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Limits on anisotropy and inhomogeneity from the cosmic background radiation

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We consider directly the equations by which matter imposes anisotropies on freely propagating background radiation, leading to a new way of using anisotropy measurements to limit the deviations of the Universe from a Friedmann-Robertson-Walker (FRW) geometry. This approach is complementary to the usual Sachs-Wolfe approach: the limits obtained are not as detailed, but they are more model independent. We also give new results about combined matter-radiation perturbations in an almost-FRW universe, and a new exact solution of the linearized equations.

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I. INTRODUCTION

The cosmic background radiation (CBR) is the keystone of modern cosmological analysis, in particular through use of the results of the Cosmic Background Explorer (COBE) [1,2] and other [3,4] measurements of anisotropies in the CBR to help us understand the nature of inhomogeneities in the Universe (see [5] for a review of these observations). This paper presents a way of analyzing this relationship that is an alternative to the usual analyses based largely on the Sachs-Wolfe effect (modified by astrophysical effects). Our analysis proceeds from slightly different assumptions than those usually made (though essentially compatible with them); it is to a considerable degree more model independent than they are.

In [6] we established a theoretical framework for investigating the direct implications of CBR anisotropies (i.e., those that follow without assuming particular inflationary or other evolutionary models for the Universe). In that paper, we set up fully covariant and gauge invariant evolution and constraint equations governing the perturbations of the photon distribution and metric. These equations were then used to show that if **(A1)** all fundamental observers measure the CBR to be almost isotropic in some domain, then it follows that **(A2)** the spacetime geometry is almost Friedmann-Robertson-Walker (FRW) type; i.e., the shear, vorticity, spatial gradients, and Weyl tensor are almost zero, and the metric may be put into perturbed-FRW form in that domain. The latter is the assumption which underlies the usual Sachs-Wolfe analyses, for it is the starting point used to set up the Sachs-Wolfe equations.

In this paper, we use the formalism of [6], extending its results to find quantitative limits on the anisotropy and inhomogeneity of spacetime set by anisotropies in the CBR, *without* assuming a particular model for the origin of such perturbations. We end up with a series of estimates (Sec. IV) relating the inhomogeneity and anisotropy of the Universe directly to the background radiation anisotropies. In addition, we find some new exact results on perturbations in almost-FRW universes with both matter and radiation (Secs. II and V). For a class of almost flat FRW universes, we reduce the full set of linearized dynamical equations to two linear ordinary differential equations (ODE's), and give a new exact solution at late times (Sec. V).

The difference from the more usual analyses is that they consider observations of the CBR made at one spacetime event P ("here and now"), relating them to assumed perturbations of the Universe at the time of decoupling, these in turn taken to arise from some particular evolutionary history or other. Here we make no such evolutionary assumptions; however [cf. (A1)], we assume that the nature of CBR anisotropies is known not only at our own spacetime position P, but also throughout an open domain containing that event (eventually chosen to represent the period between last scattering and the present, in a region containing our world line and past light cone).

At first glance this seems to make our analysis far more dependent on unobservable data than the standard approach. However, this is an illusion, for that approach builds in equivalent assumptions at the outset, but in a rather hidden way, because it assumes (A2), which cannot be *proved* on the basis of observational evidence alone [7,8]. It can only be *deduced* from such data on the basis of some kind of Copernican assumption such as (A1), which is *not* directly testable [6–8]. Our approach helps make clear precisely what these hidden assumptions are, and thereby enables one to test how weak they can be and still allow deduction of the desired results.

In the covariant approach we work in the real non-FRW spacetime, rather than starting from an assumed FRW

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model and perturbing away from it. The former approach considers the full set of dynamical equations that govern the real variables. The latter approach risks missing certain effects [9,10] or masking underlying assumptions. We provide an example of the second kind by showing that the Boltzmann equation imposes constraints on the photon distribution if the monopole moment is assumed to be Planckian to first order. The covariant formalism for analyzing fluid inhomogeneities [9,11,12] is a development of Hawking's approach [13], and gives a gauge-invariant alternative to Bardeen's formalism [14]. The covariant approach to the photon distribution function in this paper and [6] is based on [15,16], and is an alternative to the application of a Bardeen-type formalism, as presented in [17,18] (in different contexts from that of this paper).

Despite the success of the standard inflationary models with dark matter and critical density [19,20], current CBR observations are consistent with alternative models, and do not by themselves give independent tests of inflation [21,22]. Furthermore, future observations could produce problems for the standard models. The covariant approach provides a clear and direct relation between observational and theoretical quantities, unobscured by particular gauge choices or by the complexities of harmonically determined variables. Furthermore, we do not impose any specific model to generate fluctuations in the CBR. Thus we investigate, as far as possible, what is determined directly by observations of the CBR made by the family of fundamental observers. Where we are forced to make additional assumptions, they are made about observational quantities, and are thus in principle falsifiable by observation, provided we make some kind of Copernican assumption such as (A1), stating that all fundamental observers see the same kind of thing (the nature of the required assumptions is clarified below). In this sense, we provide a framework for comparing and testing various models, in which there is as clear as possible a distinction between observed and assumed properties.

Notation. The metric g_{ab} has the signature $(-,+,+,+)$. Einstein's gravitational constant, the speed of light in vacuum, and Planck's constant are 1. Parentheses on indices denote symmetrization, square brackets antisymmetrization. ∇_a is the covariant derivative defined by g_{ab}. Given a four-velocity u^a, the associated projection tensor is $h_{ab} = g_{ab} + u_a u_b$, and the comoving time derivative and spatial gradient are

$$\dot{Q}_{a\cdots b} \equiv u^c \nabla_c Q_{a\cdots b}\,,$$

$$\widehat{\nabla}_c Q_{a\cdots b} \equiv h_c{}^d h_a{}^e \cdots h_b{}^f \nabla_d Q_{e\cdots f}$$

for any tensor $Q_{a\cdots b}$ (in [6] we used ${}^3\nabla_a$ for $\widehat{\nabla}_a$). If the tensor is spatial, we define

$$|\, Q_{a\cdots b} \,| \equiv (Q_{a\cdots b} Q^{a\cdots b})^{1/2}\,.$$

Given a smallness parameter ϵ, $O[N]$ denotes $O(\epsilon^N)$ and $A \simeq B$ means $A - B = O[2]$ (i.e., these variables are equivalent to $O[1]$). When $A \simeq 0$ we shall regard A as vanishing (for it is zero to the accuracy of the first-order calculations that are the concern of this paper).

II. COVARIANT AND GAUGE-INVARIANT ANALYSIS

The fundamental observers are assumed to be comoving with cosmological matter, which is modeled by dust with mass-energy density ρ, and which is noninteracting with radiation (as we are considering the epoch after last scattering). The physically preferred four-velocity u^a of this matter is a suitable average over peculiar velocities (which are small). The matter flow is characterized by u^a and its rate of expansion Θ ($= 3H = 3\dot{S}/S > 0$, where H is the Hubble parameter and S the scale factor), shear σ_{ab}, and vorticity ω_{ab} are all nonzero in general; however, the flow lines are geodesic: $\dot{u}^a = 0$ (consequent on the vanishing of the matter pressure).

