# Predicting the masses of heavy hadrons without an explicit Hamiltonian

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There are striking regularities in the masses and mass differences of known hadrons. Some of these regularities can be understood from known general properties of the interactions of quarks without a need to specify the explicit form of the Hamiltonian. The Feynman-Hellmann theorem is one of the tools providing this understanding. If the mass regularities are exploited, predictions can be made of the masses of as yet undiscovered hadrons. In particular, it is found that the mass of the $B_c^*$ is 6320 ± 20 MeV. Predictions concerning (i) excited vector mesons, (ii) pseudoscalar mesons, (iii) $P$-wave mesons, and (iv) ground-state spin-$\frac{1}{2}$ and -$\frac{3}{2}$ baryons are also made.

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## I. INTRODUCTION

Thus far, the quark potential model has been the most successful tool enabling physicists to calculate the masses of normal mesons and baryons containing heavy quarks. We call attention to several reviews on the subject [1–8]. However, potential models suffer from the fact that, although motivated from QCD, so far, they cannot be derived from that theory. In this paper we make predictions about hadron masses with complementary methods which use general properties of the potential (or, more generally, of the interaction) but not its specific form. Among the complementary methods are (1) obtaining constraints on hadron (and quark) masses [1,9,10] from the Feynman-Hellman theorem [11,12], (2) using theorems which relate the ordering of bound-state energy levels to certain properties of potentials [13], and (3) taking advantage of regularities in known hadron masses to obtain estimates of as-yet undiscovered hadrons using either interpolation [9] or semi empirical mass formulas [14,15].

We exploit these methods to obtain constraints on quark and hadron masses. We also provide new theoretical justification for the methods we use and make predictions for the masses of as-yet undiscovered mesons and baryons. We devote considerable effort to making what we believe is a good prediction for the mass of the $B_c^*$ vector meson, using the Feynman-Hellmann theorem. We also discuss in some detail the ground-state pseudoscalar mesons and the ground-state baryons (both spin $\frac{3}{2}$ and spin $\frac{1}{2}$), because additional issues arise in these cases. We discuss only briefly the excited vector mesons and $P$-wave mesons (tensors, axial vectors, and scalars), although we give some predictions in these cases as well.

## II. THE FEYNMAN-HELLMANN THEOREM

Some years ago, Feynman [11] and Hellmann [12] independently showed that if a Hamiltonian $H$ depends on a parameter $\lambda$, then the bound-state energy eigenvalues $E(\lambda)$ vary with $\lambda$ according to the formula

$$\partial E/\partial\lambda = \langle \partial H/\partial\lambda\rangle \ , \tag{1}$$

where the expectation value is taken with respect to the normalized eigenfunction belonging to $E$. The Feynman-Hellmann theorem was applied to quarkonium physics [1,10], with $\lambda = \mu$, the reduced mass of the system. For example, Quigg and Rosner [1] applied the theorem to the nonrelativistic Hamiltonian $H = p^2/(2\mu) + V$, where $V$ is an interaction which is assumed to be flavor independent, and therefore independent of $\mu$. Then

$$\partial E/\partial\mu = -\langle p^2\rangle/(2\mu^2) < 0 \ ; \tag{2}$$

i.e., $E$ decreases monotonically as $\mu$ increases because $p^2$ is a positive definite operator. Of course, if $V$ depends on $\mu$ but $\langle \partial V/\partial\mu\rangle \leq 0$, then $\partial E/\partial\mu < 0$ still remains valid.

Even in the case of some many-body Hamiltonians with relativistic kinematics, the Feynman-Hellmann theorem may be applied to give useful information about how eigenenergies change when constituent masses $m_i$ change [16]. As an example, we consider a Hamiltonian $H$, given by

$$H = \sum_i [(\mathbf{p}_i^2 + m_i^2)^{1/2} - m_i] + V(\mathbf{r}_1, \ldots, \mathbf{r}_n; m_1, \ldots, m_n) \ , \tag{3}$$

where we have let the interaction $V$ depend explicitly on the $m_i$ ($i = 1, 2, \ldots, n$). The mass $M$ of a hadron is given in terms of the energy eigenvalue $E$ of this Hamiltonian by

$$M = E + \sum m_i \ . \tag{4}$$

Taking the partial derivative with respect to $m_i$ and using (1) we obtain

$$\partial_E/\partial m_i = \langle m_i/(p_i^2 + m_i^2)^{1/2}\rangle - 1 + \langle \partial V/\partial m_i\rangle \ . \tag{5}$$

We can see from Eq. (5) that if

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$$\langle \partial V/\partial m_i \rangle \leq 0 \ , \tag{6}$$

then

$$\partial E/\partial m_i < 0, \quad m_i = 1, 2, \ldots, n \ . \tag{7}$$

These inequalities are a generalization of (2).

In the remainder of this section we restrict ourselves to the case in which (6) and (7) both hold, so that an increase in one or more $m_i$ leads to a decrease in $E$. We generalize the definition of $\mu$ to be

$$\mu^{-1} = \sum m_i^{-1} \ . \tag{8}$$

We now note that an increase in one or more $m_i$ results in an increase in $\mu$ as well as a decrease in $E$. *Under these circumstances*, a change $\delta\mu$ results in a change $\delta E$ in the opposite direction. (We use the symbol $\delta$ to indicate these changes because we are not specifying precisely what quantities are held constant as $\mu$ varies.) It follows that $E$ will be monotonically decreasing with increasing $\mu$, provided the increase in $\mu$ arises from an increase in one or more $m_i$. We can state this result in the form

$$\delta E/\delta\mu < 0 \ , \tag{9}$$

*if the changes in the $m_i$ are all in the same direction.* The inequality (9) turns out to be a powerful tool for obtaining constraints on quark and hadron masses. As we shall see in Sec. V, this inequality holds empirically for vector mesons even in the absence of the restriction that all $m_i$ change in the same direction.

For a two-body Hamiltonian of the form (3) with a flavor-independent potential, we can say something more. Consider $E$ as a function of $\mu$ and $\mathcal{M}$, where $\mathcal{M} = m_1 + m_2$. We can then show explicitly that

$$\partial E/\partial\mu < 0, \quad \partial E/\partial\mathcal{M} < 0 \ . \tag{10}$$

It follows that even if $m_1$ and $m_2$ change in opposite directions, if $\mu$ increases and $\mathcal{M}$ either remains constant or increases, then $E$ decreases. This result holds even in the presence of some flavor-dependent interactions, including a colormagnetic interaction for vector mesons, as we shall see in the next section.

