Angular distribution functions in the decays of ψ' and ψ'' directly produced in unpolarized $\bar{p}p$ collisions

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(Received 17 August 1994)

We calculate the combined angular distribution of the two photons and of the electron in the triple cascade process $\bar{p}p \to \psi', \psi'' \to \chi_J + \gamma_1 \to (\psi\gamma_2) + \gamma_1 \to (e^+e^-) + \gamma_2 + \gamma_1$ $(J = 0,1,2)$ when $\bar{p}$ and p are unpolarized. The answer is given in the ψ', ψ'' rest frame or the $\bar{p}p$ c.m. frame. By measuring this angular distribution one can determine the magnitudes as well as the relative phases of the angular momentum helicity amplitudes in the radiative decay processes ψ', $\psi'' \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ $(J = 0,1,2)$ as well as the relative magnitudes of the angular momentum helicity amplitudes in the processes $\bar{p}p \to \psi'\psi''$ and $\psi \to e^+e^-$.

PACS number(s): 13.40.Hq, 12.39.Pn, 14.40.Gx

Recently we have shown [1] that by studying the angular distribution of the two photons in the cascade process originating from unpolarized $\bar{p}p$ collisions, namely,

$$\bar{p}p \to \psi', \psi'' \to \chi_J + \gamma_1 \to (\psi\gamma_2) + \gamma_1 \quad (J = 0,1,2) \ ,$$

one can extract the magnitudes of all the angular momentum helicity amplitudes as well as the cosines of the relative phases of these amplitudes in the radiative decays $\psi', \psi'' \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$. By including the angular distribution of the electron (e^-) in the final decay process $\psi \to e^+e^-$ we now show that we can determine the relative phases unambiguously by determining both their cosines and the sines. This is very important [2,3] since previous potential model calculations have shown that the angular momentum helicity amplitudes in the radiative decays are in general complex and hence their relative phases are nontrivial. Our final result expresses the combined angular distribution of the final stable products e^-, γ_1, and γ_2 in terms of the angles measured in the $\bar{p}p$ c.m. frame or the ψ', ψ'' rest frame. This is the frame where the analysis of experimental results should be most convenient. A brief derivation of our result follows.

Since all calculations and results are the same for the ψ' and ψ'' cases, henceforth, we shall refer to both as ψ'. The amplitude for the sequential process can be written as a product of the amplitudes for the individual processes, so we can write the probability amplitude in the ψ' rest frame as

$$T^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} = \sum_{\delta,\sigma}^{-1,0,1} \sum_{\nu}^{-J\to+J} {}_{\psi'}\langle e^-\alpha_1, e^+\alpha_2|C|\psi\sigma\rangle_{\psi'} \times {}_{\psi'}\langle\psi\sigma, \gamma_2\kappa|E|\chi_J\nu\rangle_{\psi'}\, {}_{\psi'}\langle\chi_J\nu, \gamma_1\mu|A|\psi'\delta\rangle_{\psi'}\, {}_{\psi'}\langle\psi'\delta|B|\bar{p}\lambda_1, p\lambda_2\rangle_{\psi'} \ . \tag{1}$$

In Eq. (1) the greek symbols following the particle symbols represent either the helicities or the Z component of the spin if the particles are at rest. Only the two photons γ_1 and γ_2 and the electron positron pair are finally observed. The transition amplitude depends on the helicities of these particles and those of the initial particles, namely, λ_1, λ_2, μ, κ, α_1, and α_2. In Eq.(1) we sum over the helicities and the spin indices of the unobserved intermediate particles. The symbol ψ' attached to the bra or the ket vector indicates that each individual amplitude is evaluated in the ψ' rest frame. The symbols B, A, E, and C represent the appropriate transition operators. Except for the last matrix element $\langle e^-\alpha_1, e^+\alpha_2|C|\psi\sigma\rangle$, the individual amplitudes are equal to their values evaluated in the rest frame of the decaying particles or the created particle for the case of ψ' formation from $\bar{p}p$ collisions.

