Decays $\eta_c \to \gamma\gamma$ and $\eta_b \to \gamma\gamma$: A test for potential models

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We use a simple perturbation theory argument and measurements of charmonium leptonic widths $\Gamma(\psi_{NS} \to e^+e^-)$ to estimate the ratio $R_0 \equiv |\Psi_{\eta_{c1S}}(0)|^2/|\Psi_{\psi_{1S}}(0)|^2$ in the general context of nonrelativistic potential models. We obtain $R_0 = 1.4 \pm 0.1$. We then apply well known potential model formulas, which include lowest order QCD corrections, to find $\Gamma(\eta_c \to \gamma\gamma)/\Gamma(\psi_{1S} \to e^+e^-) \approx 2.2 \pm 0.2$. The central value for $\Gamma(\psi_{1S} \to e^+e^-)$ in the 1992 Particle Data Group Tables then leads to a (nonrelativistic) prediction $\Gamma(\eta_c \to \gamma\gamma) \approx 11.8 \pm 0.8$ keV. This prediction is in good agreement with a recent measurement by the ARGUS collaboration, is consistent with a recent measurement by the L3 Collaboration, but is significantly higher than several other measurements and previous theoretical estimates, which usually assume $R_0 = 1$. The correction to $R_0 = 1$ is estimated to be smaller but non-negligible for the $b\bar{b}$ system. Using the current central measurement for $\Gamma(\Upsilon_{1S} \to e^+e^-)$ we find $\Gamma(\eta_b \to \gamma\gamma) \approx 0.58 \pm 0.03$ keV. A rough estimate of relativistic corrections reduces the expected two photon rates to about 8.8 and 0.52 keV for the η_c and η_b mesons, respectively. Such corrections, however, are not expected to significantly affect our estimates of R_0. An estimate of $\Gamma(\eta_c(2S) \to \gamma\gamma)$ is given as well.

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Potential models [nonrelativistic (NR) as well as "relativized" versions] have been successfully used to describe many properties of quarkonium ($c\bar{c}$ and $b\bar{b}$) states [1]. Relativistic effects and beyond lowest order QCD corrections are expected to be more important in $c\bar{c}$ mesons than in $b\bar{b}$ mesons but it is hard to devise model-independent tests (i.e., that do not depend strongly on the particular form of the potential being used) that may pinpoint some properties of charmonium spectroscopy or decays where a NR description clearly fails.

In the present paper we make a prediction for $\Gamma(\eta_c \to \gamma\gamma)$ [$\Gamma(\eta_b \to \gamma\gamma)$] which relies solely on (i) a NR description of the $c\bar{c}$ [$b\bar{b}$] system, (ii) experimental data of leptonic charmonium [bottomium] decays, (iii) approximate validity of lowest order perturbation theory for the color-hyperfine splitting interaction, and (iv) approximate validity of lowest order QCD radiative corrections. At the end of the paper we make an estimate of relativistic corrections to our results.

As a starting point we make use of a previously derived result [2] relying on a NR description of quarkonium systems which includes lowest order QCD corrections:

$$\frac{\Gamma(\eta_c \to \gamma\gamma)}{\Gamma(\psi_{1S} \to e^+e^-)} = \frac{4}{3}\left(1 + 1.96\frac{\alpha_s}{\pi}\right)\frac{M_\psi^2}{(2m_c)^2}\frac{|\Psi_{\eta_c}(0)|^2}{|\Psi_{\psi_{1S}}(0)|^2} \;, \tag{1}$$

where α_s should be evaluated at the charm scale. We will use $\alpha_s(m_c) \approx 0.28 \pm 0.02$ [2]. For consistency of the NR description of the system and since the hyperfine splitting will be included only perturbatively, one should set $2m_c \approx M_\psi$ [3].