## III. APPLICATION TO MESON AND BARYON EIGENENERGIES

We adopt a constituent quark picture, assuming that a meson is composed of a quark and an antiquark, and a baryon is composed of three quarks. We confine ourselves to hadrons containing $u$, $d$, $s$, $c$, and $b$ quarks. Furthermore, we neglect any violation of isospin and let $m_u = m_d = m_q$. As usual, we assume that the quark masses $m_i$ satisfy the inequalities

$$m_q < m_s < m_c < m_b \ . \tag{11}$$

In order to apply (7), and therefore (9), to hadrons, we need to discuss for which hadrons (6) is likely to hold. The interaction $V$ can be written as $V_0 + V'$, where $V_0$ is independent of quark flavors and $V'$ depends on flavor. The term $V_0$ is the static quark-antiquark potential, which is commonly assumed [17] to contain a Coulomb-like term, an approximately linear confining term, and a constant term, all independent of flavor. In the Fermi-Breit approximation, $V'$ contains both spin-dependent and spin-independent terms which are explicitly functions of flavor through the quark masses. However, most phenomenological treatments of quarkonia have not needed the Fermi-Breit spin-independent term [5], and we neglect it here. In states with zero orbital angular momentum, the expectation values of the tensor and spin-orbit terms of the Fermi-Breit interaction vanish, leaving the colormagnetic interaction as the only spin-dependent term. We write the colormagnetic term $V_{\rm cm}$ in the form [18]

$$V_{\rm cm} = -\sum_{i<j} \lambda_i \cdot \lambda_j f(r_{ij}) \sigma_i \cdot \sigma_j / (m_i m_j) \ , \tag{12}$$

where $\sigma_i, \sigma_j$ are Pauli spin matrices, $\lambda_i, \lambda_j$ are Gell-Mann SU(3) matrices, and $f(r_{ij})$ ($r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$) are positive definite operators.

If $V = V_0 + V_{\rm cm}$ we obtain

$$\langle \partial V/\partial m_i \rangle = \sum_{j\neq i} \langle f(r_{ij}) \rangle \langle \lambda_i \cdot \lambda_j \rangle \langle \sigma_i \cdot \sigma_j \rangle / (m_i^2 m_j) \ . \tag{13}$$

The quantity $\langle \lambda_i \cdot \lambda_j \rangle$ is negative for a quark-antiquark pair in a meson ($-\frac{16}{3}$) and for all quark pairs in a baryon ($-\frac{8}{3}$). Also, $\langle \sigma_i \cdot \sigma_j \rangle$ is $-3$ if two quarks (or a quark and antiquark) are in a spin-zero state, and 1 if they are in a spin-one state. Now in vector mesons and spin-$\frac{3}{2}$ baryons we have $\langle \sigma_i \cdot \sigma_j \rangle = 1$ for all quark pairs. Then we see from (13) that the eigenenergies of these hadrons satisfy (6). We therefore expect the energy eigenvalues of vector mesons and spin-$\frac{3}{2}$ baryons to satisfy (6), and therefore (7) as well. In the two-body case (vector mesons), (12) can be written

$$V_{\rm cm} = \tfrac{16}{3} f(r_{12}) / (\mu\mathcal{M}) \ . \tag{14}$$

We can see explicitly from (14) that $\langle \partial V/\partial\mu \rangle < 0$, $\langle \partial V/\partial\mathcal{M} \rangle < 0$, so that (10) holds, as we stated in the previous section.

For pseudoscalar mesons the sign of the colormagnetic term is negative, and the interaction violates (6). For spin-$\frac{1}{2}$ baryons, the three colormagnetic terms in (12) are either all $\leq 0$ or one term is positive and two are negative, so that (6) is sometimes violated. Thus, we expect that pseudoscalar mesons and spin-$\frac{1}{2}$ baryons might violate (7) for small $m_i$, where the contribution from (13) is large and positive.

In addition to $V_{\rm cm}$, terms arising from instantons may contribute to $V'$. These terms are apparently important in states in which two quarks (or a quark and an antiquark) have spin and orbital angular momentum zero (pseudoscalar mesons and spin-$\frac{1}{2}$ baryons) [19]. Instantons tend to mix the wave functions of certain mesons, like the $\eta$ and $\eta'$ (which contain both $q\bar{q}$ and $s\bar{s}$ in their wave functions, and perhaps some glueball admixture as well). Such states are unsuitable for our scheme, as, in order to compute the reduced mass of a system, we must know its quark content.

If we confine ourselves to mesons containing $q(=u,d)$, $s$, $c$, and $b$ quarks, then 10 different ground-state vector mesons and 20 different ground-state baryons of spin $\frac{3}{2}$ exist. Of the former, 9 are experimentally known; of the latter, only 4 are known, none of which contains a heavy quark.

Let a particular energy eigenvalue of a meson containing a quark $i$ and an antiquark $j$ be $E_{ij}$, where we suppress an index labeling which eigenvalue we are referring to. Likewise, we denote an energy eigenvalue of a baryon by $E_{ijk}$. If we replace a single quark by a heavier quark, then, because of (7), the eigenenergy of the new hadron will be smaller than that of the old one. By continuing this process we obtain chains of inequalities among the $E_{ij}$ and also among the $E_{ijk}$. We consider the *longest chains* of inequalities for which (7) and (9) hold. A longest chain arises when we start with the lightest hadron (containing only $q$-type quarks) and replace each of its quarks one at a time by the next heavier quark, each time obtaining a different hadron (i.e., we replace $q$ by $s$, $s$ by $c$, and $c$ by $b$). A longest meson chain contains 7 eigenenergies, and a longest baryon chain contains 10. If there are $n+1$ different kinds of quarks taken $\nu$ at a time ($\nu = 2$ for mesons and 3 for baryons), then a longest chain contains $n\nu + 1$ eigenenergies. In our case, $n+1 = 4$ because we have omitted the $t$ quark and have not distinguished between the $u$ and $d$ quarks.

There are five different longest chains for mesons, all of which have the same first two and last two eigenenergies. At least one of the three intermediate eigenenergies differs in each of the five chains. One longest chain for mesons is

$$E_{bb} < E_{bc} < E_{cc} < E_{cs} < E_{ss} < E_{sq} < E_{qq} \ . \tag{15}$$

The four other longest chains have intermediate eigenenergies

$$\begin{aligned} &E_{cc}, E_{cs}, E_{cq}, \quad E_{bs}, E_{cs}, E_{ss} \ , \\ &E_{bs}, E_{cs}, E_{cq}, \quad E_{bs}, E_{bq}, E_{cq} \ . \end{aligned} \tag{16}$$

It follows that (9) holds for the mesons with eigenenergies as in (15) or (16), whereas this may not necessarily be the case when one quark mass increases and the other decreases.

