00 \\ -0.67 \end{pmatrix}$	$\begin{pmatrix} 52.56 \\ 0.00 \\ 0.00 \\ -1.37 \end{pmatrix}$	$\begin{pmatrix} 56.40 \\ 0.00 \\ 0.00 \\ -2.09 \end{pmatrix}$	$\begin{pmatrix} 60.72 \\ 0.00 \\ 0.00 \\ -2.74 \end{pmatrix}$
72°	$\begin{pmatrix} 47.54 \\ 0.00 \\ 0.00 \\ -0.12 \end{pmatrix}$	$\begin{pmatrix} 50.12 \\ 0.00 \\ 0.00 \\ 0.04 \end{pmatrix}$	$\begin{pmatrix} 54.47 \\ 0.00 \\ 0.00 \\ 1.02 \end{pmatrix}$	$\begin{pmatrix} 60.36 \\ 0.00 \\ 0.00 \\ 3.70 \end{pmatrix}$

TABLE II. The γ^α values (shown as column vectors) for the second two-parameter family of string loops.

θ/ϕ	18°	36°	54°	72°
18°	$\begin{pmatrix} 100.85 \\ -2.93 \\ 0.88 \\ -8.77 \end{pmatrix}$	$\begin{pmatrix} 90.65 \\ -6.34 \\ -0.34 \\ -12.02 \end{pmatrix}$	$\begin{pmatrix} 74.48 \\ -3.17 \\ -0.81 \\ -6.40 \end{pmatrix}$	$\begin{pmatrix} 64.80 \\ -1.55 \\ -0.64 \\ -3.72 \end{pmatrix}$
36°	$\begin{pmatrix} 82.00 \\ 3.06 \\ 1.17 \\ -2.01 \end{pmatrix}$	$\begin{pmatrix} 76.01 \\ -1.79 \\ 0.88 \\ -5.24 \end{pmatrix}$	$\begin{pmatrix} 70.97 \\ -2.46 \\ 0.27 \\ -6.31 \end{pmatrix}$	$\begin{pmatrix} 64.82 \\ -1.42 \\ -0.03 \\ -4.58 \end{pmatrix}$
54°	$\begin{pmatrix} 72.61 \\ 5.16 \\ 0.93 \\ 0.06 \end{pmatrix}$	$\begin{pmatrix} 66.51 \\ 0.44 \\ 0.83 \\ -1.45 \end{pmatrix}$	$\begin{pmatrix} 63.68 \\ -0.77 \\ 0.54 \\ -2.56 \end{pmatrix}$	$\begin{pmatrix} 61.54 \\ -0.64 \\ 0.33 \\ -2.28 \end{pmatrix}$
72°	$\begin{pmatrix} 67.04 \\ 6.10 \\ 0.49 \\ 0.91 \end{pmatrix}$	$\begin{pmatrix} 61.24 \\ 1.38 \\ 0.46 \\ 0.08 \end{pmatrix}$	$\begin{pmatrix} 59.12 \\ -0.06 \\ 0.38 \\ -0.59 \end{pmatrix}$	$\begin{pmatrix} 58.80 \\ -0.26 \\ 0.30 \\ -0.27 \end{pmatrix}$

TABLE III. The γ^α values (shown as column vectors) for the third two-parameter family of string loops.

θ/ϕ	18°	36°	54°	72°
18°	$\begin{pmatrix} 64.15 \\ 0.00 \\ 0.00 \\ 1.00 \end{pmatrix}$	$\begin{pmatrix} 66.04 \\ 0.00 \\ 0.00 \\ 2.30 \end{pmatrix}$	$\begin{pmatrix} 69.28 \\ 0.00 \\ 0.00 \\ 4.37 \end{pmatrix}$	$\begin{pmatrix} 74.52 \\ 0.00 \\ 0.00 \\ 8.45 \end{pmatrix}$
36°	$\begin{pmatrix} 54.99 \\ 0.00 \\ 0.00 \\ 0.27 \end{pmatrix}$	$\begin{pmatrix} 57.78 \\ 0.00 \\ 0.00 \\ 0.54 \end{pmatrix}$	$\begin{pmatrix} 62.31 \\ 0.00 \\ 0.00 \\ 0.70 \end{pmatrix}$	$\begin{pmatrix} 67.63 \\ 0.00 \\ 0.00 \\ 0.44 \end{pmatrix}$
54°	$\begin{pmatrix} 48.74 \\ 0.00 \\ 0.00 \\ 0.03 \end{pmatrix}$	$\begin{pmatrix} 52.02 \\ 0.00 \\ 0.00 \\ -0.03 \end{pmatrix}$	$\begin{pmatrix} 57.78 \\ 0.00 \\ 0.00 \\ -0.53 \end{pmatrix}$	$\begin{pmatrix} 66.36 \\ 0.00 \\ 0.00 \\ -2.94 \end{pmatrix}$
72°	$\begin{pmatrix} 45.30 \\ 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}$	$\begin{pmatrix} 48.74 \\ 0.00 \\ 0.00 \\ -0.03 \end{pmatrix}$	$\begin{pmatrix} 54.98 \\ 0.00 \\ 0.00 \\ -0.10 \end{pmatrix}$	$\begin{pmatrix} 64.04 \\ 0.00 \\ 0.00 \\ -0.16 \end{pmatrix}$

TABLE IV. The γ^α values (shown as column vectors) for the fourth two-parameter family of string loops.