The frame defined by u^a defines an invariant 3+1 splitting of tensors [23]. In particular, for a photon four-momentum,

$$p^a = E(u^a + e^a)\,, \qquad e_a u^a = 0\,, \qquad e_a e^a = 1\,, \tag{1}$$

where E is the photon energy and e_a the direction of photon momentum, relative to fundamental observers. Then the CBR distribution function may be expanded as [15]

$$f(x^c, E, e^d) = F(x^c, E) + F_a(x^c, E)e^a + F_{ab}(x^c, E)e^a e^b + \cdots\,, \tag{2}$$

where the covariant harmonics (multipole moments) $F_{a_1\cdots a_L}(x^c, E)$ for $L \geq 1$ are symmetric trace-free tensors orthogonal to u^a that provide a measure of the deviation of f from exact isotropy (as measured by u^a). If the CBR is almost isotropic after last scattering for all fundamental observers, then [6]

$$F,\ \dot{F} = O[0]\,, \quad F_{a_1\cdots a_L},\ \nabla_b F_{a_1\cdots a_L} = O[1]\,, \quad L \geq 1\,. \tag{3}$$

Energy integrals of the first three harmonics define the radiation energy density, energy flux, and anisotropic stress [15]:

$$\mu = 4\pi \int_0^\infty E^3 F dE = 3p = O[0]\,, \tag{4}$$

$$q_a = \frac{4\pi}{3} \int_0^\infty E^3 F_a dE = O[1]\,, \tag{5}$$

$$\pi_{ab} = \frac{8\pi}{15} \int_0^\infty E^3 F_{ab} dE = O[1] \tag{6}$$

(in [6] we used μ_R for μ). We will also need the integral of the third harmonic:

$$\xi_{abc} = \frac{8\pi}{35} \int_0^\infty E^3 F_{abc} dE = O[1]\,. \tag{7}$$

Note that the total energy-momentum tensor is made up of matter and radiation contributions:

$$T_{ab} = (\rho + \mu)u_a u_b + \tfrac{1}{3}\mu h_{ab} + \pi_{ab} + 2u_{(a}q_{b)}\,.$$

A. Covariant linearized harmonics of the Boltzmann equation

The Boltzmann equation in curved spacetime,

$$p^a\left(\frac{\partial}{\partial x^a}-\Gamma^c{}_{ab}p^b\frac{\partial}{\partial p^c}\right)f(x^d,p^e)=C[f],$$

may be decomposed into covariant harmonic equations via (1) and (2). The full (exact and nonlinear) results are given in [16, p. 501]. For the collision-free and zero acceleration case, the linearized zero, first, and second harmonic equations are

$$E\dot F-\tfrac{1}{3}E^2\Theta\frac{\partial F}{\partial E}+\tfrac{1}{3}E\widehat\nabla_aF^a\simeq 0\,, \tag{8}$$

$$E\dot F_a-\tfrac{1}{3}E^2\Theta\frac{\partial F_a}{\partial E}+E\widehat\nabla_aF+\tfrac{2}{5}E\widehat\nabla_bF^b{}_a\simeq 0\,, \tag{9}$$

$$E\dot F_{ab}-\tfrac{1}{3}E^2\Theta\frac{\partial F_{ab}}{\partial E}-E^2\sigma_{ab}\frac{\partial F}{\partial E}+E\widehat\nabla_{(a}F_{b)} -\tfrac{1}{3}h_{ab}\widehat\nabla_cF^c+\tfrac{3}{7}E\widehat\nabla_cF^c{}_{ab}\simeq 0\,. \tag{10}$$

These are the fundamental (covariant) equations governing the dynamics of radiation at a microscopic level.

B. Covariant linearized evolution and constraint equations

If (8)–(10) are multiplied by E^2 and integrated over all photon energies, then they produce the radiation conservation equations governing μ and q_a, as well as the crucial evolution equation for π_{ab} (given for the first time in [6], in full nonlinear form). These equations and the remaining (linearized) conservation, Einstein, Ricci, and Bianchi equations are as follows, obtained from [6] and the general exact equations of [23] (with $\dot u_a=0$ but allowing for an imperfect energy-momentum tensor).

(a) Matter and radiation energy and momentum conservation:

$$\dot\rho+\Theta\rho=0\,,\qquad \dot u_a=0\,, \tag{11}$$

$$\dot\mu+\tfrac{4}{3}\Theta\mu+\widehat\nabla_aq^a\simeq 0\,, \tag{12}$$

$$\dot q_a+\tfrac{4}{3}\Theta q_a+\tfrac{1}{3}\widehat\nabla_a\mu+\widehat\nabla_b\pi^b{}_a\simeq 0\,. \tag{13}$$

(b) Evolution of radiation anisotropic stress tensor:

$$\dot\pi_{ab}+\tfrac{4}{3}\Theta\pi_{ab}+\tfrac{8}{15}\mu\sigma_{ab}+2\widehat\nabla_{(a}q_{b)}-\tfrac{2}{3}h_{ab}\widehat\nabla_cq^c+\widehat\nabla_c\xi^c{}_{ab}\simeq 0\,. \tag{14}$$

(c) Einstein, Ricci, and Bianchi propagation equations:

$$\dot\Theta+\tfrac{1}{3}\Theta^2+\mu+\tfrac{1}{2}\rho\simeq 0\,, \tag{15}$$

$$\dot\sigma_{ab}+\tfrac{2}{3}\Theta\sigma_{ab}+E_{ab}-\tfrac{1}{2}\pi_{ab}\simeq 0\,, \tag{16}$$

$$\dot\omega_{ab}+\tfrac{2}{3}\Theta\omega_{ab}\simeq 0\,, \tag{17}$$

$$\dot E_{ab}+\Theta E_{ab}+\widehat\nabla^dH_{(a}{}^c\varepsilon_{b)cd}+(\tfrac{1}{2}\rho+\tfrac{2}{3}\mu)\sigma_{ab} +\tfrac{1}{2}\dot\pi_{ab}+\tfrac{1}{6}\Theta\pi_{ab}+\tfrac{1}{2}\widehat\nabla_{(a}q_{b)}-\tfrac{1}{6}h_{ab}\widehat\nabla_cq^c\simeq 0\,, \tag{18}$$

$$\dot H_{ab}+\Theta H_{ab}-\widehat\nabla^dE_{(a}{}^c\varepsilon_{b)cd}+\tfrac{1}{2}\widehat\nabla^d\pi_{(a}{}^c\varepsilon_{b)cd}\simeq 0\,. \tag{19}$$

(d) Einstein, Ricci, and Bianchi constraint equations:

$$q_a-\tfrac{2}{3}\widehat\nabla_a\Theta+\widehat\nabla^b(\sigma_{ba}+\omega_{ba})=0\,, \tag{20}$$

$$\widehat\nabla_a\omega^a=0\,, \tag{21}$$

$$H_{ab}+\widehat\nabla^d[\sigma_{(a}{}^c+\omega_{(a}{}^c]\varepsilon_{b)cd}\simeq 0\,, \tag{22}$$

$$\widehat\nabla_bE^b{}_a-\tfrac{1}{3}\widehat\nabla_a(\mu+\rho)+\tfrac{1}{2}\widehat\nabla_b\pi^b{}_a+\tfrac{1}{3}\Theta q_a\simeq 0\,, \tag{23}$$

$$\widehat\nabla_bH^b{}_a-(\rho+\tfrac{4}{3}\mu)\omega_a+\tfrac{1}{2}\varepsilon_{abc}\widehat\nabla^bq^c\simeq 0\,, \tag{24}$$

where E_{ab} and H_{ab} are the electric and magnetic parts of the Weyl tensor, respectively, and $\varepsilon_{abc}\equiv\eta_{abcd}u^d$, with η_{abcd} the spacetime permutation tensor.

It follows [6] from these equations and (3)–(7) that

$$\psi,\ \dot\psi=O[0]\,,\qquad \widehat\nabla_a\psi=O[1]\,,\qquad \psi\equiv\mu,\ \Theta,\ \rho\,, \tag{25a}$$

$$\sigma_{ab},\ \omega_{ab},\ E_{ab},\ H_{ab}=O[1]\,. \tag{25b}$$

These qualitative results from [6] will be made more quantitative in Sec. IV.