Thus, one obtains

$$\frac{\Gamma(\eta_c \to \gamma\gamma)}{\Gamma(\psi_{1S} \to e^+e^-)} \approx 1.57\frac{|\Psi_{\eta_c}(0)|^2}{|\Psi_{\psi_{1S}}(0)|^2} \;, \tag{2}$$

and analogously, using $\alpha_s(m_b) \approx 0.18 \pm 0.01$ [2],

$$\frac{\Gamma(\eta_b \to \gamma\gamma)}{\Gamma(\Upsilon_{1S} \to e^+e^-)} \approx 0.37\frac{|\Psi_{\eta_b}(0)|^2}{|\Psi_{\Upsilon_{1S}}(0)|^2} \;. \tag{3}$$

It is important to note that even if we were far more conservative with the uncertainty in the value of $\alpha_s(m_c)$ and $\alpha_s(m_b)$, the resulting uncertainty in the numerical coefficients of Eqs. (2) and (3) would only be of the order of a few percent.

A widely used approximation at this point is to set $|\Psi_{\eta_c}(0)|^2/|\Psi_{\psi_{1S}}(0)|^2 \approx 1$: [$|\Psi_{\eta_b}(0)|^2/|\Psi_{\Upsilon_{1S}}(0)|^2 \approx 1$], leading to $\Gamma(\eta_c \to \gamma\gamma)/\Gamma(\psi_{1S} \to e^+e^-) \approx 1.6$ [$\Gamma(\eta_b \to \gamma\gamma)/\Gamma(\Upsilon_{1S} \to e^+e^-) \approx 0.37$]. We will show below that for the $c\bar{c}$ case this commonly used assumption is off by more than 30% (and could be off by as much as 50%). For the $b\bar{b}$ case the correction to this approximation is estimated to be smaller but still significant.

In the context of NR potential models, the interaction Hamiltonian responsible for the splitting between the $1\,^3S_1$ states and the $1\,^1S_0$ state of a $Q\bar{Q}$ meson is given by $\hat{\mathbf{H}} = \hat{\mathbf{H}}_{S_{12}} + \hat{\mathbf{H}}_{\rm SO}$ where

$$\hat{\mathbf{H}}_{S_{12}} = \frac{3\boldsymbol{\mu}_1\cdot\hat{\mathbf{r}}\boldsymbol{\mu}_2\cdot\hat{\mathbf{r}} - \boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2\delta^3(\mathbf{r})$$
$$= \hat{\mathbf{H}}^r_{S_{12}} + \hat{\mathbf{H}}^\delta_{S_{12}} \;. \tag{4}$$

$\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are, respectively, the static color-magnetic moment operator for the quark and the antiquark and $\mathbf{r}$ is the relative position vector. Color indices have been omitted and electromagnetic interactions have been ignored. The spin-orbit part of the Hamiltonian, $\hat{\mathbf{H}}_{\rm SO}$, is included for completeness. However, it does not contribute to lowest order to either the energy splitting [Eq. (5) below] or the shifts of the wave functions [Eqs. (6)–(9) below] for S states.

We now study the effects of the Hamiltonian $\hat{\mathbf{H}}$ to *lowest* order in perturbation theory. The energy splitting between the $1\,^3S_1$ and $1\,^1S_0$ states is given to lowest order by

$$\begin{aligned}\Delta E^{(1)} &= E^{(1)}_{1\,^3S_1} - E^{(1)}_{1\,^1S_0}\\ &= \langle\Psi^{(0)}_{1\,^3S_1}|\hat{\mathbf{H}}_{S_{12}}|\Psi^{(0)}_{1\,^3S_1}\rangle - \langle\Psi^{(0)}_{1\,^1S_0}|\hat{\mathbf{H}}_{S_{12}}|\Psi^{(0)}_{1\,^1S_0}\rangle\\ &= \langle\Psi^{(0)}_{1\,^3S_1}|\hat{\mathbf{H}}^{\delta}_{S_{12}}|\Psi^{(0)}_{1\,^3S_1}\rangle - \langle\Psi^{(0)}_{1\,^1S_0}|\hat{\mathbf{H}}^{\delta}_{S_{12}}|\Psi^{(0)}_{1\,^1S_0}\rangle\ ,\end{aligned} \tag{5}$$

where the last step follows after angular integration from the fact that the relevant states have $\ell = 0$. Experimentally, $\Delta E_{c\bar{c}} = M_{\psi_{1S}} - M_{\eta_{c1S}} = 118 \pm 2$ MeV [4]. We estimate $\Delta E_{b\bar{b}} = M_{\Upsilon_{1S}} - M_{\eta_{b1S}} = 45 \pm 15$ MeV, using the measured value of $\Delta E_{c\bar{c}}$ and NR potential model formulas which include lowest order QCD corrections.