For the matrix element of the process $\chi_J(J = 0,1,2) \to \psi + \gamma_2$, we find

$$\begin{aligned} {}_{\psi'}\langle\psi\sigma, \gamma_2\kappa|E|\chi_J\nu\rangle_{\psi'} &= {}_{\chi_J}\langle p_{\psi_{\chi_J}}\theta', \phi', \sigma\kappa|U^\dagger_\Lambda(\psi', \chi)EU_\Lambda(\psi', \chi)|\chi_J, \nu\rangle_{\chi_J} \\ &= {}_{\chi_J}\langle p_{\psi_{\chi_J}}\theta', \phi'; \sigma\kappa|E|\chi_J, \nu\rangle_{\chi_J} \ . \end{aligned} \tag{2}$$

$U_\Lambda(A,B)$, is the unitary operator corresponding to the Lorentz transformation of the helicity type [4] which takes us from the B rest frame to the A rest frame, and the state vector $|p_{\psi_\chi}, \theta', \phi'; \sigma\kappa\rangle$ is the two-particle $\psi\gamma$ helicity state in the χ_J rest frame with $(p_{\psi_\chi}, \theta', \phi')$ giving the three-momentum of ψ in that frame. In Eq. (2) we have made use of the fact that the transition operator E is invariant under Lorentz transformations. We will choose the positive

Z axis of our coordinate system along the direction of motion of χ_J in the ψ' rest frame. The X and Y axes are arbitrary in our discussions of this paper. The experimentalist can choose them according to his or her convenience. In the χ_J rest frame the index ν is the Z component of the total angular momentum of χ_J. So after expanding the $\psi\gamma$ two-particle helicity state in the c.m. frame, in terms of the angular momentum states [4], we find, in the usual way [5,6]

$$\begin{aligned} {}_\chi\langle p_{\psi\chi}\theta',\phi';\sigma\kappa|E|\chi_J\nu\rangle_\chi &= \sum_{J'M'}\sqrt{\frac{2J'+1}{4\pi}}D^{J'^*}_{M',\sigma-\kappa}(\phi',\theta',-\phi')_\chi\langle J'M';\sigma\kappa|E|J\nu\rangle_\chi \\ &= \sqrt{\frac{2J+1}{4\pi}}D^{J^*}_{\nu,\sigma-\kappa}(\phi',\theta',-\phi')E^J_{\sigma\kappa}\ . \end{aligned} \tag{3}$$

The relations between θ',ϕ' and the angles $(\tilde{\theta}',\tilde{\phi}')$ representing the direction of ψ in the ψ' rest frame are given later.

The matrix-elements for the processes $\bar{p}p\to\psi'$ and $\psi'\to\chi_J+\nu$ in the ψ' rest frame are easily evaluated in terms of the angular momentum helicity amplitudes and the Wigner D^J functions:

$${}_{\psi'}\langle\bar{p}\lambda_1,p\lambda_2|B|\psi'\delta\rangle_{\psi'} = \sqrt{3/4\pi}D^{1^*}_{\delta\lambda}(\phi,\theta,-\phi)B_{\lambda_1\lambda_2}\ , \tag{4}$$

where (θ,ϕ) gives the directions of $\bar{p}$ momentum in the $\bar{p}p$ c.m. frame or the ψ' rest frame, and

$$\lambda=\lambda_1-\lambda_2 \tag{5}$$

the symbol $B_{\lambda_1\lambda_2}$ represents the angular momentum helicity amplitudes of this process. We also have

$$\begin{aligned} {}_{\psi'}\langle\chi_J\nu,\gamma_1\mu|A|\psi'\delta\rangle_{\psi'} &= \sqrt{3/4\pi}A^J_{\nu\mu}D^{1^*}_{\delta,\nu-\mu}(0,0,0) \\ &= \sqrt{3/4\pi}A^J_{\nu\mu}\delta_{\delta,\nu-\mu} \end{aligned} \tag{6}$$

since χ_J is moving along the positive Z axis.