C. Integrability conditions and conserved quantities

In the case of zero acceleration, the linearized form of the identity [11] governing the commutation of time and spatial derivatives is

$$\widehat\nabla_a\dot\Psi\simeq(\widehat\nabla_a\Psi)^{\cdot}+\tfrac{1}{3}\Theta\widehat\nabla_a\Psi, \tag{26}$$

where Ψ is any $O[1]$ spatial tensor or any scalar with $O[1]$ gradient. The commutation of spatial derivatives themselves is given by the projected Ricci identities [11], which imply the exact identity

$$\widehat\nabla_{[a}\widehat\nabla_{b]}\psi=-\dot\psi\omega_{ab}, \tag{27}$$

where ψ is any scalar, and for a nearly FRW spacetime, the linearized conditions

$$\widehat{\nabla}_{[a}\widehat{\nabla}_{b]}\psi_c \simeq \frac{k}{S^2}h_{c[a}\psi_{b]}, \tag{28}$$

where ψ_a is any $O[1]$ spatial vector and $k = 0, 1, -1$ is the spatial curvature index of the limiting (background) spacetime, and

$$\widehat{\nabla}_{[a}\widehat{\nabla}_{b]}\psi_{cd} \simeq -\frac{k}{S^2}(h_{c[a}\psi_{b]d} - \psi_{c[a}h_{b]d}), \tag{29}$$

where ψ_{ab} is any $O[1]$ spatial tensor. These identities could be overlooked in an approach that starts from a FRW background solution and perturbs away from it. The integrability conditions implicit in (27)–(29) lead to the following new results.

(I) *If the covariant vector perturbations are spatially homogeneous to first order, then the vorticity vanishes to first order (i.e., nonzero terms are at most second order), and either all vector perturbations vanish to first order, or the spacetime has a flat FRW background.*

(II) *If the covariant tensor perturbations are spatially homogeneous to first order, then the spacetime is either FRW to first order, or it has a flat FRW background.*

The first result follows from (27), which shows that $\omega_{ab} \simeq 0$ since $\widehat{\nabla}_a\psi$ is a vector perturbation for $\psi = \mu, \rho$, and from (28), which implies $kS^{-2}\psi_a \simeq 0$ for $\psi_a = q_a$ or any other vector perturbation. The second result follows from (28) and (29), which imply

$$kS^{-2}\psi_a \simeq 0 \simeq kS^{-2}\psi_{ab}$$

for any vector perturbation ψ_a (since $\widehat{\nabla}_a\psi_b$ is a tensor) or tensor perturbation ψ_{ab}. Thus either $k/S^2 \simeq 0$ (the background FRW model is flat to the accuracy we are working), or

$$\widehat{\nabla}_a\mu \simeq 0 \simeq \widehat{\nabla}_a\rho\,, \quad q_a \simeq 0\,, \quad \pi_{ab} \simeq 0\,, \quad \widehat{\nabla}^c\xi_{cab} \simeq 0\,.$$

In the latter case, (27) implies $\omega_{ab} \simeq 0$, (14) implies $\sigma_{ab} \simeq 0$, and then (20) gives $\widehat{\nabla}_a\Theta \simeq 0$, while (16) and (22) give $E_{ab} \simeq 0 \simeq H_{ab}$. Thus the spacetime is FRW to first order (i.e., it differs from FRW at most by second order-terms). □

We can derive further linearized integrability results that hold in general (i.e., without assuming homogeneity of vector or tensor perturbations), by covariant differentiation of the dynamical equations. For example, taking the gradient of (24), and using (21) and $\widehat{\nabla}_{[a}\widehat{\nabla}_b q_{c]} \simeq 0$ [which follows from (28)], we get

$$\widehat{\nabla}_a\widehat{\nabla}_b H^{ab} \simeq 0\,. \tag{30}$$

Similarly, using the contraction of (29) for $\psi_{cd} = \omega_{cd}$, the gradient of (20) gives

$$\widehat{\nabla}_a\widehat{\nabla}_b\sigma^{ab} + \widehat{\nabla}_a q^a - \tfrac{2}{3}\widehat{\nabla}^2\Theta \simeq 0\,, \tag{31}$$

while (23) yields

$$\widehat{\nabla}_a\widehat{\nabla}_b(E^{ab} + \tfrac{1}{2}\pi^{ab}) + \tfrac{1}{3}\Theta\widehat{\nabla}_a q^a - \tfrac{1}{3}\widehat{\nabla}^2(\mu + \rho) \simeq 0\,. \tag{32}$$

Finally, we note the existence of various quantities that are conserved to first order along the matter flow. For example, (17) immediately shows that

$$(S^2\omega_{ab})^{\cdot} \simeq 0\,,$$

while (26), (19), and (16) imply that

$$\dot{H}^*_{ab} + \Theta H^*_{ab} \simeq 0\,, \qquad H^*_{ab} \equiv H_{ab} + \widehat{\nabla}^d\sigma_{(a}{}^c\varepsilon_{b)cd}\,. \tag{33}$$

Then (33) and (11) give

$$\left(\frac{H_{ab} + \widehat{\nabla}^d\sigma_{(a}{}^c\varepsilon_{b)cd}}{\rho}\right)^{\cdot} \simeq 0\,.$$

It therefore follows that if either the vorticity or H^*_{ab}/ρ are negligible (i.e., $O[2]$) at last scattering, they remain so at all subsequent times.

III. TEMPERATURE ANISOTROPY OF THE CBR

It is important to realize that the covariant dipole moment F_a of the CBR distribution [see (2)], although dependent upon the choice of u^a, cannot be set to zero by this choice, since it is frequency dependent and its vanishing implies special conditions on the anisotropy of f. Since the u^a frame is physically defined by matter, and is already assumed to have been corrected for local peculiar velocities, F_a represents a possible residual intrinsic frequency-dependent dipole moment of the CBR distribution relative to matter, with invariant significance. This distributional dipole, however, contains much more information than the dipole of CBR *temperature* anisotropy, which is in fact proportional to the energy flux q_a given by (5) [see (41) below].

Even for the temperature dipole, however, one cannot separate the intrinsic dipole from that induced by peculiar velocity of the observer [19,24]. It is standard to assume that the intrinsic temperature dipole is negligible after correction for a peculiar velocity. This is equivalent to the nontrivial assumption that the average four-velocity of matter coincides with the energy-frame [12] four-velocity of radiation. Although it can be justified for adiabatic perturbations within the standard model [24], we will not make this special assumption, so that we allow for an intrinsic dipole in the temperature (i.e., nonzero q_a) after correction for local peculiar velocities. This approach accommodates future improvement of observational results for the peculiar velocity which are *independent* of the CBR observations, and which may reveal a nonnegligible residual temperature dipole. From this point of view, the current limits on the intrinsic dipole should be related to the current uncertainties in the local peculiar velocity.

Another aspect of the dipole moment F_a which appears not to have been previously recognized is its link to deviations from a thermal Planck spectrum in the monopole moment F. This aspect is hidden if one starts from a background FRW solution and then perturbs, rather than considering the real non-FRW solution. The latter approach shows via the Boltzmann equation that nontrivial constraints are imposed on the dipole if F is Planckian to first order (as strongly indicated by observations [5]). For suppose that

$$F(x^a, E) \simeq 2\left[\exp\left(\frac{E}{kT(x^a)}\right) - 1\right]^{-1}, \tag{34}$$

where k is Boltzmann's constant. Then (34) and the Boltzmann monopole harmonic equation (8) imply

$$\hat{\nabla}_a F^a \simeq \left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right) \frac{3E/kT}{1 - \cosh(E/kT)}. \tag{35}$$

Thus the dipole moment is subject to the restriction (35) if the monopole moment is Planckian to $O[1]$.