We next turn to the spin-$\frac{3}{2}$ baryons and denote their energy eigenvalues by $E_{ijk}$. From our discussion we can write down a number of longest chains. One such chain is

$$E_{bbb} < E_{bbc} < E_{bcc} < E_{bcs} < E_{bss} < E_{css} < E_{sss} < E_{ssq} < E_{sqq} < E_{qqq} \ . \tag{17}$$

All longest distinct baryon chains, of which we have counted 42, contain the same first two and last two eigenenergies.

In both the meson and baryon cases, each longest chain can be considered to be a path in a simple tree diagram, which depends on the number of different quarks $n+1$ and the number of quarks $\nu$ in the hadron. One can obtain the number $N(n,\nu)$ of distinct longest chains by counting the different paths in each diagram. More generally, it can be shown that $N(n,\nu) = N(\nu,n)$ and is equal to the dimensionality of an irreducible representation of the permutation group corresponding to a Young Tableau with $\nu$ rows, each containing $n$ boxes. The explicit formula is

$$N(n,\nu) = (n\nu)! \frac{\prod_{i=1}^{\nu}(\nu-i)!}{\prod_{i=1}^{\nu}(n+\nu-i)!} \ . \tag{18}$$

## IV. CONSTRAINTS ON QUARK MASS DIFFERENCES

As has been pointed out by a number of authors [1,2,20,21] Eq. (4) can be used together with (9) to obtain constraints on quark mass differences in the form of inequalities. What is new in our treatment is our justification of the use of (9) for longest chains of vector meson and spin-$\frac{3}{2}$ baryon eigenenergies. In particular, this means that, unlike some other authors, we do not need to use as input spin-averaged hadron masses, which are normally calculated in a perturbative approximation. In this connection, Lipkin [10] has pointed out that, strictly speaking, the Feynman-Hellmann theorem should not be applied to spin-averaged masses because they are not eigenvalues of the Hamiltonian.

In order to obtain inequalities among quark masses we use the experimental values of hadron masses, including the $\rho$, $K^*$, $\phi$, $D^*$, $D_s^*$, $B^*$, $B_s^*$, $J/\psi$, and $\Upsilon$ mesons (of the 10 vector mesons, only the $B_c^*$ is missing) and the $\Delta$, $\Sigma^*$, $\Xi^*$, and $\Omega$ baryons (only 4 of the 20 spin-$\frac{3}{2}$ baryons are known). When the isospin of a state is greater than zero we average over the members of the isospin multiplet. For the $\bar{q}q$ state we believe it is better to choose the $\rho$ than the $\omega$ because the latter might have a small admixture of $\bar{s}s$, whereas such an admixture in the $\rho$ violates isospin. Note that in our scheme the $\omega$ is degenerate with the $\rho$, although it is actually 15 MeV heavier. In fact, in not distinguishing between $u$ and $d$ quarks, we consider the $\omega$ and $\rho$ together as just a single one of the ten distinct vector mesons in our scheme, which is concerned only with masses. We have given an *a priori* reason for choosing the $\rho$ instead of the $\omega$. In the next paragraph we give an *a posteriori* reason. However, if instead of choosing the $\rho$, we choose the $\omega$, our predictions are not appreciably affected. All the meson masses are taken from the Particle Data Group [22], except the mass of the $B_s^*$, which comes from two recent measurements [23,24] of the mass of the $B_s$ plus a measurement of $m(B_s^*) - m(B_s)$ (which needs confirmation) quoted in [22].

Using the masses of the observed ground-state vector mesons we obtain the inequalities

$$\begin{aligned} m_s - m_q &> M(K^*) - M(\rho) = 126 \pm 4 \text{ MeV} , \\ m_c - m_s &> M(D^*) - M(K^*) = 1115 \pm 4 \text{ MeV} , \\ m_b - m_c &> M(B^*) - M(D^*) = 3316 \pm 6 \text{ MeV} . \end{aligned} \quad (19)$$

The errors are partly experimental and partly due to our assumption of isospin invariance. If we substitute the $\omega$ for the $\rho$, the right-hand side of the first of these inequalities becomes smaller by 15 MeV. However, we can regain the stronger inequality (larger right-hand side) by considering

$$m_s - m_q > M(\phi) - M(K^*) = 125 \pm 4 \text{ MeV} . \quad (20)$$

This fact gives an *a posteriori* justification of our decision to use the $\rho$, rather than the $\omega$, in our set of vector mesons.

Apparently stronger inequalities have been obtained previously with spin-averaged values for vector and pseudoscalar mesons. The results in MeV are [20]

$$\begin{aligned} m_s - m_q &> 184 \pm 4, \\ m_c - m_s &> 1180 \pm 4, \\ m_b - m_c &> 3343 \pm 4 . \end{aligned} \quad (21)$$

However, aside from Lipkin's [10] objection to using spin-averaged masses, the spin averaging process relies on a perturbative treatment of the spin-dependent forces, which may not be justified for light mesons. In fact, we shall see in Sec. VI that when we vary the quark masses to obtain a best fit to the data, the quark mass differences violate (21) but satisfy (19). Still other authors [21] have obtained inequalities among quark masses using a variety of assumptions, but not if the Hamiltonian contains both relativistic kinematics and a flavor-dependent interaction.

As we have remarked, only four ground-state baryons of spin $\frac{3}{2}$ have been observed thus far. These lead only to inequalities for $m_s - m_q$. The strongest of these is

$$m_s - m_q > M(\Sigma^*) - M(\Delta) = 153 \pm 4 \text{ MeV} . \quad (22)$$

This inequality is stronger than the corresponding inequality we obtained from mesons. As we shall see in Sec. VIII, in order to obtain a best fit to the baryon data we must use quark mass differences for baryons which are up to 35 MeV larger than the corresponding quark mass differences for mesons.

## V. SATISFYING THE QUARK MASS CONSTRAINTS

Many different sets of quark constituent masses are used in the literature, most of them obtained from fits to spectroscopic data. We show in Table I a selection of these sets [7,9,14,15,17,18,21,25–28], including in the first row the set that we use in this work for vector masses. (In the next section we explain how we arrive at this mass set). We can see from Table I that the sets of masses given in the first seven rows satisfy the inequalities given in (19), whereas the sets of masses given in the last five rows violate one or more of these inequalities. Therefore, if one calculates the vector meson masses with any one of the last five sets and a potential which is flavor independent except for a conventional colormagnetic term, one will obtain results in disagreement with experiment. This disagreement will occur independently of whether one uses a nonrelativistic Schrödinger equation or a wave equation with relativistic kinematics of the form given in Eq. (3).