The lowest order correction to the radial and orbital ground state wave functions at a point $\mathbf{r}$ due to the Hamiltonian $\hat{\mathbf{H}}$ is given (in common bra-ket notation) by

$$\begin{aligned}\Delta^{(1)}\Psi_{1\,^3S_1}(\mathbf{r}) &= \Delta^{(1)}\langle\mathbf{r}|\Psi_{1\,^3S_1}\rangle\\ &= \sum_{\substack{n\\ E^{(0)}_n\neq E^{(0)}_{1S}}} \frac{\langle\mathbf{r}|\Psi^{(0)}_n\rangle\langle\Psi^{(0)}_n|\hat{\mathbf{H}}_{S_{12}}|\Psi^{(0)}_{1\,^3S_1}\rangle}{E^{(0)}_{1S} - E^{(0)}_n}\ ,\end{aligned} \tag{6}$$

$$\begin{aligned}\Delta^{(1)}\Psi_{1\,^1S_0}(\mathbf{r}) &= \Delta^{(1)}\langle\mathbf{r}|\Psi_{1\,^1S_0}\rangle\\ &= \sum_{\substack{n\\ E^{(0)}_n\neq E^{(0)}_{1S}}} \frac{\langle\mathbf{r}|\Psi^{(0)}_n\rangle\langle\Psi^{(0)}_n|\hat{\mathbf{H}}_{S_{12}}|\Psi^{(0)}_{1\,^1S_0}\rangle}{E^{(0)}_{1S} - E^{(0)}_n}\ .\end{aligned} \tag{7}$$

But the wave function at the origin is zero for all non-S states and therefore only S states contribute to the sum for $\mathbf{r} = 0$. Thus, the *shifts in the wave functions at the origin* are given as

$$\begin{aligned}\Delta^{(1)}\Psi_{1\,^3S_1}(\mathbf{0}) &= \sum_{N>1} \frac{\langle\mathbf{0}|\Psi^{(0)}_{NS}\rangle\langle\Psi^{(0)}_{NS}|\hat{\mathbf{H}}_{S_{12}}|\Psi^{(0)}_{1\,^3S_1}\rangle}{E^{(0)}_{1S} - E^{(0)}_{NS}}\\ &= \sum_{N>1} \frac{\langle\mathbf{0}|\Psi^{(0)}_{NS}\rangle\langle\Psi^{(0)}_{NS}|\hat{\mathbf{H}}^{\delta}_{S_{12}}|\Psi^{(0)}_{1\,^3S_1}\rangle}{E^{(0)}_{1S} - E^{(0)}_{NS}}\\ &= \sum_{N>1} \frac{|\Psi^{(0)}_{NS}(\mathbf{0})|^2\langle\Psi^{(0)}_{1\,^3S_1}|\hat{\mathbf{H}}^{\delta}_{S_{12}}|\Psi^{(0)}_{1\,^3S_1}\rangle}{(E^{(0)}_{1S} - E^{(0)}_{NS})\Psi^{(0)}_{1\,^3S_1}(\mathbf{0})^*}\ .\end{aligned} \tag{8}$$

The last step is possible because the matrix element has support only at the origin. The previous step follows from the fact that only S states are involved in the sum. Note that only one state per radial excitation contributes to the sum. The wave function in the denominator is a shorthand notation to mean that multiplication on the left by $\langle\Psi^{(0)}_{1^3S_1}|\mathbf{0}\rangle$ will cancel it.