One consequence of (35) is

$$\frac{\dot{T}}{T} = -\frac{\dot{S}}{S} + O[1]. \tag{36}$$

In contrast with many other treatments (where $TS = \text{const}$), T is *not* a fictitious background temperature, but is the gauge-invariant average temperature in the actual spacetime. Now observations of the CBR measure temperatures in different directions on the sky. The full-sky average temperature $T(x^a)$ at event x^a is determined by the monopole harmonic of the photon distribution

$$\mu(x^a) = a[T(x^a)]^4 = 4\pi \int_0^\infty E^3 F(x^a, E) dE \tag{37}$$

on using (4), where a is the Stefan-Boltzmann constant. A directional temperature is determined by *all* the harmonics (2) via the directional energy density per unit solid angle that is defined by the integrated (bolometric) brightness [25] (see also [26–28])

$$\begin{aligned} I(x^a, e^b) &= \int_0^\infty E^3 f(x^a, E, e^b) dE \\ &= \frac{a}{4\pi}[T(x^a) + \delta T(x^a, e^b)]^4, \end{aligned} \tag{38}$$

which defines the gauge-invariant fluctuation $\delta T(x^a, e^b)$.

The covariant multipole moments $\tau_{a_1 \cdots a_L}(x^b)$ ($L \geq 1$) of temperature anisotropy are trace-free, symmetric tensors orthogonal to u^a, defined by

$$\tau \equiv \frac{\delta T}{T} = \tau_a e^a + \tau_{ab} e^a e^b + \tau_{abc} e^a e^b e^c + \cdots \ . \tag{39}$$

By (38), (37), (2), and (4) they are given in general, to a good approximation, as normalized integrals of the covariant distribution harmonics:

$$\tau_{a_1 \cdots a_L} \simeq \left(\frac{1}{4\mu}\right) 4\pi \int_0^\infty E^3 F_{a_1 \cdots a_L} dE. \tag{40}$$

These moments give a covariant and gauge-invariant description of the CBR temperature variation, with spectral information integrated out. By (5)–(7), (40) gives the dipole, quadrupole and octopole as

$$\tau_a \simeq \frac{3q_a}{4\mu}, \qquad \tau_{ab} \simeq \frac{15\pi_{ab}}{8\mu}, \qquad \tau_{abc} \simeq \frac{35\xi_{abc}}{8\mu}. \tag{41}$$

The $\tau_{a_1 \cdots a_L}$ are a frame-independent alternative to the usual multipole coefficients A_{LM} in an expansion in spherical harmonics Y_{LM}. If we choose a standard triad in the rest space of u^a such that $e^a = (0, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, then the two approaches are linked (cf. [15]) by

$$\begin{aligned} \tau(x, \mathbf{e}) &= \sum_{L=1}^{\infty} \sum_{M=-L}^{L} A_{LM}(x) Y_{LM}(\theta, \phi) \\ &= \sum_{L=1}^{\infty} \tau_{a_1 \cdots a_L}(x) e^{a_1} \cdots e^{a_L}. \end{aligned}$$

The correlation function

$$C(\alpha) = \langle \tau(x, \mathbf{e}) \tau(x, \mathbf{e}') \rangle, \quad e^a e'_a = \cos\alpha,$$

where the angular brackets denote a statistical average, is the key quantity in actual observations. In this paper we will not consider the detailed statistical analysis of the correlation function, which is given, for example, in [19–21,24,26,29–31], where an inflationary model for perturbations is assumed. Our concern here is with the underlying principles of how to relate observational limits to properties of the spacetime geometry in a model-independent way.

Current CBR observations place limits on $\tau_{a \cdots b}(t_0, \mathbf{y}|_C)$ where t_0 is the proper time along our world line C since last scattering and $\mathbf{y}|_C$ are comoving coordinates of C. By (A1), these limits may be extended to hold on each world line $\mathbf{y}$ at a proper time t_0 along the world line after last scattering:

$$|\ \tau_{a_1 \cdots a_L}(t_0, \mathbf{y})\ | < \epsilon_L(t_0). \tag{42}$$

As in [6], we assume that anisotropies are $O[1]$ back to last scattering. Then we extend (42) to hold for all times $0 < t \leq t_0$, and obtain the assumption **(B1)** there exist $O[1]$ constants ϵ_L such that $\epsilon_L(t) \leq \epsilon_L$. Hence, for any event x^b after last scattering,

$$|\ \tau_{a_1 \cdots a_L}(x^b)\ | < \epsilon_L. \tag{43}$$

In principle (and possibly not too far off in practice), observations place limits on the comoving time derivatives of the multipoles, and as before we assume **(B2)** there exist $O[1]$ constants ϵ'_L such that

$$|\ \dot{\tau}_{a_1 \cdots a_L}(x^b)\ | < \epsilon'_L\, \Theta(x^b), \tag{44}$$

where we have normalized the derivatives relative to the expansion of the Universe (recall that $\Theta > 0$ is $O[0]$). Since it is effectively impossible for us to move cosmological distances off C, there will not be direct observations of the spatial derivatives of the multipoles. However, we will need limits on these quantities, and so assume also, on general plausibility grounds (given the basic Copernican assumption), that additionally **(B3)** there exist $O[1]$ constants ϵ''_L such that

$$|\,\widehat{\nabla}_c\tau_{a_1\cdots a_L}(x^b)\,| < \epsilon''_L\Theta(x^b)\,. \qquad (45)$$

One should note that we expect all the quantities ϵ defined here to be *very* small: probably at most 10^{-4}. The exact isotropic case considered by Ehlers, Geren, and Sachs [32] corresponds to $\epsilon_L = \epsilon'_L = \epsilon''_L = 0$, and is a special case of what follows.

IV. MODEL-INDEPENDENT LIMITS ON SPACETIME ANISOTROPY AND INHOMOGENEITY

The observationally based limits (43)–(45) on temperature anisotropy lead directly via (41) to the limits

$$|q_a| < \tfrac{4}{3}\mu\epsilon_1\,, \qquad |\pi_{ab}| < \tfrac{8}{15}\mu\epsilon_2\,, \qquad |\xi_{abc}| < \tfrac{8}{35}\mu\epsilon_3 \quad (46)$$

on the radiation anisotropy tensors. Differentiating the harmonics (41) and using radiation energy conservation (12), (5)–(7), and (25a) we get, to $O[1]$,

$$|\dot{q}_a| < \tfrac{4}{3}\mu\Theta(\tfrac{4}{3}\epsilon_1 + \epsilon'_1)\,, \qquad |\dot{\pi}_{ab}| < \tfrac{8}{15}\mu\Theta(\tfrac{4}{3}\epsilon_2 + \epsilon'_2)\,, \quad (47)$$

$$|\widehat{\nabla}_a q_b| < \tfrac{4}{3}\mu\Theta\epsilon''_1\,, \quad |\widehat{\nabla}_a\pi_{bc}| < \tfrac{8}{15}\mu\Theta\epsilon''_2\,,$$
$$|\widehat{\nabla}_a\xi_{bcd}| < \tfrac{8}{35}\mu\Theta\epsilon''_3\,. \qquad (48)$$

Now we can use radiation momentum conservation (13) and the evolution equation (14) for π_{ab}, together with (46)–(48), to derive the limits imposed directly by observations on the gradient of energy density [equivalently average temperature, by (37)] and on the shear:

$$\frac{|\widehat{\nabla}_a\mu|}{\mu} = 4\frac{|\widehat{\nabla}_a T|}{T} < H(8\epsilon_1 + 12\epsilon'_1 + \tfrac{72}{5}\epsilon''_2)\,, \qquad (49)$$

$$\frac{|\sigma_{ab}|}{\Theta} < \tfrac{8}{3}\epsilon_2 + \epsilon'_2 + 5\epsilon''_1 + \tfrac{9}{7}\epsilon''_3\,. \qquad (50)$$

These equations show explicitly the role of the dipole, quadrupole, and octopole: The gradient of radiation energy density or average temperature, which reflects inhomogeneous deviations from FRW spacetime, is bounded by the limits on both the dipole and quadrupole of temperature anisotropy; the shear, which reflects anisotropic deviations from FRW spacetime, is bounded by the limits on the dipole, quadrupole, and octopole.