θ/ϕ	18°	36°	54°	72°
18°	$\begin{pmatrix}84.69\\1.98\\0.76\\-1.27\end{pmatrix}$	$\begin{pmatrix}75.43\\-0.84\\0.13\\-2.31\end{pmatrix}$	$\begin{pmatrix}67.82\\-1.21\\-0.14\\-2.88\end{pmatrix}$	$\begin{pmatrix}62.44\\-0.80\\-0.29\\-3.47\end{pmatrix}$
36°	$\begin{pmatrix}77.37\\3.40\\0.97\\-0.52\end{pmatrix}$	$\begin{pmatrix}71.03\\0.28\\0.39\\-1.97\end{pmatrix}$	$\begin{pmatrix}65.71\\-0.55\\0.23\\-2.93\end{pmatrix}$	$\begin{pmatrix}61.86\\-0.63\\0.28\\-3.92\end{pmatrix}$
54°	$\begin{pmatrix}70.11\\4.76\\0.77\\0.93\end{pmatrix}$	$\begin{pmatrix}65.69\\1.68\\0.54\\0.12\end{pmatrix}$	$\begin{pmatrix}62.72\\0.69\\0.72\\-0.38\end{pmatrix}$	$\begin{pmatrix}60.42\\0.06\\0.82\\-0.84\end{pmatrix}$
72°	$\begin{pmatrix}64.41\\5.50\\0.42\\1.81\end{pmatrix}$	$\begin{pmatrix}61.37\\2.74\\0.26\\1.74\end{pmatrix}$	$\begin{pmatrix}61.13\\2.35\\0.65\\2.93\end{pmatrix}$	$\begin{pmatrix}61.33\\1.41\\1.25\\4.46\end{pmatrix}$

TABLE V. The γ^α values (shown as column vectors) for the fifth two-parameter family of string loops.

θ/ϕ	18°	36°	54°	72°
18°	$\begin{pmatrix}114.46\\-4.06\\-0.32\\-2.21\end{pmatrix}$	$\begin{pmatrix}94.04\\0.72\\-0.32\\4.07\end{pmatrix}$	$\begin{pmatrix}80.22\\1.21\\0.03\\5.06\end{pmatrix}$	$\begin{pmatrix}68.84\\0.61\\0.08\\3.32\end{pmatrix}$
36°	$\begin{pmatrix}93.52\\-2.28\\0.37\\-2.47\end{pmatrix}$	$\begin{pmatrix}82.49\\-1.88\\-0.32\\-1.40\end{pmatrix}$	$\begin{pmatrix}72.06\\-0.11\\-0.20\\0.70\end{pmatrix}$	$\begin{pmatrix}65.40\\0.14\\-0.09\\1.04\end{pmatrix}$
54°	$\begin{pmatrix}77.15\\0.47\\0.36\\-0.55\end{pmatrix}$	$\begin{pmatrix}72.94\\-1.48\\0.03\\-1.56\end{pmatrix}$	$\begin{pmatrix}67.22\\-0.34\\-0.10\\-0.44\end{pmatrix}$	$\begin{pmatrix}62.74\\0.04\\0.02\\-0.06\end{pmatrix}$
72°	$\begin{pmatrix}67.11\\1.50\\0.15\\0.18\end{pmatrix}$	$\begin{pmatrix}64.76\\-0.40\\0.11\\-0.41\end{pmatrix}$	$\begin{pmatrix}64.05\\0.30\\0.06\\0.55\end{pmatrix}$	$\begin{pmatrix}62.98\\0.58\\0.26\\1.89\end{pmatrix}$

The trajectories chosen for this catalog are the same five two-parameter families of loops used in the original catalog. The **a** and **b** loops for these trajectories are piecewise linear loops composed of a small number of segments. The relative orientation of the **a** and **b** loops is defined by two angles ϕ and θ. For a full description of these trajectories, the reader is referred to Ref. [6].

The first set of trajectories we consider are defined by **a** and **b** loops consisting of two and three segments, respectively. The three-segment **b** loop is an equilateral triangle. Values of γ^α for the trajectories defined by these **a** and **b** loops are given in Table I for several values of the angles ϕ and θ. It should be noted that the one-parameter family of loops obtained by setting $\theta = 90°$ is equivalent to the family of loops for which the analytic formulas (3.1) and (3.2) are given in Sec. III.

The second set of trajectories we consider are defined by **a** and **b** loops which are both equilateral triangles. Values of γ^α for this family of loop trajectories are given in Table II for several values of the angles ϕ and θ.

The third set of trajectories we consider are defined by **a** and **b** loops consisting of two and five segments, respectively. The five-segment **b** loop is a pentagon. Values of γ^α for the trajectories defined by these **a** and **b** loops are given in Table III for several values of the angles ϕ and θ.

The fourth set of trajectories we consider are defined by **a** and **b** loops consisting of five and three segments, respectively. The five-segment **a** loop is a pentagon and the three-segment **b** loop is an equilateral triangle. Values of γ^α for the trajectories defined by these **a** and **b** loops are given in Table IV for several values of the angles ϕ and θ.

The final set of trajectories we consider are defined by **a** and **b** loops consisting of five segments each. Both loops are pentagons. Values of γ^α for the trajectories defined by these **a** and **b** loops are given in Table V for several values of the angles ϕ and θ.

The γ^α values given in this appendix should provide a convenient reference against which any future numerical or analytical methods may be tested. Additional tests are provided by comparison to the analytic results given in Sec. III and to the large number of simple analytical results (for $\dot{E}$) given in a recent paper by the authors [10]. Finally, comparisons may be made, for any loop, to the results given by the computer implementation of our general formula.

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