In the same way,

$$\Delta^{(1)}\Psi_{1\,^1S_0}(\mathbf{0}) = \sum_{N>1} \frac{|\Psi^{(0)}_{NS}(\mathbf{0})|^2\langle\Psi^{(0)}_{1\,^1S_0}|\hat{\mathbf{H}}^{\delta}_{S_{12}}|\Psi^{(0)}_{1\,^1S_0}\rangle}{(E^{(0)}_{1S} - E^{(0)}_{NS})\Psi^{(0)}_{1\,^1S_0}(\mathbf{0})^*}\ . \tag{9}$$

Therefore, to *lowest* order in perturbation theory,

$$\begin{aligned}|\Psi_{1\,^3S_1}(\mathbf{0})|^2 - |\Psi_{1\,^1S_0}(\mathbf{0})|^2 &= \left\{\Psi^{(0)}_{1\,^3S_1}(\mathbf{0})^*\left[\Delta^{(1)}\Psi_{1\,^3S_1}(\mathbf{0})\right] + \left[\Delta^{(1)}\Psi_{1\,^3S_1}(\mathbf{0})^*\right]\Psi^{(0)}_{1\,^3S_1}(\mathbf{0})\right\}\\ &\quad - \left\{\Psi^{(0)}_{1\,^1S_0}(\mathbf{0})^*\left[\Delta^{(1)}\Psi_{1\,^1S_0}(\mathbf{0})\right] + \left[\Delta^{(1)}\Psi_{1\,^1S_0}(\mathbf{0})^*\right]\Psi^{(0)}_{1\,^1S_0}(\mathbf{0})\right\}\\ &= 2\Delta E^{(1)}\sum_{N>1}\frac{|\Psi^{(0)}_{NS}(\mathbf{0})|^2}{E^{(0)}_{1S} - E^{(0)}_{NS}}\ ,\end{aligned} \tag{10}$$

where $\Delta E^{(1)}$ is given by Eq. (5). Strictly speaking, within any specific potential model, the sum on the right-hand side (RHS) of Eq. (10) is logarithmically divergent because $|\Psi_N(0)|^2 \propto \frac{dE_N}{dN}$ for large values of N [5]. The appearance of this divergence in our perturbative expansion is due to the singular nature of the contact term $\hat{\mathbf{H}}^{\delta}_{S_{12}}$ [see Eq. (4)]. In the physical situation, however, the Dirac δ function in $\hat{\mathbf{H}}^{\delta}_{S_{12}}$ should be replaced by a "smeared" peak [e.g., a Gaussian of finite width $\Delta r = O(1/M)$ where M is m_c or m_b] which will significantly suppress the contributions for large N, where $\Psi_N(\mathbf{r})$ has many nodes within the "smearing radius" Δr. Because we estimate this sum (see below) by using experimental data (which contains the full dynamics, including the smearing), we expect that the series converges rapidly so that an estimate using the first few terms will be physically meaningful.

Equation (10) can be rewritten in the form

$$\begin{aligned}\frac{|\Psi_{1\,^1S_0}(\mathbf{0})|^2}{|\Psi_{1\,^3S_1}(\mathbf{0})|^2} &= 1 + 2\Delta E^{(1)}\sum_{N>1}\frac{|\Psi^{(0)}_{NS}(\mathbf{0})|^2}{|\Psi_{1\,^3S_1}(\mathbf{0})|^2}\frac{1}{E^{(0)}_{NS} - E^{(0)}_{1S}}\\ &\approx 1 + 2\left(M_{1\,^3S_1} - M_{1\,^1S_0}\right)\sum_{N>1}\frac{|\Psi_{N\,^3S_1}(\mathbf{0})|^2}{|\Psi_{1\,^3S_1}(\mathbf{0})|^2}\frac{1}{M_{N\,^3S_1} - M_{1\,^3S_1}}\ ,\end{aligned} \tag{11}$$

where the last step (which is consistent to lowest order) was taken to express the result in terms of experimentally measurable quantities.