Note that if we follow the usual assumption that the CBR temperature dipole is negligible (after correction for local peculiar velocities), then by (41) and (46)–(48) it follows that

$$\tau_a \simeq 0 \;\Leftrightarrow\; q_a \simeq 0 \;\Leftrightarrow\; \epsilon_1 = \epsilon'_1 = \epsilon''_1 = 0\,. \qquad (51)$$

In this case, the bounds in (49) and (50) are reduced, so that a negligible dipole reduces the limits on anisotropy and inhomogeneity.

Using (12) and (37), (49) may be rewritten

$$\frac{|\widehat{\nabla}_a T|}{|\dot{T}|} < \tfrac{1}{2}\epsilon_1 + 3\epsilon'_1 + \tfrac{18}{5}\epsilon''_2\,,$$

from which it follows that

$$\frac{|\widehat{\nabla}_a T|}{T} \ll \frac{|\dot{T}|}{T} \;\Rightarrow\; d_R \gg t_R, \qquad (52)$$

where

$$d_R = \frac{T}{|\widehat{\nabla}_a T|}\,, \qquad t_R = \frac{T}{|\dot{T}|} \qquad (53)$$

are characteristic length and time scales defined by the CBR.

Note that the only dynamical equation that has been used to determine the bounds (49) and (50) is the Boltzmann equation [which implies the conservation equations (12) and (13) as well as the evolution equation (14)]. An inspection of the remaining dynamical equations (11) and (15)–(24) shows that no further bounds, e.g., on vorticity and the Weyl tensor, can be deduced from these equations, without making further assumptions. The problem is that we are unable to deduce directly bounds on the derivatives of the shear and other tensors.

There is another aspect of the results (47)–(50): In practice ϵ'_L and, especially ϵ''_L are not known from observations. In order to produce more useful versions of the results, we need a reasonable estimate of these quantities. First, we make the reasonable assumption **(C1)** the spatial gradients of the temperature harmonics are not greater than their time derivatives: $\epsilon''_L \le \epsilon'_L$. Next, we can estimate ϵ'_L by invoking the characteristic time t_R defined by (53), leading to assumption **(C2)** the bounds on the time derivatives of the temperature harmonics are estimated by $\Theta\,\epsilon'_L \simeq \epsilon_L/t_R$. Collecting the resulting estimates

$$\epsilon''_L \;\le\; \epsilon'_L \;\simeq\; \tfrac{1}{3}\epsilon_L, \qquad (54)$$

where we have used (36) and (3). We can now use (54) to recast (47)–(50) in terms of the observationally realistic ϵ_L:

$$\frac{|\dot{q}_a|}{\Theta} < \tfrac{20}{9}\mu\epsilon_1\,, \quad \frac{|\dot{\pi}_{ab}|}{\Theta} < \tfrac{8}{9}\mu\epsilon_2\,,$$
$$\frac{|\widehat{\nabla}_a q_b|}{\Theta} < \tfrac{4}{9}\mu\epsilon_1\,, \quad \frac{|\widehat{\nabla}_a\pi_{bc}|}{\Theta} < \tfrac{8}{45}\mu\epsilon_2\,, \qquad (55)$$

$$\frac{|\widehat{\nabla}_a\mu|}{\mu} = 4\frac{|\widehat{\nabla}_a T|}{T} < H(12\epsilon_1 + \tfrac{24}{5}\epsilon_2)\,, \qquad (56)$$

$$\frac{|\sigma_{ab}|}{\Theta} < \tfrac{5}{3}\epsilon_1 + 3\epsilon_2 + \tfrac{3}{7}\epsilon_3\,. \qquad (57)$$

If the dipole is neglected, then by (51) we can set $\epsilon_1 = 0$ in (56) and (57). Note also that for the μ and H on the right sides of (55) and (56), we may use the $O[0]$ values, i.e., the values they take in the limiting (background) FRW spacetime, which has noninteracting dust and isotropic radiation; exact solutions are given in [33]. In particular, if (46) and (55)–(57) are evaluated here and now, then $\mu_0 = aT_0^4 = 3H_0^2(\Omega_R)_0$, where $T_0 = 2.7$ K, H_0 is the Hubble constant, and $\Omega_R \equiv \mu/3H^2$ is the radiation density parameter; the ϵ_L follow from current CBR observations.

The bounds (46) and (55)–(57) are the main results of the quest for a direct link from feasible observational limits on the CBR to limits on the deviations of spacetime from FRW. However, they do not extend to the vorticity and the all-important further issue of limiting the matter inhomogeneities from the radiation anisotropy. To attempt this, we use these bounds in the evolution and constraint equations (16) and (23) to derive

$$|\widehat{\nabla}_a\rho| < \mu\Theta(\tfrac{16}{3}\epsilon_1 + \tfrac{12}{5}\epsilon_2) + 9|\widehat{\nabla}_a E_{bc}|\,, \tag{58}$$

$$|E_{ab}| < \Theta^2(\tfrac{10}{9}\epsilon_1 + 2\epsilon_2 + \tfrac{2}{7}\epsilon_3) + \tfrac{4}{15}\mu\epsilon_2 + |\dot{\sigma}_{ab}|\,. \tag{59}$$

In addition, the identity (27) for μ together with (12) implies

$$|\omega_{ab}| \le \left(\frac{3}{4\mu\Theta}\right)|\widehat{\nabla}_a\widehat{\nabla}_b\mu|\,. \tag{60}$$

In order to place bounds on the derivative terms on the right hand sides of (58)–(60), we make a further assumption motivated by (46), (54), (55), and (17). We postulate that **(C3)** there exist constants α, β, γ of the order of 1 such that

$$|\dot{\sigma}_{ab}| \le \alpha\Theta|\sigma_{ab}|\,, \quad |\widehat{\nabla}_a\widehat{\nabla}_b\mu| \le \beta\Theta|\widehat{\nabla}_a\mu|\,,$$
$$|\widehat{\nabla}_a E_{bc}| \le \gamma\Theta|E_{ab}|\,. \tag{61}$$

Using (61) and (34) in (58)–(60) leads to the further bounds

$$\frac{|E_{ab}|}{\Theta} < H(2+3\alpha)(\tfrac{5}{3}\epsilon_1 + 3\epsilon_2 + \tfrac{3}{7}\epsilon_3) + \tfrac{4}{15}H\Omega_R\,\epsilon_2\,, \tag{62}$$

$$\frac{|\widehat{\nabla}_a\rho|}{\rho} < \frac{\Omega_R}{\Omega_M}H(16\epsilon_1 + \tfrac{36}{5}\gamma\epsilon_2) + \frac{H}{\Omega_M}\gamma(2+3\alpha) \times(45\epsilon_1 + 81\epsilon_2 + \tfrac{81}{7}\epsilon_3)\,, \tag{63}$$

$$\frac{|\omega_{ab}|}{\Theta} \le \beta(3\epsilon_1 + \tfrac{6}{5}\epsilon_2)\,, \tag{64}$$

where $\Omega_M \equiv \rho/3H^2$ is the matter density parameter. We thus obtain (46), (55)–(57), and (62)–(64) as plausible limits on the deviation from a FRW model in the time since last scattering (within our past light cone), based on the reasonable assumptions (B1)–(B3) and (C1)–(C3). These results give for the first time a direct relation between CBR observational limits and limits to anisotropy and inhomogeneity, without assuming a specific evolutionary model, or the curvature index k of the background.