For the matrix element of the final process $\psi\to e^+e^-$, the situation is more involved. We have

$$\begin{aligned} {}_{\psi'}\langle e^-\alpha_1,e^+\alpha_2|C|\psi\sigma\rangle_{\psi'} &= {}_\psi\langle e^-\alpha_1,e^+\alpha_2|U^\dagger_\Lambda(\psi',\psi)CU_\Lambda(\psi',\chi_J)U_\Lambda(\chi_J,\psi)|\psi\sigma\rangle_\psi \\ &= {}_\psi\langle e^-\alpha_1,e^+\alpha_2|U^\dagger_\Lambda(\psi',\psi)CU_\Lambda(\psi',\psi)U^\dagger_\Lambda(\psi',\psi)U_\Lambda(\psi',\chi_J)U_\Lambda(\chi_J,\psi)|\psi\sigma\rangle_\psi \\ &= {}_\psi\langle e^-\alpha_1,e^+\alpha_2|CU^\dagger_\Lambda(\psi',\psi)U_\Lambda(\psi',\chi_J)U_\Lambda(\chi_J,\psi)|\psi\sigma\rangle_\psi\ . \end{aligned} \tag{7}$$

In the first equality of Eqs. (7) we made use of the fact that the single-particle state $|\psi\sigma\rangle_{\psi'}$ was also part of the two-particle helicity state of Eq. (2). It was obtained by successively performing two unitary operations corresponding to two Lorentz transformations, the first taking the ψ state from its rest frame to the χ_J rest frame and the second taking it from the χ_J rest frame to the ψ' rest frame. In the last equality of Eqs. (7) we now make use of the fact that

$$U_\Lambda(\psi',\chi_J)U_\Lambda(\chi_J,\psi)=U_\Lambda(\psi',\psi)U_{R_W}\ , \tag{8}$$

where U_{R_W} is a unitary operator corresponding to a pure rotation, usually called a "Wigner rotation." Using Eq. (8) and the unitarity of U_Λ, Eq. (7) now leads to

$$\begin{aligned} {}_{\psi'}\langle e^-\alpha_1,e^+\alpha_2|C|\psi\sigma\rangle_{\psi'} &= {}_\psi\langle e^-\alpha_1,e^+\alpha_2|CU_{R_W}|\psi\sigma\rangle_\psi \\ &= {}_\psi\langle e^-\alpha_1,e^+\alpha_2|U_{R_W}U^\dagger_{R_W}CU_{R_W}|\psi\sigma\rangle_\psi \\ &= {}_\psi\langle e^-\alpha_1,e^+\alpha_2|U_{R_W}C|\psi\sigma\rangle_\psi \end{aligned} \tag{9}$$

since

$$U_{R^\dagger_W}CU_{R_W}=C\ . \tag{10}$$

Using the expansion of the two-particle helicity state in terms of the angular momentum states, we can write the right-hand side of Eq. (9) as

$$\begin{aligned} {}_\psi\langle e^-\alpha_1,e^+\alpha_2|U_{R_W}C|\psi\sigma\rangle_\psi = \sqrt{3/4\pi}D^{1^*}_{\sigma,\alpha_1-\alpha_2} \\ \times(R^{-1}_W\hat{\mathbf{e}}_\psi)C_{\alpha_1\alpha_2}\ , \end{aligned} \tag{11}$$

where $\hat{\mathbf{e}}_\psi$ is a unit vector in the direction of e^- in the ψ rest frame and R_W is the (3×3) rotation matrix and $C_{\alpha_1\alpha_2}$ are the angular momentum helicity amplitudes.