Notice that since we are working in the context of NR potential models we can express the ratio of the wave functions at the origin for S vector states in terms of ratios of leptonic widths and masses [2]:

$$\frac{|\Psi_{N\,^3S_1}(0)|^2}{|\Psi_{1\,^3S_1}(0)|^2} = \frac{\Gamma(N\,^3S_1 \rightarrow e^+e^-)M^2_{N\,^3S_1}}{\Gamma(1\,^3S_1 \rightarrow e^+e^-)M^2_{1\,^3S_1}} \ . \tag{12}$$

Our result is then

$$\frac{|\Psi_{1\,^1S_0}(0)|^2}{|\Psi_{1\,^3S_1}(0)|^2} = 1 + 2\,(M_{1\,^3S_1} - M_{1\,^1S_0}) \sum_{N>1} \frac{\Gamma(N\,^3S_1 \rightarrow e^+e^-)M^2_{N\,^3S_1}}{\Gamma(1\,^3S_1 \rightarrow e^+e^-)M^2_{1\,^3S_1}} \frac{1}{M_{N\,^3S_1} - M_{1\,^3S_1}} \ . \tag{13}$$

We calculate this ratio using $M_{\psi_{1S}} - M_{\eta_{c1S}} = 118\pm2$ MeV as well as the information on masses and leptonic widths for the $^3S_1$ states $\psi(3097)$, $\psi(3685)$, $\psi(3770)$, $\psi(4040)$, $\psi(4160)$, and $\psi(4415)$ given in the 1992 Particle Data Group (PDG) tables [4]. We obtain

$$\frac{|\Psi_{\eta_c 1S}(0)|^2}{|\Psi_{\psi 1S}(0)|^2} = 1.4 \pm 0.1 \ . \tag{14}$$

The error was estimated from the given experimental uncertainties [4] . We have ignored possible contributions from "continuum" states above the $D\bar{D}$ threshold because these are outside the potential model Hilbert space. We are assuming implicitly that the discrete spectrum constitutes a complete set of states of the relevant NR Schrödinger equation and thus including (physical) continuum state contributions would amount to double counting.

We note that over half of the contribution of the sum over states in Eq. (13) to our result in Eq. (14) is due to the lowest excitation $\psi(3685)$ (≈ 0.23) while the contribution of the highest observed radial excitation $\psi(4415)$ is quite small (≈ 0.03). Therefore, we expect that the five radial excitations of ψ_{1S} observed so far saturate the sum to a good approximation.

Although we have no reliable estimate of the corrections to the result of Eq. (14) due to effects that are higher order in $\hat{\mathbf{H}}_{S_{12}}$, the magnitude of the lowest order correction ($\approx 40\%$) can be used as an indication that such corrections are not likely to change our result significantly (i.e., by more than $\sim$10%). Such corrections should be viewed as a systematic uncertainty of our approximations and will be ignored hereafter.

Combining our results in Eqs. (2) and (14), we obtain the prediction

$$\frac{\Gamma(\eta_c \rightarrow \gamma\gamma)}{\Gamma(\psi_{1S} \rightarrow e^+e^-)} \approx 2.2 \pm 0.2 \ . \tag{15}$$

Using the value for the ψ_{1S} leptonic width quoted in Ref. [4],

$$\Gamma(\psi \rightarrow e^+e^-) = 5.36 \pm^{0.29}_{0.28} \ \text{keV}, \tag{16}$$

we obtain an absolute estimate

$$\Gamma(\eta_c \rightarrow \gamma\gamma) = 11.8 \pm 0.8 \pm 0.6 \ \text{keV}, \tag{17}$$

where the first uncertainty stems from our main result Eq. (15), while the second reflects the experimental uncertainty in Eq. (16).

We point out that if we had used the common approximation $|\Psi_{\eta_c}(0)|^2/|\Psi_{\psi}(0)|^2 = 1$ instead of our estimate [Eq. (14)], we would have obtained a rate $\Gamma(\eta_c \rightarrow \gamma\gamma) \approx 8.4 \pm 0.5$ keV instead of our result in Eq. (17). This is in line with most previous theoretical predictions (around 7 keV) [6] and also agrees within errors with the experimental average [$\Gamma(\eta_c \rightarrow \gamma\gamma) = 6.6 \pm^{2.4}_{2.1}$ keV] quoted in Ref. [4] (see also Refs. [7], [8], and [9]). A more recent measurement by the ARGUS Collaboration [10], on the other hand, is $\Gamma(\eta_c \rightarrow \gamma\gamma) = 12.6 \pm 4.0$ keV, which is centered closer to our prediction [Eq. (17)]. Finally, a recently published result by L3 [11], $\Gamma(\eta_c \rightarrow \gamma\gamma) = 8.0 \pm 2.3 \pm 2.4$ keV, is centered closer to the estimate based on the assumption $R_0 = 1$ but is still consistent with our prediction [Eq. (17)]. A compilation of experimental measurements of $\Gamma(\eta_c \rightarrow \gamma\gamma)$ is given in Table I for comparison with our NR prediction [Eq. (17)] and our result that includes an estimate of likely relativistic corrections, Γ_{rel}, given later in the paper.