TABLE I. Values of quark constituent masses in MeV for calculating meson energy eigenvalues from experimental values of their masses. We show in the first row the (rounded off) values of the quark masses used in this work and, for comparison, values used by some other authors in subsequent rows.

| Reference | $m_q$ | $m_s$ | $m_c$ | $m_b$ |
|---|---|---|---|---|
| This work | 300 | 440 | 1590 | 4920 |
| [9][a] | 263 | 404 | 1543 | 4876 |
| [15] | 300 | 500 | 1800 | 5200 |
| [18] | 220 | 419 | 1628 | 4977 |
| [21] | 310 | 620 | 1910 | 5270 |
| [25] | 336 | 510 | 1680 | 5000 |
| [26] | 337 | 600 | 1870 | 5259 |
| [7] | 350 | 500 | 1500 | 4700 |
| [14][a] | 270 | 600 | 1700 | 5000 |
| [17] | 335 | 450 | 1840 | 5170 |
| [27] | 330 | 550 | 1650 | 4715 |
| [28][a] | 325 | 602 | 1320 | 4749 |

[a]One of several sets of quark masses in this reference.

We use the *experimental values of* the vector meson masses together with the input values of the quark masses to calculate the meson eigenenergies $E_{ij}$ from Eq. (4). We can plot these eigenenergies as a function of $\mu$. For illustrative purposes we use four different sets of input quark masses from Table I, and show in Fig. 1 how $E$ varies as a function of $\mu$ for the ground-state vector mesons. For completeness we include mesons containing $q\bar{b}$, $q\bar{c}$, and $s\bar{b}$ as well as $s\bar{c}$, $s\bar{s}$, and $c\bar{c}$ in order to see how $E$ varies with $\mu$ when $m_1$ increases and $m_2$ decreases. In Figs. 1(a) and 1(b), the quark masses satisfy all the inequalities (19), whereas in Figs. 1(c) and 1(d), the quark masses violate at least one of these inequalities.

We see from Figs. 1(a) and 1(b) that, when the quark masses satisfy (19), $E$ appears indeed to be a monotonically decreasing function of $\mu$ for the vector mesons. From Figs. 1(c) and 1(d) we see that, with quark masses violating (19), $E$, as obtained from the *observed meson masses*, is not monotonically decreasing as a function of $\mu$. The sets of quark masses used in (c) and (d) seem *a priori* as reasonable as those of sets (a) and (b). However, if we use the quark masses of (c) or (d) with a flavor-independent potential and a conventional colormagnetic term, the *calculated* values of $E$ must be monotonically decreasing and therefore in disagreement with experiment.

We see from Figs. 1(a) and 1(b) that, not only is $E$ monotonically decreasing as a function of $\mu$, but is also concave upward. We cannot prove concavity from the

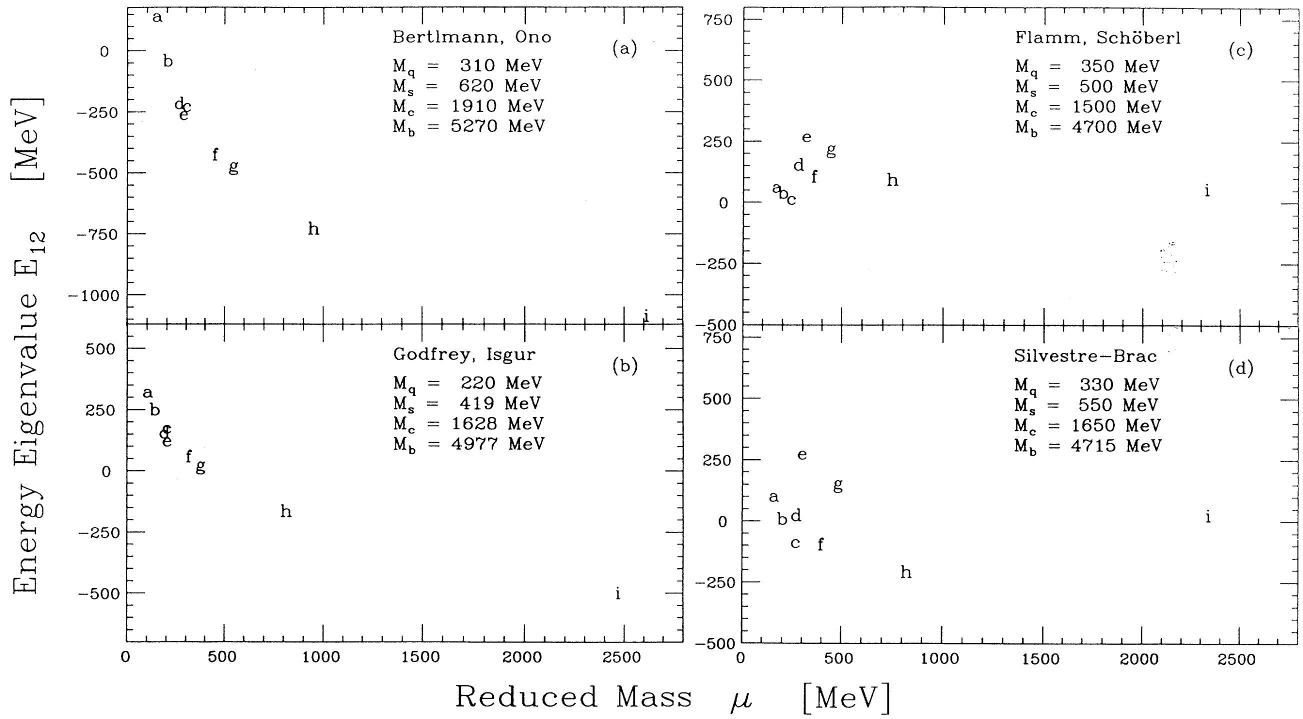


FIG. 1. Energy eigenvalues of vector mesons using experimental masses from Refs. [22–24] in conjunction with four sets of quark masses from Table I: (a) from Ref. [21], (b) from Ref. [18], (c) from Ref. [7], and (d) from Ref. [27]. In the figure, the letters $a$ through $i$ stand for $\rho, K^*, \phi, D^*, B^*, D_s^*, B_s^*, J/\psi$, and $\Upsilon$, respectively.