The Wigner-rotated unit vector $R^{-1}_W\hat{\mathbf{e}}_\psi$ can be obtained in the following way. If R represents the (4×4) matrix whose spatial part gives the (3×3) matrix R_W mentioned above, we know, from the definition of Eq. (8),

$$R=\Lambda^{-1}(\psi',\psi)\Lambda(\psi',\chi_J)\Lambda(\chi_J,\psi)\ , \tag{12}$$

where the Λ's are the (4×4) Lorentz transformation matrices. Now we note that the electron is extremely relativistic in the ψ rest frame and its four-momentum vector p_{e_ψ} can be represented to a very good approximation as

$$p_{e_\psi} = \frac{M_\psi}{2}(1, \hat{\mathbf{e}}_\psi) \ , \tag{13}$$

$$\begin{aligned} R^{-1}p_{e_\psi} &= \Lambda^{-1}(\chi_J, \psi)\Lambda^{-1}(\psi', \chi_J)\Lambda(\psi', \psi)p_{e_\psi} \\ &= \Lambda^{-1}(\chi_J, \psi)\Lambda^{-1}(\psi', \chi_J)\Lambda(\psi', \psi)\Lambda^{-1}(\psi', \psi)p_{e_{\psi'}} \\ &= \Lambda^{-1}(\chi_J, \psi)\Lambda^{-1}(\psi', \chi_J)p_{e_{\psi'}} \ , \end{aligned} \tag{14}$$

where $p_{e_{\psi'}}$ is the four-momentum vector of e^- in the ψ' frame:

$$p_{e_{\psi'}} = E_{e_{\psi'}}(1, \hat{\mathbf{e}}) \ , \tag{15}$$

$$\begin{aligned} R^{-1}p_{e_\psi} &= \frac{M_\psi}{2}(1, R_W^{-1}\hat{\mathbf{e}}_\psi) \\ &= \Lambda^{-1}(\chi_J, \psi)\Lambda^{-1}(\psi', \chi_J)p_{e_{\psi'}} \\ &= \Lambda^{-1}(\chi_J, \psi)\Lambda^{-1}(\psi', \chi_J)E_{e_{\psi'}}(1, \hat{\mathbf{e}}') \ . \end{aligned} \tag{16}$$

The spatial part of the right-hand side of Eq. (16) gives, within a normalization factor, the rotated unit vector $\tilde{\hat{\mathbf{e}}} = R_W^{-1}\hat{\mathbf{e}}_\psi$ in terms of the angles $(\tilde{\theta}'', \tilde{\phi}'')$ measured in the ψ' frame. The explicit relations will be given later.

The transition probability amplitude of Eq. (1) now becomes

$$\begin{aligned} T^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} = \frac{3\sqrt{3(2J+1)}}{(4\pi)^2} &\sum_{\delta,\sigma}^{-1,0,1} C_{\alpha_1\alpha_2}E^J_{\sigma\kappa}B_{\lambda_1\lambda_2}A^J_{\mu+\delta,\mu} \\ &\times D^{1^*}_{\sigma,\alpha_1-\alpha_2}(R_W^{-1}\hat{\mathbf{e}}_\psi)D^{J^*}_{\mu+\delta,\sigma-\kappa}(\phi', \theta', -\phi') \\ &\times D^{1^*}_{\delta\lambda}(\phi, \theta, -\phi) \ . \end{aligned} \tag{17}$$

The C and the P invariances of the transition operators lead to the following relations [4] among the angular momentum helicity amplitudes:

$$\begin{aligned} B_{\lambda_1\lambda_2} &\overset{C}{=} B_{\lambda_2\lambda_1} \overset{P}{=} B_{-\lambda_1,-\lambda_2} \ , \\ A^J_{\nu\mu} &\overset{P}{=} (-1)^J A^J_{-\nu,-\mu} \ , \\ E^J_{\sigma\kappa} &\overset{P}{=} (-1)^J E^J_{-\sigma-\kappa} \ , \end{aligned} \tag{18}$$

and

$$C_{\alpha_1\alpha_2} \overset{C}{=} C_{\alpha_2\alpha_1} \overset{P}{=} C_{-\alpha_1,-\alpha_2} \ .$$