We can also use the above formalism to estimate the rate $\eta_c(2S) \rightarrow \gamma\gamma$ which may be measured in the near future. Equation (1) remains unchanged with the obvious replacement $\eta_c \rightarrow \eta_c(2S)$ and $\psi_{1S} \rightarrow \psi_{2S}$. In Eq. (13) we replace $1S \rightarrow 2S$ everywhere and the sum is now over $N \neq 2$. Using essentially the same experimental information (masses and leptonic width) as required above, supplemented by $M_{\eta_c(2S)} = 3594.0 \pm 5.0$ [4], we obtain $|\Psi_{\eta_c 2S}(0)|^2/|\Psi_{\psi 2S}(0)|^2 \approx 1.2 \pm 0.1$. Using the measured leptonic width $\Gamma(\psi_{2S} \rightarrow e^+e^-) = 2.14 \pm 0.21$ [4] we then obtain, from the analogue of Eq. (1), $\Gamma(\eta_c(2S) \rightarrow \gamma\gamma) = 5.7 \pm 0.5 \pm 0.6$ keV.

TABLE I. Experimental results for $\Gamma(\eta_c \rightarrow \gamma\gamma)$.

Experiment	$\Gamma(\eta_c \rightarrow \gamma\gamma)$ (keV)
PLUTO (1986) [7]	28 ± 15
TPC (1988) [8]	$6.4^{+5.0}_{-3.4}$
CLEO (1990) [9]	$5.9^{+2.1}_{-1.8} \pm 1.9$
PDG AVE. (1992) [4]	$6.6^{+2.4}_{-2.1}$
ARGUS (1992) [10]	12.6 ± 4.0
L3 (1993) [11]	$8.0 \pm 2.3 \pm 2.4$
E-760 (1993) [12]	$5.0^{+3.8}_{-2.8} \pm 2.3$ (preliminary)
CLEO II (1993) [13]	$5.73 \pm 1.34 \pm 1.57$ (preliminary)

Although the effect is smaller for the $b\bar{b}$ mesons, we repeat the same procedure for completeness. Using our above mentioned estimate for the energy splitting $\Delta E_{b\bar{b}} = M_{\Upsilon_{1S}} - M_{\eta_{b1S}} = 45 \pm 15$ MeV and the experimental information about Υ states (masses and leptonic widths) [4], we obtain an estimate, analogous to the result in Eq. (14), of

$$\frac{|\Psi_{\eta_{b1S}}(0)|^2}{|\Psi_{\Upsilon_{1S}}(0)|^2} \approx 1.16 \pm 0.06 \ . \tag{18}$$

Using our results, Eqs. (3) and (18), we obtain

$$\frac{\Gamma(\eta_b \to \gamma\gamma)}{\Gamma(\Upsilon_{1S} \to e^+e^-)} \approx 0.43 \pm 0.02 \ . \tag{19}$$

The current experimental value for $\Gamma(\Upsilon_{1S} \to e^+e^-)$ [4] then leads to the prediction

$$\Gamma(\eta_b \to \gamma\gamma) = 0.58 \pm 0.03 \pm 0.02 \ \text{keV} \ , \tag{20}$$

where the first error stems from Eq. (19) and the second one from the present experimental uncertainty in $\Gamma(\Upsilon_{1S} \to e^+e^-)$ [4].

To summarize, the main point of the present paper is to show that within the general context of NR potential models one can (using available experimental information) make a reliable estimate for the ratio $R_0 = |\Psi_{1\,^1S_0}(0)|^2/|\Psi_{1\,^3S_1}(0)|^2$ for $c\bar{c}$ and $b\bar{b}$ systems. A commonly used approximation is $R_0 = 1$. Our estimates [Eqs. (14) and (18)] imply that $R_0 = 1$ is a bad approximation for the $c\bar{c}$ system and requires moderate corrections for the $b\bar{b}$ system.