Feynman-Hellmann theorem. However, in the two-body nonrelativistic case, we can show for a power-law potential of the form

$$V = \alpha r^{\beta}, \quad \alpha\beta > 0, \quad \beta > -2 , \tag{23}$$

that the concavity condition

$$\partial^2 E/\partial\mu^2 > 0 \tag{24}$$

is true. This result follows directly from the scaling property of the Schrödinger equation with a power-law potential [1]. The fact that (24) holds for a power-law potential is relevant because, as has been emphasized by Martin [29,30], the quarkonium static potential can be well approximated by a power law. If we include a Fermi-Breit term in the potential of form given by Eq. (12), (24) remains valid. Of course, we are interested in the curvature of $E$ as a function of $\mu$ without concerning ourselves with how $\mathcal{M}$ varies. In the presence of the colormagnetic interaction (12) we can say that the curvature is positive, provided that as $\mu$ increases, $\mathcal{M}$ does not decrease. It has been shown [31,32] that in the two-body nonrelativistic case with a flavor-independent potential, the ground-state energy satisfies

$$\partial^2 E/\partial\lambda^2 < 0 , \tag{25}$$

where $\lambda = 1/\mu$. Not only is $E$ concave upward as a function of $\mu$ in a class of nonrelativistic two-body models, but it turns out that

$$\delta^2 E/\delta\mu^2 > 0 \tag{26}$$

holds empirically for vector mesons [see Figs. 1(a) and 1(b)] and also for spin-$\frac{3}{2}$ baryons.

## VI. PREDICTING THE $B_C^*$ MASS

Of the 10 vector mesons in our scheme, only the $B_c^*$ has not yet been seen. We can use the inequalities (9) and (26) together with the experimental vector meson masses and Eq. (4) to estimate the mass of the $B_c^*$ by interpolation. We do this by assuming that $E(\mu)$ can be approximated by a simple curve containing only a few parameters. The simplest curve (containing only two parameters) is a straight line, but a straight line violates the concavity condition (26). We therefore approximate $E(\mu)$ with a three-parameter curve. We emphasize that the functional form of the curve has no theoretical significance other than that it satisfies the inequalities (9) and (26). In addition to the three parameters of the curve we have four additional parameters, namely, the quark masses $m_q$, $m_s$, $m_c$, and $m_b$. Using a given three-parameter curve we vary the seven parameters in order to obtain a best fit to the eigenenergies $E_{ij}$. We then obtain the value of $E_{bc}$ from the fitted curve and use the fitted masses $m_c$ and $m_b$ to obtain $M(B_c^*)$.

We at once encounter a difficulty in our scheme: namely, that a longest chain of meson eigenenergies contains only seven members, and one of these ($E_{bc}$) is unknown. Therefore, if we use a longest chain, we have seven parameters and only six data points so that the parameters are not uniquely determined. We overcome this difficulty by using all nine known meson eigenenergies, assuming that (9) and (26) hold even in this case. We find that our assumption is consistent with experiment: namely, we can find a set of quark masses such that the meson eigenenergies, as calculated from the experimental values of the meson masses with the aid of (4), satisfy (9) and (26). The procedure of including all nine known meson masses constrains the parameters much more than using only six masses. However, even when we use nine data points, it turns out that the quark mass differences are much more constrained than the quark masses themselves.

Kwong and Rosner [9] previously used the interpolation method with two three-parameter curves (quadratic and Padé), although without any theoretical justification of the inequalities (9) and (26). We have used three different three-parameter curves: an exponential

$$E = a\ \exp(-\mu/b) - c\ , \tag{27}$$

a quadratic

$$E = a + b\mu + c\mu^2\ , \tag{28}$$

and a hyperbolic (or Padé)

$$E = a/(\mu + b) - c\ , \tag{29}$$

where $a$, $b$, and $c$ are parameters to be varied. In principle, these parameters are functions of the sum of the quark masses as well as a function of $\mu$, but from our previous discussion, we expect $E$ to be decreasing when plotted as a function of $\mu$ with $a$, $b$, and $c$ independent of any other masses in the problem. We obtain comparable fits to the data with all three curves, and the meson energies are quite stable to our choice of functions.

We show in Fig. 2 our fit to the vector meson energies, with an exponential curve as an example, and the set of quark masses given in the first row of Table I. These quark masses are rounded to the nearest 10 MeV and are based on a somewhat arbitrary choice of 300 MeV for the mass of the $u$ and $d$ quarks. We can get comparable fits to the data for quark masses which differ from our choices by 100 MeV or more, but the mass differences are much more constrained. With the exponential fit, the values of the parameters with our quark masses are

$$\begin{aligned} a &= 754\ \text{MeV}\ , \\ b &= 1375\ \text{MeV}\ , \\ c &= 506\ \text{MeV}\ . \end{aligned} \tag{30}$$

Our result for the $B_c^*$ mass is

$$M(B_c^*) = 6320 \pm 20\ \text{MeV}\ . \tag{31}$$

We have estimated the theoretical error partly from the spread in values obtained using the different functional forms for $E(\mu)$ given in Eqs. (27)–(29) and partly from values obtained with different longest chains and various constraints on the quark masses. Our quoted errors in Eq. (31) and our subsequent predictions reflect the stability inherent in our procedures. Our predicted value of the $B_c^*$ mass is given in Table II in the first row of column 4.

Our value of $M(B_c^*)$ is more stable to the choice of curve than the result of Kwong and Rosner [9]: $6284 < M(B_c^*) < 6349$ MeV. Perhaps one reason for this is that we differ from those authors in the choice of the function $\chi^2$ to be minimized. We choose

$$\chi^2 = \sum [E(\mu) - E(\text{expt})]^2/(\Delta M)^2\ , \tag{32}$$

where $E(\mu)$ is obtained from one of the three curves, $E(\text{expt})$ are the experimental eigenenergies obtained with the help of (4), and $\Delta M$ are the experimental errors in the meson masses, except that we have taken a minimum error of 1 MeV and increased some errors to take isospin mass splittings into account.

Other authors, using potential models, have obtained similar values of the $B_c^*$ mass. For example, Martin [30], using a power-law potential, obtained a value of

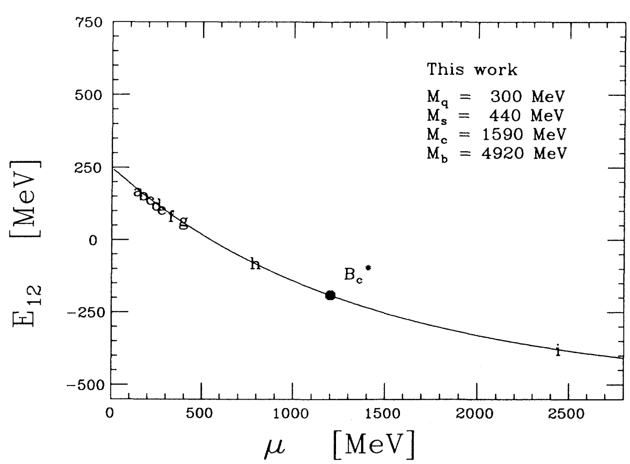


FIG. 2. Energy eigenvalues of vector mesons using our quark masses from the first row of Table I and the experimental masses from Refs. [22–24]. The letters stand for the same mesons as in Fig. 1, and the solid circle is our prediction for the $B_c^*$. The solid line is a fit to the vector meson data with an exponential form, Eq. (27), with parameters given in Eq. (30).