Making use of the symmetry relations of Eqs. (18) we relabel the independent angular momentum helicity amplitudes as follows:

TABLE I. Expressions for the nonzero coefficients $\beta^{L_1L_2}_d$ in terms of the angular momentum helicity amplitudes A^J_ν ($J = 0, 1, 2$; $\nu = 0 \to J$). The expressions for $\gamma^{L_3L_2}_{d'}$ are identical except for the fact that the helicity amplitudes A^J_ν are replaced by E^J_ν in the expressions for $\beta^{L_1L_2}_d$. In all cases $\beta^{L_1L_2}_0 = 0$ for odd L_2. We also assumed the following normalization conventions: $\sum_{\nu=0}^J |A\nu|^2 = \sum_{\nu=0}^J |E_\nu|^2 = 1$.

$J = 0$	$\beta^{00}_0 = \frac{2}{\sqrt{3}}$
	$\beta^{20}_0 = \sqrt{2/3}$
$J = 1$	$\beta^{00}_0 = -2/3$
	$\beta^{20}_0 = -\frac{\sqrt{2}}{3}(\lvert A_0\rvert^2 - 2\lvert A_1\rvert^2)$
	$\beta^{02}_0 = \frac{\sqrt{2}}{3}(2\lvert A_0\rvert^2 - \lvert A_1\rvert^2)$
	$\beta^{22}_0 = \frac{2}{3}$
	$\beta^{21}_1 = i\,\mathrm{Im}(A_1A_0^*)$
	$\beta^{22}_1 = \mathrm{Re}(A_1A_0^*)$.
$J = 2$	$\beta^{00}_0 = \frac{2}{\sqrt{15}}$
	$\beta^{20}_0 = \sqrt{2/15}(\lvert A_0\rvert^2 - 2\lvert A_1\rvert^2 + \lvert A_2\rvert^2)$
	$\beta^{02}_0 = -\sqrt{2/21}(2\lvert A_0\rvert^2 + \lvert A_1\rvert^2 - 2\lvert A_2\rvert^2)$
	$\beta^{04}_0 = \sqrt{2/105}(6\lvert A_0\rvert^2 - 4A_1\rvert^2 + \lvert A_2\rvert^2)$
	$\beta^{22}_0 = -\frac{2}{\sqrt{21}}(\lvert A_0\rvert^2 - \lvert A_1\rvert^2 - \lvert A_2\rvert^2)$
	$\beta^{24}_0 = \frac{1}{\sqrt{105}}(6\lvert A_0\rvert^2 + 8\lvert A_1\rvert^2 + \lvert A_2\rvert^2)$
	$\beta^{21}_1 = \frac{-i}{\sqrt{5}}[\sqrt{3}\mathrm{Im}(A_1A_0^*) - \sqrt{2}\mathrm{Im}(A_2A_1^*)]$
	$\beta^{23}_1 = \frac{i}{\sqrt{5}}[\sqrt{2}\mathrm{Im}(A_1A_0^*) + \sqrt{3}\mathrm{Im}(A_2A_1^*)]$
	$\beta^{23}_2 = i\sqrt{2}\mathrm{Im}(A_2A_0^*)$
	$\beta^{22}_1 = -\frac{1}{\sqrt{7}}[\mathrm{Re}(A_1A_0^*) - \sqrt{6}\mathrm{Re}(A_2A_1^*)]$
	$\beta^{22}_2 = \sqrt{8/7}\mathrm{Re}(A_2A_0^*)$
	$\beta^{24}_1 = \frac{1}{\sqrt{7}}[\sqrt{6}\mathrm{Re}(A_1A_0^*) + \mathrm{Re}(A_2A_1^*)]$
	$\beta^{24}_2 = \sqrt{6/7}\mathrm{Re}(A_2A_0^*)$