The main measurable consequence of this study is that $\Gamma(\eta_c \to \gamma\gamma)$ is expected to be significantly larger (by between 30% and 50%) than predictions based on the $R_0 = 1$ assumption. In fact, the NR "potential model" predictions for the rates of all important decay modes of the η_c [relative to $\Gamma(\psi \to e^+e^-)$] are enhanced by the same amount with respect to the predictions based on the $R_0 = 1$ assumption. Thus, if $R_0 = 1$ is used for such predictions, the error made would have to be compensated by using unphysically large values for the strong coupling α_s at the tree level.

From the theoretical viewpoint, precise measurement of $\Gamma(\eta_c \to \gamma\gamma)$ favoring lower values (e.g., around or below 7 keV, which agrees well with the central value in Ref. [4] and also the recent measurements Refs. [11–13]) would put into question the validity of NR potential model assumptions for the description of $c\bar{c}$ systems. The only way that we see to escape this conclusion would be to argue that the ratio $\left(\frac{M_\psi}{2m_c}\right)$ in Eq. (1) is significantly smaller than 1. However, this seems to go against the weak-binding assumption that is needed for self-consistency of NR potential models. Of course, a naive NR estimate of $\Gamma(\eta_c \to \gamma\gamma)$, where perturbative QCD corrections are ignored and $R_0 = 1$ is assumed, would indeed give such a low value because in such an approximation $\Gamma(\eta_c \to \gamma\gamma) \approx 4/3\Gamma(\psi \to e^+e^-)$ [see Eq. (1)]. Our main result [Eq. (17)] shows that this approximation is unwarranted.

On the other hand, if $\Gamma(\eta_c \to \gamma\gamma)$ turns out to be close to our prediction [Eq. (17)] (see also the measured value in Ref. [10]), we could state that NR potential model descriptions of charmonium have passed yet another test successfully. Because we know that a fully relativistic description of these systems is in fact the correct one, this "success" would just mean that relativistic effects are either unimportant or are well mimicked by a NR description, for the relevant processes.

All the above results and discussions are based on a strictly NR description of the $c\bar{c}$ and $b\bar{b}$ system. As stated at the beginning of the paper, relativistic corrections may be significant, especially for the $c\bar{c}$ system. We conclude our paper with a brief discussion of the likely effects of such corrections on our predictions. We think that our results for the ratios R_0 [Eqs. (14) and (18)] should be *essentially unaffected* by relativistic corrections, because the main input for those estimates consists of actual experimental data (leptonic widths and mass splittings) which of course include full relativistic corrections. The argument used for these estimates [Eqs. (4)–(13)] relies more on lowest order perturbation theory than on strictly NR dynamics. On the other hand, Eq. (1) and its analogue for $b\bar{b}$ systems does rely more directly on the NR description. The main relativistic correction to that ratio comes from the fact that the contribution of the quark propagator between the two photons in the η_c (η_b) decays is sensitive to the momentum distribution of the quarks in the decaying meson (see, for example, Refs. [14] and [15]). We roughly estimate these effects by using commonly quoted values for $\frac{v^2}{c^2}|_{\text{ave}} \equiv \langle\beta^2\rangle$ for the orbital and

TABLE II. Decay rate $\Gamma(\eta_c \to \gamma\gamma)$, based on $\Gamma(\psi \to e^+e^-) \approx 5.36$ [4], for different theoretical assumptions. The last two rows representing our predictions [Eq. (17)] and $\Gamma_{\text{rel}}(\eta_c \to \gamma\gamma)$.