From these limits we can obtain conservative estimates of present-time bounds on the anisotropy and inhomogeneity of the Universe. Let

$$\epsilon \equiv \max(\epsilon_1, \epsilon_2, \epsilon_3)$$

denote the upper limit of currently observed anisotropy in the CBR temperature variation, and take $\alpha = \beta = \gamma = 1$. Then (56), (57), and (64) imply

$$\left(\frac{|\widehat{\nabla}_a\mu|}{\mu}\right)_0 < 17H_0\epsilon\,, \quad \left(\frac{|\sigma_{ab}|}{\Theta}\right)_0 < 6\epsilon\,, \quad \left(\frac{|\omega_{ab}|}{\Theta}\right)_0 < 5\epsilon\,. \tag{65}$$

The remaining limits (62) and (63) depend on the current values of the density parameters. Taking $(\Omega_R)_0 \ll 1$, (62) gives

$$\left(\frac{|E_{ab}|}{\Theta}\right)_0 < 26H_0\epsilon, \tag{66}$$

while (63) implies

$$\left(\frac{|\widehat{\nabla}_a\rho|}{\rho}\right)_0 < C(\Omega)H_0\epsilon\,, \qquad C(\Omega) \equiv \frac{688}{(\Omega_M)_0}\,. \tag{67}$$

The latter is a relatively poor limit, although possibly still significant in view of the small values expected for the ϵ_L. If we are willing to assume that $(\Omega_M)_0 \simeq 1$ today, we get a reasonably tight limit from (67). However, the observational evidence points towards a range of values between 0.1 and 0.3 as more plausible [34]. Including the lowest limits implied by nucleosynthesis, we can represent the full range of possibilities by a table of values:

$(\Omega_M)_0$	0.02	0.1	0.3	1
$C(\Omega)$	34393	6880	2293	688

As we go back in time towards last scattering, whatever value it has today, Ω_M will rapidly approach 1.

We note that if the dipole is neglected, then by (51) we can set $\epsilon_1 = 0$ in (56), (57), and (62)–(64) to obtain the revised "dipole-free" estimates:

$$\left(\frac{|\widehat{\nabla}_a\mu|}{\mu}\right)_0 < 5H_0\epsilon\,, \quad \left(\frac{|\sigma_{ab}|}{\Theta}\right)_0 < 4\epsilon\,, \quad \left(\frac{|\omega_{ab}|}{\Theta}\right)_0 < 2\epsilon\,, \tag{65$'$}$$

$$\left(\frac{|E_{ab}|}{\Theta}\right)_0 < 18H_0\epsilon\,, \quad \left(\frac{|\widehat{\nabla}_a\rho|}{\rho}\right)_0 < \frac{463}{(\Omega_M)_0}H_0\epsilon\,, \tag{66$'$}$$

which are considerably better. We recover the Ehlers-Geren-Sachs (EGS) result [32] on setting $\epsilon = 0$ (exact isotropy implies an exact FRW solution).

V. ALMOST FLAT FRW SOLUTIONS

As already pointed out, we are unlikely to have direct observational information about the spatial gradients of the temperature harmonics, i.e., about the ϵ''_L bounds defined by (45). In Sec. IV we used the estimate (C1) that the spatial gradients are not greater than the time derivatives [see (54)]. Here we consider a more stringent, but apparently still reasonable, assumption on the spatial gradients. In effect we extend to the spatial gradient bounds the assumption (C2) already made on the time derivative bounds; i.e., we replace assumptions (C1)–(C3) by **(D)** the bounds on both time and spatial derivatives of the temperature harmonics are determined via the characteristic scales (53) of the CBR:

$$\begin{aligned} \Theta\epsilon'_L &\simeq \frac{\epsilon_L}{t_R} \;\Rightarrow\; \epsilon'_L \simeq \tfrac{1}{3}\epsilon_L \,, \\ \Theta\epsilon''_L &\simeq \frac{\epsilon_L}{d_R} \;\Rightarrow\; \epsilon''_L < (\epsilon_1 + \tfrac{2}{5}\epsilon_2)\epsilon_L \,, \end{aligned} \tag{68}$$

where we have used (36) and (56). It immediately follows from (68) that $\epsilon''_L = O[2]$, and hence, by (45) and (41),

$$\widehat{\nabla}_a q_b \simeq 0\,, \quad \widehat{\nabla}_a \pi_{bc} \simeq 0\,, \quad \widehat{\nabla}_a \xi_{bcd} \simeq 0\,, \quad \ldots \,. \tag{69}$$

Thus, from an apparently reasonable assumption on the spatial gradients, we are led to the vanishing at first order of vector and tensor inhomogeneities in the radiation. The point is that the radiation time scale t_R is a zero-order quantity (it exists for exactly isotropic CBR), while the radiation length scale d_R is first order — it is only finite when there are inhomogeneities in the CBR [see also (52)]. If inhomogeneities in the temperature fluctuation τ are determined by the characteristic inhomogeneity scale, then, as shown by (68), they are negligible in comparison with anisotropies. This in turn leads to a very restrictive condition: *if the radiation energy flux and anisotropic stress are homogeneous to first order,* i.e., if (69) holds, which will follow if (D) is true, then either *the spacetime is FRW to first order*, or *the spatial curvature vanishes to first order and so the background has a flat FRW geometry.*

This follows as a special case of the results (I) and (II) derived in Sec. II C. Note that it may also be derived (but with greater effort) by considering the integrability conditions of the constraint equations (20)–(24) in the case that (69) holds.

This result may be viewed as providing an alternative motivation for the almost flat FRW model of the Universe. Furthermore, we are able to reduce the system of dynamical equations in this model to a pair of linear evolution equations, for the shear and energy flux. Using the $O[1]$ identity (26) with (69), the spatial gradients of (13) and (14) imply

$$\widehat{\nabla}_a\widehat{\nabla}_b\mu \simeq 0 \;\Rightarrow\; \omega_{ab} \simeq 0\,, \quad \widehat{\nabla}_a\sigma_{bc} \simeq 0\,, \tag{70}$$

where we have used (27). Then (70) reduces (22) and the spatial gradient of (16) to

$$H_{ab} \simeq 0\,, \qquad \widehat{\nabla}_a E_{bc} \simeq 0\,. \tag{71}$$

By (69)–(71), the system of dynamical equations (11)–(24) closes at $O[1]$ and reduces to a subsystem for σ_{ab}, π_{ab}, E_{ab},

$$\dot{\sigma}_{ab} + \tfrac{2}{3}\Theta\sigma_{ab} + E_{ab} - \tfrac{1}{2}\pi_{ab} \simeq 0\,, \tag{72}$$

$$\dot{\pi}_{ab} + \tfrac{4}{3}\Theta\pi_{ab} + \tfrac{8}{15}\mu\sigma_{ab} \simeq 0\,, \tag{73}$$

$$\dot{E}_{ab} + \Theta E_{ab} + (\tfrac{1}{2}\rho + \tfrac{2}{5}\mu)\sigma_{ab} - \tfrac{1}{2}\Theta\pi_{ab} \simeq 0\,, \tag{74}$$

and a subsystem for q_a, $\widehat{\nabla}_a\psi$ ($\psi \equiv \mu, \rho, \Theta$),

$$\dot{q}_a + \tfrac{4}{3}\Theta q_a + \tfrac{1}{3}\widehat{\nabla}_a\mu \simeq 0\,, \tag{75}$$

$$2\widehat{\nabla}_a\Theta - 3q_a \simeq 0\,, \tag{76}$$

$$\widehat{\nabla}_a(\rho+\mu) - \Theta q_a \simeq 0\,. \tag{77}$$

These subsystems represent a *decoupling of the anisotropy and inhomogeneity,* since (72)–(74) contain no spatial gradients. In the special case where we assume that the temperature dipole is negligible, (41) and (75)–(77) show

$$q_a \simeq 0 \;\Rightarrow\; \widehat{\nabla}_a\mu \simeq \widehat{\nabla}_a\rho \simeq \widehat{\nabla}_a\Theta \simeq 0,$$

which, together with (69)–(71), shows that the spacetime is homogeneous to first order. Thus, in the dipole-free case when (D) is assumed, the spacetime is Bianchi type I to the accuracy of the calculation. Conversely, *if there is inhomogeneity in the radiation and matter, and if the CBR characteristic length scale determines the CBR temperature inhomogeneity [i.e., if (D) holds], then the dipole cannot be negligible after correction for peculiar velocities.*