$$\begin{aligned} B_\lambda &= B_{(\lambda_1-\lambda_2)} = \sqrt{2}B_{\lambda_1\lambda_2} \ , \\ A_\nu &= A^J_{\nu,1} = (-1)^J A^J_{-\nu,-1} \quad (\nu = 0 \to J) \ , \\ E_\sigma &= E^J_{\sigma-1,-1} = (-1)^J E^J_{-\sigma+1,1} \ , \\ C_\alpha &= C_{(\alpha_1-\alpha_2)} = \sqrt{2}C_{\alpha_1\alpha_2} \ . \end{aligned} \tag{19}$$

When $\bar{p}$ and p are unpolarized, the normalized function describing the angular distribution of the two photons and of the electron in the final state can be written as

$$W_J = N_J \sum_{\alpha_1,\alpha_2}^{\pm 1/2} \sum_{\mu,\kappa}^{\pm 1} \sum_{\lambda_1,\lambda_2}^{\pm 1/2} T^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} T^{*\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} \ , \tag{20}$$

where N_J is a normalization constant so chosen that the angular distribution function W integrated over all the directions of the three particles will give the value one. After a lengthy algebra, using Eqs. (17)–(20) and making use of the Clebsch-Gordan series relation for the Wigner D^J functions, namely,

$$D^{j_1}_{m_1m_2}D^{j_2}_{m_1'm_2'} = \sum_{J=|j_1-j_2|}^{j_1+j_2} \langle j_1j_2m_1m_1'|J, m_1+m_1'\rangle\langle j_1j_2m_2m_2'|J, m_2+m_2'\rangle D^J_{m_1+m_1',m_2+m_2'} \ , \tag{21}$$

we obtain the normalized angular distribution function in the form

$$W_J = \frac{3(2J+1)}{4(4\pi)^3} \sum_{L_1,L_3}^{0,2} \varepsilon_{L_3}\alpha_{L_1} \sum_{L_2}^{0\to 2J} \sum_{d,d'}^{0\to d_m,d'_m} \beta_d^{L_1L_2}\gamma_{d'}^{L_3L_2}\mathcal{Y}_{dd'}^{L_1L_3L_2} , \tag{22}$$

where

$$\begin{aligned} d_m &= \mathrm{Min}(L_1,L_2,J) , \\ d'_m &= \mathrm{Min}(L_3,L_2,J) , \\ \alpha_0 &= |B_0|^2+|B_1|^2 = 1 , \\ \alpha_2 &= \frac{1}{\sqrt{2}}(|B_1|^2-2|B_0|^2) , \\ \varepsilon_0 &= |C_0|^2+|C_1|^2 = 1 , \\ \varepsilon_2 &= \frac{1}{\sqrt{2}}(|C_1|^2-2|C_0|^2) . \end{aligned} \tag{23}$$

It should be noted that C_0 is expected to be of the order of m_e/E_e of C_1 and therefore should be negligible compared to C_1. The coefficients $\beta_d^{L_1L_2}$ and $\gamma_{d'}^{L_3L_2}$ are given in terms of the angular momentum helicity amplitudes A_ν and E_ν of the radiative decays $\psi' \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$, respectively:

$$\begin{aligned} \beta_d^{L_1L_2} = &\sum_{s}^{d,d+2,\ldots,2J-d} [A_{(s+d)/2}A^*_{(s-d)/2} + (-1)^{L_2}A^*_{(s+d)/2}A_{(s-d)/2}] \\ &\times \left\langle JJ; \frac{s+d}{2}, -\frac{s-d}{2}\middle| L_2 d\right\rangle \left\langle 11; \frac{s+d-2}{2}, -\frac{s-d-2}{2}\middle| L_1 d\right\rangle , \end{aligned} \tag{24}$$

$$\gamma_{d'}^{L_3L_2} = \beta_{d'}^{L_3L_2}(A\to E) . \tag{25}$$

The explicit expressions for the nonzero $\beta_d^{L_1L_2}$ are listed for $J = 0$, 1, and 2 in Table I.