$\Gamma(\eta_c \to \gamma\gamma)$ (keV)	QCD correction $\alpha_s(m_c) = 0.28$	$\left\lvert\frac{\Psi_\eta(0)}{\Psi_\psi(0)}\right\rvert^2 \approx 1.4$	Relativistic correction $\langle\beta^2\rangle \approx 0.25$
7.1	x	x	x
5.4	x	x	√
10	x	√	x
8.4	√	x	x
7.5	x	√	√
6.3	√	x	√
11.8	√	√	x
8.8	√	√	√

TABLE III. Sample of previous theoretical estimates of $\Gamma(\eta_c \to \gamma\gamma)$ and $\Gamma(\eta_b \to \gamma\gamma)$ together with our predictions (NR and relativistically corrected).

Reference	$\Gamma(\eta_c \to \gamma\gamma)$ (keV)	$\Gamma(\eta_b \to \gamma\gamma)$ (keV)
[16]	$0.0014 \times \Gamma(\eta_c) \approx 14$	—
[17]	7.7	0.45
[18]	8.4	0.50
[19]	6.8	0.38
[20]	4.8	0.17
[21]	15	0.17
[22]	5.6	—
[23]	4.6	—
This work (strictly NR)	11.8	0.58
This work (incl. relat. corr.)	8.8	0.52

radial ground state $c\bar{c}$ and $b\bar{b}$ quarkonium: $\langle\beta^2\rangle_\psi \approx 0.25$ and $\langle\beta^2\rangle_\Upsilon \approx 0.1$. The β dependence in the integrands of Eqs. (13) and (15) in Ref. [14] is "pulled out" of the integrals by replacing β^2 by $\langle\beta^2\rangle$. This procedure results in a relativistic correction factor of

$$\frac{\frac{1-\langle\beta^2\rangle}{\sqrt{\langle\beta^2\rangle}}\ln\left[\frac{1+\sqrt{\langle\beta^2\rangle}}{\sqrt{1-\langle\beta^2\rangle}}\right]}{\frac{1}{3}\left[2+\sqrt{1-\langle\beta^2\rangle}\right]}$$

to the RHS of Eq. (1) and its $b\bar{b}$ analogue. Using the above values for $\langle\beta^2\rangle$ leads to a numerical factor of about $\frac{1}{1.34}$ for the $c\bar{c}$ system and $\frac{1}{1.11}$ for the $b\bar{b}$ system. Using our NR central estimates given in Eqs. (17) and (20) above, we obtain "relativistically corrected" central estimates of $\Gamma_{\rm rel}(\eta_c \to \gamma\gamma) \approx 8.8$ keV, $\Gamma_{\rm rel}(\eta_c(2S) \to \gamma\gamma) \approx 4.25$ keV, and $\Gamma_{\rm rel}(\eta_b \to \gamma\gamma) \approx 0.52$ keV, where we assumed $\langle\beta^2\rangle_{\psi_{1S}} \approx \langle\beta^2\rangle_{\psi_{2S}}$. If an accurate experimental measurement would give $\Gamma(\eta_c \to \gamma\gamma) \approx 8-9$ keV, caution again has to be applied in the interpretation. The present paper would imply that this value is the result of a near cancellation of two effects, namely, $R_0 \approx 1.4$ and relativistic corrections. On the other hand, a strictly NR description that included QCD corrections, but would (incorrectly) set $R_0 = 1$, would produce a similar answer. In order to facilitate interpretation of experimental results in terms of different possible theoretical assumptions, we summarize the corresponding predictions for $\Gamma(\eta_c \to \gamma\gamma)$ in Table II. For completeness, we present in Table III a sample of previous theoretical estimates of $\Gamma(\eta_c \to \gamma\gamma)$ and $\Gamma(\eta_b \to \gamma\gamma)$ based on nonrelativistic and relativistic potential models as well as QCD sum rule techniques. For comparison, we include also the predictions in this paper (stricltly NR and with expected relativistic correction).

Finally, we would like to remark that our main result for the $c\bar{c}$ system, namely, that $R_0 \equiv |\Psi_{\eta_{c1S}}(0)|^2/|\Psi_{\psi_{1S}}(0)|^2$ is significantly larger than 1 [see Eq. (14)], could help to explain the experimentally observed suppression of the rate for $\psi_{1S} \to \eta_c\gamma$ relative to theoretical estimates that use identical spatial wave functions for ψ_{1S} and η_c. The rate for $\psi_{2S} \to \eta_c\gamma$ will also be affected by our result.

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