From now on we will assume that the dipole is not negligible, i.e., $q_a \neq O[2]$. In (72)–(77), the coefficients μ, ρ, and Θ may be given their FRW zero-order forms, which are in fact the solutions of (11), (12) [using (69)], and (15). In particular, these equations imply the $O[1]$ Friedmann equation [6]

$$\Theta^2 - 3(\rho+\mu) \simeq 0\,. \tag{78}$$

Before we consider the decoupling and solving of (72)–(74) and (75)–(77), we give the limits that they imply, using (46), (47), (68), and (69),

$$\begin{aligned} \frac{|\widehat{\nabla}_a\mu|}{\mu} &< 12H\epsilon_1\,, \quad \frac{|\widehat{\nabla}_a\rho|}{\rho} < 16\left(\frac{\Omega_R}{\Omega_M}\right)H\epsilon_1\,, \\ \frac{|\widehat{\nabla}_a\Theta|}{\Theta} &< 2\Omega_R H\epsilon_1\,, \quad \frac{|\sigma_{ab}|}{\Theta} < 3\epsilon_2\,. \end{aligned} \tag{79}$$

Note that a bound on E_{ab} does not follow directly. These results sharpen the bounds given by (56), (57), and (63) [recalling that (C1)–(C3) have been replaced by (D)]. In fact, the bound on the matter inhomogeneity is drastically sharpened at late times, when $\Omega_R \ll \Omega_M$. The source of this is the disappearance of the gradient of the electric Weyl tensor from (58) — the term which produced the weak bound of (63). This gradient, along with all tensor gradients (i.e., gradients of vectors and tensors), drops out by virtue of (D). In other words, *if only scalar inhomogeneities occur (i.e., all vectors are gradients of scalars), then the matter inhomogeneities are rapidly suppressed at late times.*

A potential problem arising from (79) is that the limits suggest that the matter inhomogeneity is much less than the radiation inhomogeneity at late times. However, the upper limits do not force this to occur; they simply allow the possibility. Below we will give an example of a late-time solution where the matter inhomogeneity is greater than the radiation inhomogeneity.

By taking $u^a\nabla_a$ derivatives and using (15) and (78), the subsystems (69)–(71) and (72)–(74) may be decoupled:

$$\dddot{\sigma}_{ab} + \Theta^{-1}(\tfrac{10}{3}\Theta^2 + \mu + \tfrac{1}{2}\rho)\ddot{\sigma}_{ab} + (\tfrac{36}{55}\mu + \tfrac{43}{6}\rho)\dot{\sigma}_{ab} + \Theta^{-1}(\tfrac{44}{45}\mu\Theta^2 + \tfrac{7}{6}\rho\Theta^2 - \tfrac{44}{15}\mu^2 - \tfrac{9}{4}\rho^2 - \tfrac{26}{5}\mu\rho)\sigma_{ab} \simeq 0\,, \tag{80}$$

$$\ddot{q}_a + 3\Theta\dot{q}_a + (\tfrac{16}{9}\Theta^2 - 2\mu - \tfrac{2}{3}\rho)q_a \simeq 0\,. \tag{81}$$

In principle we can obtain the $O[1]$ solution for the almost flat FRW model after last scattering as follows. The solutions of (11), (12), and (78) imply

$$\mu \simeq rS^{-4}\,, \qquad \rho = mS^{-3}\,, \qquad \Theta \simeq S^{-2}[3(r+mS)]^{1/2}\,, \qquad \dot{r} \simeq 0 = \dot{m}, \tag{82}$$

which allow us to reduce (80) and (81) to linear ODE's in S. We write

$$\sigma_{ab} \simeq A^{(I)}_{ab}\Sigma_{(I)}(S)\,, \qquad I = 1,2,3, \tag{83}$$

$$q_a \simeq B^{(\Lambda)}_a Q_{(\Lambda)}(S)\,, \qquad \Lambda = 1,2, \tag{84}$$

with $\dot{A}^{(I)}_{ab} \simeq 0 \simeq \dot{B}^{(\Lambda)}_a$. Then $\Sigma_{(I)}$ are linearly independent solutions of

$$\Sigma''' + \left[\frac{2(8r+9mS)}{S(r+mS)}\right]\Sigma'' - \left[\frac{r(94r+89mS)}{S^2(r+mS)^2}\right]\Sigma' - \left[\frac{(480r^2+1006rmS+525m^2S^2)}{20S^3(r+mS)^2}\right]\Sigma \simeq 0 \tag{85}$$

and $Q_{(\Lambda)}$ are linearly independent solutions of

$$Q'' + \left[\frac{3(4r+5mS)}{2S(r+mS)}\right]Q' + \left[\frac{2(5r+7mS)}{S^2(r+mS)}\right]Q \simeq 0\,. \tag{86}$$

Given the solution $\Sigma(S)$ of (85), we find from (72) and (73) that

$$\pi_{ab} \simeq \frac{C_{ab}}{S^4} - \left(\frac{8r}{5S^4}\right)A^{(I)}_{ab}\int\left(\frac{S\Sigma_{(I)}}{[3(r+mS)]^{1/2}}\right)dS\,, \tag{87}$$

$$E_{ab} \simeq \frac{C_{ab}}{2S^4} - A^{(I)}_{ab}\left[\left(\frac{[3(r+mS)]^{1/2}}{3S}\right)\Sigma'_{(I)} + \left(\frac{4r}{5S^4}\right)\int\left(\frac{S\Sigma_{(I)}}{[3(r+mS)]^{1/2}}\right)dS\right], \tag{88}$$

where $\dot{C}_{ab} \simeq 0$.

Similarly, given the solution $Q(S)$ of (86), we find from (75)–(77) that

$$\widehat{\nabla}_a\mu \simeq -[3(r+mS)]^{1/2}B^{(\Lambda)}_a\left[Q'_{(\Lambda)} + \frac{4}{S}Q_{(\Lambda)}\right], \tag{89}$$

$$\widehat{\nabla}_a\Theta \simeq \tfrac{3}{2}B^{(\Lambda)}_a Q_{(\Lambda)}\,, \tag{90}$$

$$\widehat{\nabla}_a\rho \simeq -[3(r+mS)]^{1/2}B^{(\Lambda)}_a\left[Q'_{(\Lambda)} + \frac{5}{S}Q_{(\Lambda)}\right]. \tag{91}$$

Note that because two time derivatives were needed in the decoupling that led to (80), there will be a consistency condition imposed on the integration constants $A^{(I)}_{ab}$, C_{ab} via (74).