The symbol $\mathcal{Y}_{dd'}^{L_1L_3L_2}$ is a function of all the angles describing the directions of ψ, e^-, and of $\bar{p}$ in various frames. The direction of γ_1 is opposite to that of χ_J in the ψ' rest frame and the direction of γ_2 is opposite to that of ψ in the χ_J rest frame:

$$\begin{aligned} \mathcal{Y}_{dd'}^{L_1L_3L_2} = &\left(1-\frac{\delta_{d0}}{2}\right)\left(1-\frac{\delta_{d'0}}{2}\right)[(D_{d0}^{L_1}D_{d'0}^{L_3^*}D_{dd'}^{L_2^*} + D_{d0}^{L_1^*}D_{d'0}^{L_3}D_{dd'}^{L_2} \\ &+(-1)^{L_2}(D_{-d0}^{L_1}D_{d'0}^{L_3^*}D_{-d,d'}^{L_2^*} + D_{-d0}^{L_1^*}D_{d'0}^{L_3}D_{-d,d'}^{L_2})] . \end{aligned} \tag{26}$$

The arguments of the Wigner D^J functions, $D_{d0}^{L_1}$, $D_{dd'}^{L_2}$, and $D_{d'0}^{L_3}$ are, respectively, $(\phi,\theta,-\phi)$, the direction of $\bar{p}$ in the ψ' frame; $(\phi',\theta',-\phi')$, the direction of ψ in the χ_J frame; and $\hat{\tilde{\mathbf{e}}} = \mathbf{R}_{\mathbf{W}}^{+}\hat{\mathbf{e}}_\psi$, the direction of e^- in the ψ frame rotated by the adjoint or inverse of the Wigner rotation matrix.

We will now express all angles in terms of angles measured in the ψ' frame. The angles (θ',ϕ') measured in the χ_J rest frame are related to the angles $\tilde{\theta}'$ and $\tilde{\phi}'$ measured in the ψ' frame by the relations

$$\phi' = \tilde{\phi}' , \tag{27}$$

$$\cos\theta' = \left((\cos^2\tilde{\theta}'-1)\frac{\beta_2}{\beta_1} + \cos\tilde{\theta}'\sqrt{1-\beta_2^2}\sqrt{1-(\beta_2/\beta_1)^2+\cos^2\tilde{\theta}'[(\beta_2/\beta_1)^2-\beta_2^2]}\right)\frac{1}{(1-\beta_2^2\cos^2\tilde{\theta}')} . \tag{28}$$

Since $0 \le \theta' \le \pi$, $\sin\theta'$ has to be positive and so it will be given by the positive square root:

$$\sin\theta' = +\sqrt{1-\cos^2\theta'} , \tag{29}$$

where $\cos\theta'$ is given by Eq. (28). In Eq. (28), β_1 is the parameter v/c of ψ in the χ_J rest frame and β_2 is v/c of the χ_J in the ψ' rest frame. Simple algebra gives

$$\beta_1 = \frac{M_\chi^2 - M_\psi^2}{M_\chi^2 + M_\psi^2} ,$$
$$\beta_2 = \frac{M_{\psi'}^2 - M_\chi^2}{M_{\psi'}^2 + M_\chi^2} . \tag{30}$$