TABLE II. Predicted masses of as-yet unobserved $B_c(\bar{b}c)$ mesons. In column 4 we show predictions for the ground-state vector and $P$-wave mesons and upper limits for two excited vectors from interpolation of the energy eigenvalues, using the Feynman-Hellmann theorem. In column 5 we show the pseudoscalar mass obtained from a semiempirical mass formula, and excited vector and $P$-wave states from interpolation of mass differences.

| Name | Spin-parity $J^P$ | $n^{2S+1}L_J$ | Mass(MeV) | Mass (MeV) |
|---|---|---|---|---|
| $B_c^*$ | $1^-$ | $1\ ^3S_1$ | $6320 \pm 20$ | |
| $B_c$ | $0^-$ | $1\ ^1S_0$ | | $6255 \pm 30$ |
| $B_c^*$ | $1^-$ | $2\ ^3S_1$ | <6940 | $6900 \pm 20$ |
| $B_c^*$ | $1^-$ | $3\ ^3S_1$ | <7290 | $7250 \pm 20$ |
| $B_c^*$ | $0^+$ | $1\ ^3P_0$ | $6630 \pm 40$ | $6660 \pm 30$ |
| $B_c^*$ | $1^+$ | $1\ ^3P_1$ | $6730 \pm 40$ | $6740 \pm 30$ |
| $B_c^*$ | $2^+$ | $1\ ^3P_2$ | $6760 \pm 40$ | $6780 \pm 30$ |

6318 MeV, and Eichten and Quigg [33], using various potentials, obtained values between 6319 and 6343 MeV. Bagan *et al.* [34] have averaged a variety of other people's results to obtain 6330 ± 20. On the other hand, some authors have obtained quite different values of the mass of the $B_c^*$. For example, Jain and Munczek [35] find $M(B_c^*) = 6277$ MeV.

## VII. OTHER MESON MASS PREDICTIONS

As we have already remarked we do not expect the pseudoscalar meson energy eigenvalues to be monotonically decreasing as a function of $\mu$. However, once we have a prediction for the $B_c^*$ mass, we can obtain estimates for pseudoscalar meson masses in other ways. We use semiempirical mass formulas [14,15] for the splitting between vector and pseudoscalar states. These semiempirical formulas are based on the colormagnetic interaction as given by the Fermi-Breit theory. However, the formulas take into account the fact that the strong-interaction coupling constant runs. The formulas also make empirical corrections which depend on quark masses so as to get improved agreement with known data compared to the Fermi-Breit formula. The semiempirical formulas then may be used to predict mass splittings in cases where experimental data are absent.

Following [15] we take the hyperfine splitting in mesons to be given by

$$M_V - M_S = p\alpha_s(2\mu)\mu^q/(m_1 + m_2) , \qquad (33)$$

where $m_1$ and $m_2$ are the constituent quark masses, and we have determined the two free parameters $p$ and $q$ from a fit to the experimental splittings to be $q = 0.642$ and $p = 1.917$ $(\text{GeV})^{2-q}$. The running coupling constant is given by

$$\alpha_s(Q) = \frac{4\pi}{\beta_0 t + (\beta_1/\beta_0)\ln t} , \qquad (34)$$

where

$$t = \ln(Q^2/\Lambda^2_{\text{QCD}}) ,$$
$$\beta_0 = 11 - 2n_f/3 , \qquad (35)$$
$$\beta_1 = 102 - 38n_f/3 ,$$

with $\Lambda_{\text{QCD}} = 100$ MeV, and $n_f = 4$. We use the quark masses of the first row of Table I rather than the masses in [15], so that our values of the parameters $q$ and $p$ differ a little from those in [15].

We show in Fig. 3 the eigenenergies of the pseudoscalar mesons, where we use the data [22] on pseudoscalar masses as input together with the quark masses obtained for the vector mesons. We omit the $\eta$ and $\eta'$ mesons, because, as we have already remarked, they are mixed states of uncertain quark content. We also give in Fig. 3 the eigenenergies of the pseudoscalars obtained from the corresponding vectors with the aid of the semiempirical mass formula (33). We see from Fig. 3 that, except (as expected) for the pion, the eigenenergies obtained from the observed masses and from the semiempirical formulas are in remarkably good agreement. Note that the eigenenergies of the light pseudoscalars violate the condition that $E(\mu)$ be monotonically decreasing. We expect this violation because the colormagnetic term for light pseudoscalars gives a large positive contribution to $\partial E/\partial\mu$.

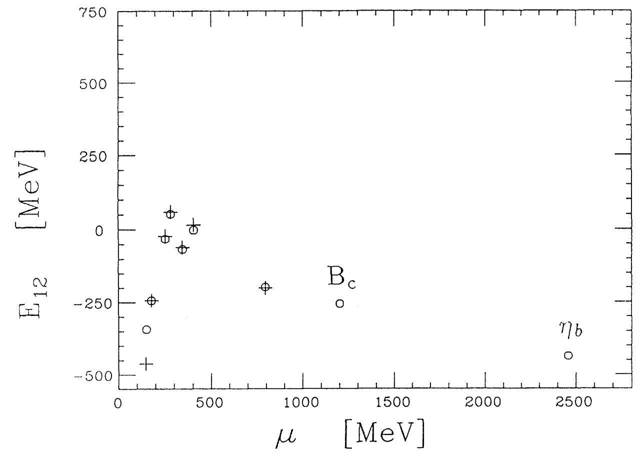


FIG. 3. Eigenenergies of the pseudoscalar mesons obtained from the masses of the Particle Data Group [22] (crosses) compared with the eigenenergies from the vector meson masses and the semiempirical mass formula of Eq. (33) (open circles). The quark masses of the first row of Table I were used to obtain eigenenergies from masses. In order of increasing $\mu$ are $\pi$, $K$, $D$, $B$, $D_s$, $B_s$, $\eta_c$, $B_c$, and $\eta_b$.

Using the semiempirical mass formula we obtain a splitting in the $B_c$ system of 65 ± 10 MeV, and in the $\bar{b}b$ system of 55 ± 10 MeV. We estimate that the masses of the $B_c$ and $\eta_b$ are

$$M(B_c) = 6255 \pm 30 \text{ MeV} ,$$
$$M(\eta_b) = 9405 \pm 15 \text{ MeV} . \qquad (36)$$

We give the mass of the $B_c$ in Table II.