Thus (78), (83),(87),(88), and (84),(89)–(91) represent *an exact solution of the linearized equations governing the metric, CBR, and matter after last scattering in a universe subject to the observational assumption (D).* The explicit analytic form of the solutions depends on finding explicitly the solutions to the ODE's [Eqs. (85) and (86)]. We can provide explicit solutions for *late times* (i.e., long after last scattering) when

$$\mu \ll \rho \simeq \tfrac{1}{3}\Theta^2 \ll 1 \tag{92}$$

using (78). By (92), (80) simplifies to

$$\dddot{\sigma}_{ab} + \tfrac{7}{2}\Theta\ddot{\sigma}_{ab} + \tfrac{43}{18}\Theta^2\dot{\sigma}_{ab} + \tfrac{5}{36}\Theta^3\sigma_{ab} \simeq 0\,. \tag{93}$$

Writing $\sigma_{ab} \simeq U_{ab}\Theta^n$, where $\dot{U}_{ab} \simeq 0$, we find from (93) that $n = \frac{1}{3}, \frac{5}{3}, 2$. However, $n = \frac{1}{3}$ violates (50) at late times. Thus an acceptable solution to (93) is

$$\sigma_{ab} \simeq U_{ab}\Theta^{5/3} + V_{ab}\Theta^2\,, \qquad \dot{U}_{ab} \simeq 0 \simeq \dot{V}_{ab}\,. \tag{94}$$

Using (92) and (94) in (72)–(74) we find that the consistency condition on the constants of integration gives $V_{ab} \simeq 0$ and the late-time solution is

$$\sigma_{ab} \simeq U_{ab}\Theta^{5/3}\,, \qquad \pi_{ab} \simeq \tfrac{1}{6}U_{ab}\Theta^{8/3} \simeq \tfrac{1}{6}\Theta\sigma_{ab}\,, \qquad E_{ab} \simeq \tfrac{1}{4}U_{ab}\Theta^{8/3} \simeq \tfrac{3}{2}\pi_{ab}\,. \tag{95}$$

Now we use (92) to simplify (81) to

$$\ddot{q}_a + 3\Theta\dot{q}_a + \tfrac{14}{9}\Theta^2 q_a \simeq 0\,. \tag{96}$$

With $q_a \simeq J_a\Theta^n$, $\dot{J}_a \simeq 0$, we get $n = \frac{7}{3}, \frac{8}{3}$, neither of which violates the limit (46). Thus a solution of (96) is

$$q_a \simeq J_a\Theta^{7/3} + K_a\Theta^{8/3}\,, \qquad \dot{J}_a \simeq 0 \simeq \dot{K}_a\,, \tag{97}$$

and this leads via (75)–(77) to

$$\begin{gathered}\widehat{\nabla}_a\mu \simeq -\tfrac{1}{2}J_a\Theta^{10/3}\,, \quad \widehat{\nabla}_a\Theta \simeq \tfrac{3}{2}q_a\,, \\ \widehat{\nabla}_a\rho \simeq \tfrac{3}{2}J_a\Theta^{10/3} + K_a\Theta^{11/3}\,.\end{gathered} \tag{98}$$

Thus we have presented an exact and explicit late-time solution [Eqs. (94), (95), (97), and (98)], which in particular provides a quantitative measure for the rate of approach towards the limiting background solution. This solution also allows us to tighten the limit in (79) on shear at late times, and to give a late-time limit on the Weyl tensor: By (46) and (95) we have

$$\frac{|\sigma_{ab}|}{\Theta} < \left(\frac{16\Omega_R}{15\Omega_M}\right)\epsilon_2\,, \qquad \frac{|E_{ab}|}{\Theta} < \tfrac{4}{5}\Omega_R H\epsilon_2\,. \tag{99}$$

Furthermore, the solution contains spacetimes in which the late-time radiation inhomogeneity is negligible in comparison with the matter inhomogeneity. This arises when J_a and K_a in (98) are chosen to ensure that

$$\frac{|\widehat{\nabla}_a\mu|}{\mu} \ll \frac{|\widehat{\nabla}_a\rho|}{\rho} < 16\left(\frac{\Omega_R}{\Omega_M}\right)H\epsilon_1, \tag{100a}$$

where the final inequality follows from (79), consistently with (97) and (98). The particular choice of $J_a \simeq 0$, i.e., radiation inhomogeneity vanishing to first order at late times, will achieve (100a), with

$$\widehat{\nabla}_a\mu \simeq 0\,, \quad \frac{|\widehat{\nabla}_a\rho|}{\rho} < 4\left(\frac{\Omega_R}{\Omega_M}\right)H\epsilon_1\,, \tag{100b}$$

by (97), (98), and (46).

VI. CONCLUSION

We have seen how a series of assumptions of increasing sharpness (incorporating the inevitable Copernican supposition) leads to increasingly powerful deductions from the CBR anisotropy. As emphasized before, these limits are independent of detailed assumptions about the dynamical history of matter in the Universe, and provide an alternative mode of analysis to the usual approaches. This analysis has the advantage of being both covariant and gauge invariant [11,12]. It gives somewhat less information than the usual approaches based on the Sachs-Wolfe effect and its generalizations, precisely because it is more model independent; however, this also means that its conclusions are more robust than those more standard analyses, as they do not depend so much on the assumptions of particular evolutionary models. In particular they are not dependent on whether or not inflation took place, and whether or not the density parameter Ω is near the critical value.

The analysis proceeds through a series of increasingly restrictive Copernican assumptions about the nature of CBR anisotropies in an open neighborhood about our world line, which is envisaged as including the observable region of the Universe. Such assumptions are inevitable if we wish to justify the assumption of an almost-FRW model [7,8]. The qualitative assumption (A1) (Sec. I) is sharpened to the quantitative assumptions (B1)–(B3) (Sec. II) leading to the limits (46)–(50), which can be sharpened a little if we assume that matter and radiation frames coincide [i.e., if there is no CBR temperature dipole; see (51)]. These restrictions are relatively weak; in particular, they do not bound the matter inhomogeneity or vorticity. Somewhat sharper assumptions (C1)–(C3) (Sec. IV) give better limits, leading to our estimates (65)–(67), which are further sharpened if the dipole can be neglected to give (65′) and (66′). These equations make quantitative the results of [6] [the Universe is almost FRW; see (A2)] and include as a special case the Ehlers-Geren-Sachs exact theorem [32]. They confirm previous estimates based on the CBR anisotropies that the shear and vorticity are at most about 10^{-3} of the expansion (on choosing $\epsilon = 10^{-4}$, to concur with recent CBR anisotropy measurements).

We regard these assumptions and limits as highly plausible, and believe they are useful not only in terms of the limits obtained, but also in making quite explicit the kind of Copernican assumptions one has to make in order to extract information from the CBR anisotropies [such assumptions necessarily underlie the standard Sachs-Wolfe type analyses, because these analyses assume (A2) as their starting point, but they do so in a somewhat hidden way]. More debatable are the stronger assumption (D) of Sec. V, which seems on the face of it quite plausible but then leads to very restrictive conclusions: Either the Universe is FRW to first order (that is, its difference from a FRW geometry is at most second order) or it has a flat background FRW geometry (to the accuracy of our first-order calculation). When this is true we can get explicit solutions of the equations representing the combined matter and radiation system, but they only allow spatial inhomogeneity when the dipole term cannot be neglected.

Thus some may wish to adopt these stronger assumptions, while others may feel the conclusions are too strong and therefore the assumptions should be questioned. We are open minded in this matter; the main point is that the analysis presented here makes quite clear the range of possible assumptions, and their consequences when we take the Einstein-Liouville equations (and consequent propagation and conservation equations) into account.

Finally we believe this paper shows well the utility of the covariant harmonic approach to both perturbations and to kinetic theory. In particular, in examining kinetic effects, it makes quite clear how only the first three harmonic terms in the distribution function explicitly enter the field equations; the anisotropies represented by higher harmonics can affect the geometry only by *cascading* [15]: that is, by inducing anisotropies in the lower-order harmonics through divergence or gradient terms, as in Eq. (10), or through collisions [16,35].

Note added in proof. Since this paper was completed, we have been able to improve the argument by introducing sharpened assumptions, leading to more explicit conclusions. A supplementary paper presenting these improvements is in preparation.

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