We next turn to excited vector meson states, for which the data are considerably poorer than for the ground states. Furthermore, complications might arise from possible mixing with four-quark, hybrid, and glueball states. For example, the excited $\rho(1465)$ and $\omega(1394)$ states differ in mass by about 70 MeV, although they ought to be degenerate in our model. We believe this difference indicates appreciable mixing. Similar considerations apply to light $P$-wave mesons. Therefore, we use only the charmonium and bottomonium data of the Particle Data Group [22] and confine ourselves to excited $B_c^*$ states. Because we have only two data points for each excited state (a $\psi$ and an $\Upsilon$) we cannot do better than use a linear fit to predict the masses of missing states, again using the quark masses in the first row of Table I. The problem with a linear fit is that a straight line is not concave upward, so that the predictions made in this fashion should be regarded as upper limits. We show in column 4 of Table II our predictions for the upper limits of two excited vector meson $B_c^*$ states.

Quigg [36] suggested that it might be better to interpolate between mass differences, since these are considerably smaller than the masses themselves. With this procedure we have no theoretical reason to reject a linear interpolation. We show in column 5 of Table II our predictions for the masses of vector $B_c^*$ excited states using linear interpolation of the mass differences between corresponding states in the $c\bar{c}$ and $b\bar{b}$ systems. Again the errors include not only statistical errors but an estimate of the errors associated with the procedures. Note that the predicted masses in column 5 are less than the upper limits of column 4, i.e., the latter are indeed an upper bound.

Turning to the $P$-wave mesons we are not able to show analytically that the sum of the Fermi-Breit tensor and spin-orbit interactions satisfies the inequality (6). Nevertheless, it turns out empirically that if we use the same values of quark masses as for the vectors (row 1 of Table I), the eigenenergies of the tensor ($J^P = 2^+$), axial vector ($J^P = 1^+$), and scalar ($J^P = 0^+$) mesons satisfy (9) and (26). We exploit this fact to fit separate three-parameter exponential curves to the tensors, axial vectors, and scalars so as to obtain predictions for the $B_c$ $P$-wave states. These are shown in column 4 of Table II. Our estimated errors are rather large because of deviations of the curve from the eigenenergies of the known mesons.

We can also use linear interpolation between $\bar{c}c$ and $\bar{b}b$ states to obtain the masses of $B_c$ $P$-wave states. These are also given in column 5 of Table II together with estimated errors. We see from Table II that the predictions for the $P$-wave mesons in columns 4 and 5 agree within the errors.

## VIII. BARYON MASSES

Because the masses of only four baryons of spin $\frac{3}{2}$ are known experimentally and none of these contains any heavy quarks, the Feynman-Hellmann theorem *by itself* does not enable us to make useful predictions of the masses of any baryons containing heavy quarks. However, if we use the Feynman-Hellmann theorem in conjunction with a semiempirical formula for the colormagnetic splitting in baryons [15], we are able to make some useful estimates of unknown masses. The reason is that the masses of the $\Lambda_c$ (quark content $qqc$), $\Sigma_c$ ($qqc$), $\Xi_c$ ($qsc$), and $\Lambda_b$ ($qqb$) spin-$\frac{1}{2}$ baryons are known from experiment [22,37,38], so that we can estimate the masses of the corresponding spin-$\frac{3}{2}$ baryons from a semiempirical formula for the colormagnetic splitting in baryons [15]. We are then able to use a procedure analogous to that we used for mesons in order to obtain estimates of the masses of unknown baryons.

The expression (33) for mesons can be generalized to baryons as follows [15]. We order the quarks so that if two quarks have the same flavor, they are chosen to be the first two; if all quark flavors are different, then the first two are the lightest. We denote by $M^*$ the mass of the ground-state spin-$\frac{3}{2}$ baryon, with $M_S$ the mass of the ground-state spin-$\frac{1}{2}$ baryon in which the first two quarks have spin 1, and with $M_A$ the mass of the spin-$\frac{1}{2}$ baryon whose first two quarks are in a relative spin 0 state. We take the two-quark colormagnetic matrix elements as

$$8R_{ij,k} = F_{ij,k}\, p\alpha_s(2\mu_{ij})\mu_{ij}^q/(m_i + m_j) \ , \qquad (37)$$

with

$$F_{ij,k} = [\mu_{ij} + x(\mu_{ik} + \mu_{jk})]/(\mu_{ij} + \mu_{ik} + \mu_{jk}) \qquad (38)$$

to simulate the shrinking of the wave function with increasing mass $m_k$ of the spectator quark. The expression for $F_{ij,k}$ in Eq. (38) is slightly different from that given in Ref. [15] and fits the data somewhat better. The parameters $n_f$ and $\Lambda_{\text{QCD}}$ are chosen to be the same as in the meson case, while $p$, $q$, and $x$ are adjustable.

Following [15] we write the ground-state baryon mass differences

$$\begin{aligned} M^* - M_S &= 3R_{13,2} + 3R_{23,1} \ , \\ M_S - M_A &= 4R_{12,3} - 2R_{13,2} - 2R_{23,1} \ . \end{aligned} \qquad (39)$$

Although the structure of (39) is motivated by perturbation theory we take these semiempirical mass formulas to be more generally applicable, the justification being the good agreement with observed baryon mass splittings.

A difficulty is that, unlike the meson case, where we determined the quark masses from a fit to the vector meson eigenenergies, we do not know *a priori* what input values to use for quark masses which will be best suited for baryons. Our procedure is to start with $m_q = 300$ MeV and the other quark masses taken at reasonable starting values, for example, with the values given in [15]. We then adjust the quark masses by an iteration procedure which we shall now describe.

(1) We use input quark masses and adjust the parameters $p$, $q$, and $x$ to get a best fit to the known colormagnetic splittings in baryons. We then use the semiempirical mass formula to calculate the masses of three spin-$\frac{3}{2}$ baryons ($\Sigma_c^*$, $\Xi_c^*$, and $\Sigma_b^*$) which are not known from experiment.

(2) We then use the Feynman-Hellmann theorem, analogously to the meson case; i.e., we adjust the parameters of a three-parameter curve and the quark masses $m_s, m_c$ and $m_b$ to get a best fit to the eigenenergies of the $\Delta$, $\Sigma^*$, $\Xi^*$, $\Omega$, $\Sigma_c^*$, $\Xi_c^*$, and $\Sigma_b^*$.

(3) We then use the new quark masses in the semiempirical mass formula and repeat steps (1) and (2).

In practice, this method rapidly converges. We find the best parameters of the semiempirical mass formula (39) are $p = 0.331\ (\text{GeV})^{2-q}$, $q = 0.417$, $x = 3.805$ when used with the following (rounded) quark masses for baryons in MeV.

$$m_q = 300,\ m_s = 475,\ m_c = 1640,\ m_b = 4990. \qquad (40)$$

The mass difference $m_s - m_q$ satisfies the inequality (22). The parameters of an exponential curve of form (27) turn out to be

$$a = 1307 \text{ MeV},\ b = 757 \text{ MeV},\ c = 813 \text{ MeV}. \qquad (41)$$

There are several reasons why our procedure for baryons is not as precise as that we used for mesons. First, the input "data" for baryons include three baryon masses which do not come from experiment but are only estimated from a mass formula. Second, even using these three baryons, we have only seven baryons to obtain the three parameters of a curve and the three quark masses. Third, we have to obtain unknown baryon masses by extrapolation, which is a less precise method than the interpolation method used for mesons.

Comparing the masses of Eq. (40) with those used for mesons (see the first row of Table I), we see that the quark masses which give a best fit to the baryons are (except for $m_q$, which was assumed to be the same) a little higher than those which lead to a best fit to the mesons. Because these quark masses are constituent masses, i.e., *effective* ones, there are no theoretical reasons why the masses determined from the baryons should coincide exactly with those determined from the mesons. If we insist that a single set of quark masses hold for both baryons and mesons, and vary these masses, our overall best fit to the hadron data is significantly poorer and our predictions have greater errors.

Using the baryon data only, we can predict the masses of as yet unobserved baryons from the Feynman-Hellmann theorem and the baryon semiempirical mass formulas. As we have remarked, the baryon masses, given in Table III, are obtained by extrapolation, rather than interpolation, so that the errors are larger than in the meson case. The errors in the masses of the $\Xi_b$ ($qsb$, spin $\frac{1}{2}$, antisymmetric in $qs$), $\Xi_b'$ (spin $\frac{1}{2}$, symmetric in $qs$), $\Xi_b^*$ (spin $\frac{3}{2}$), $\Omega_c$ ($ssc$, spin $\frac{1}{2}$), $\Omega_c^*$ (spin $\frac{3}{2}$), $\Omega_b$($ssb$, spin $\frac{1}{2}$), and $\Omega_b^*$ (spin $\frac{3}{2}$) arise partly because of the substantial error in the measurements to date [37,38] of the mass of the $\Lambda_b$. We believe that the following predicted mass differences (in MeV) are likely to have smaller errors than any of the masses given in Table III:

$$\begin{aligned} M(\Sigma_b) - M(\Lambda_b) &= 200 \pm 20 \ , \\ M(\Sigma_b^*) - M(\Lambda_b) &= 230 \pm 20 \ , \\ M(\Xi_b) - M(\Lambda_b) &= 190 \pm 30 \ , \\ M(\Xi_b') - M(\Lambda_b) &= 330 \pm 30 \ , \\ M(\Xi_b^*) - M(\Lambda_b) &= 360 \pm 30 \ . \end{aligned} \tag{42}$$

TABLE III. Predicted masses of as-yet unobserved baryons. In column 3 we show a prediction for a ground-state spin-$\frac{1}{2}$ baryon ($\Xi_b$) whose first two quarks have an antisymmetric spin wave function. (Antisymmetric $\Omega_c$ and $\Omega_b$ states do not exist in our picture.) Column 4 shows predictions for ground-state spin-$\frac{1}{2}$ baryons with symmetric spin wave function in the first two quarks. In column 5 we show predictions for the ground-state spin-$\frac{3}{2}$ baryons. See Eq. (42) for (we believe, more precise) estimates for baryon mass differences.

| Name | Quark content | $M_A$(MeV) | $M_S$ (MeV) | $M^*$ (MeV) |
|---|---|---|---|---|
| $\Lambda_b, \Sigma_b, \Sigma_b^*$ | $qqb$ | $5630 \pm 40$[a] | $5830 \pm 40$ | $5860 \pm 40$ |
| $\Xi_b, \Xi_b', \Xi_b^*$ | $qsb$ | $5820 \pm 40$ | $5960 \pm 40$ | $5990 \pm 40$ |
| $\Omega_c, \Omega_c^*$ | $ssc$ | — | $2710 \pm 50$ | $2770 \pm 50$ |
| $\Omega_b, \Omega_b^*$ | $ssb$ | — | $6070 \pm 60$ | $6100 \pm 60$ |

[a]Input.

It is interesting that our semiempirical mass formula makes the $\Sigma_b$ about 10 MeV heavier than the $\Xi_b$. However, the probable error is such that this is not a firm prediction. All we can really say is that the $\Sigma_b$ and $\Xi_b$ have masses which are very likely within 20 MeV of each other.

## IX. CONCLUSIONS

In conclusion, we have shown that the Feynman-Hellmann theorem leads to the inequality $\delta E/\delta\mu < 0$ for most ground-state vector mesons and spin-$\frac{3}{2}$ baryons, even in the presence of relativistic kinematics and a flavor-dependent colormagnetic interaction. We were not able to show this for hadron pairs which differ by one member of the pair containing both a heavier and a lighter quark than the other, but the result seems to be empirically true even in this case. This inequality and the concavity condition (26), provide theoretical justification for an interpolation method [9], which allows one to make a quantitative prediction about the mass of the $B_c^*$ and other mesons without assuming any specific functional form for the quark-antiquark interaction. We obtain the masses of still other mesons containing heavy quarks by using a semiempirical mass formula and by interpolating among mass differences.

For the baryons we can also use (9) and (26) to obtain predictions, but we need the semiempirical mass formula from the outset and also need to extrapolate the data in order to obtain useful results. Therefore, our predictions are not as precise as in the meson case. Our predicted baryon masses could be considerably improved if a more precise measurement were made of the mass of the $\Lambda_b$. Also, if the masses of the $\Sigma_c^*$ and $\Xi_c^*$ were measured, we would not have to rely so heavily on the semiempirical mass formula, and therefore could further improve our results. Nevertheless, because our baryon predictions require extrapolation of the data, rather than interpolation as in the meson case, our predicted baryon masses will continue to have a somewhat great uncertainty than our meson results.

In making our predictions of the values of heavy hadron masses, we have not had to assume an explicit form for the Hamiltonian, but only some general characteristics about its flavor dependence. Therefore, our results ought to have a greater generality than those based on specific models.

*Note added in proof.* In the full listings of the latest Review of Particle Properties [L. Montanet *et al.*, Phys. Rev. D **50**, 1171, (1994)], the mass of the $\Omega_c$ is given by $2710 \pm 5$ MeV, in agreement with our prediction in Table III.

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