If the direction of $\hat{\tilde{\mathbf{e}}} = R_W^{-1}\,\hat{\mathbf{e}}_\psi$ (where $\hat{\mathbf{e}}_\psi$ is the direction of e^- in the ψ frame) is given by the spherical polar angles θ'' and ϕ'', then these angles are related to the angles of e^-, $(\tilde{\theta''}, \tilde{\phi''})$, measured in the ψ' frame by the relations

$$\cos\phi'' = \frac{1}{\eta'}(\gamma_2\beta_2\sin\theta' + \cos\theta'\cos\phi'\sin\tilde{\theta''}\cos\tilde{\phi''} + \cos\theta'\sin\tilde{\theta''}\sin\tilde{\phi''} - \sin\theta'\cos\tilde{\theta''}\gamma_2) , \tag{31}$$

$$\sin\phi'' = \frac{1}{\eta'}(\cos\phi'\sin\tilde{\theta''}\sin\tilde{\phi''} - \sin\phi'\sin\tilde{\theta''}\cos\tilde{\phi''}) , \tag{32}$$

$$\cos\theta'' = [-\gamma_1\gamma_2(\beta_1 + \beta_2\cos\theta') + \gamma_1(\sin\theta'\cos\phi'\sin\tilde{\theta''}\cos\tilde{\phi''} + \sin\theta'\sin\phi'\sin\tilde{\theta''}\sin\tilde{\phi''}) + \gamma_1\gamma_2(\beta_1\beta_2 + \cos\theta')\cos\tilde{\theta''}]\frac{1}{\eta} , \tag{33}$$

$$\sin\theta'' = +\sqrt{1 - \cos^2\theta''} = \frac{\eta'}{\eta} , \tag{34}$$

where

$$\eta' = [(\gamma_2\beta_2\sin\theta' + \cos\theta'\cos\phi'\sin\tilde{\theta''}\cos\tilde{\phi''} + \cos\theta'\sin\phi'\sin\tilde{\theta''}\sin\tilde{\phi''} \\ -\sin\theta'\cos\tilde{\theta''}\gamma_2)^2 + (\cos\phi'\sin\tilde{\theta''}\sin\tilde{\phi''} - \sin\phi'\sin\tilde{\theta''}\cos\tilde{\phi''})^2]^{1/2} , \tag{35}$$

$$\eta = [\gamma_1\gamma_2(1 + \beta_1\beta_2\cos\theta') - \gamma_1\beta_1(\sin\theta'\cos\phi'\sin\tilde{\theta''}\cos\tilde{\phi''} + \sin\theta'\sin\phi'\sin\tilde{\theta''}\sin\tilde{\phi''}) - \gamma_1\gamma_2(\beta_2 + \beta_1\cos\theta')\cos\tilde{\theta''}] . \tag{36}$$

The constants γ_i $(i = 1, 2)$ are related to β_i $(i = 1, 2)$ by the relations

$$\gamma_i = \frac{1}{\sqrt{1 - \beta_i^2}} . \tag{37}$$

A little algebra starting from Eqs. (30) shows

$$\gamma_1 = \frac{M_\chi^2 + M_\psi^2}{2M_\chi M_\psi} ,$$
$$\gamma_2 = \frac{M_{\psi'}^2 + M_\chi^2}{2M_{\psi'} M_\chi} . \tag{38}$$

It is useful to note that, in Eq. (26),

$$D^L_{M0} = \sqrt{4\pi/(2L+1)}\,Y^*_{LM} . \tag{39}$$

Since the spherical harmonics in Eq. (22) are linearly independent one can completely determine the relative phases as well as the relative magnitudes of the angular momentum helicity amplitudes or equivalently of the radiative multipole amplitudes in the processes $\psi' \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ $(J = 0, 1, 2)$. For example, when enough data exist to perform the required numerical integrations one can use the equations

$$\int d\Omega d\Omega'' d\Omega' Y_{L_1 d}(\theta,\phi) Y^*_{L_3 d'}(\theta'',\phi'') D^{L_2}_{dd'}(\phi',\theta',-\phi') W_J(\theta,\phi;\theta',\phi';\theta'',\phi'')$$

$$= \left(1 - \frac{\delta_{d0}}{2}\right)\left(1 - \frac{\delta_{d'0}}{2}\right)\sqrt{4\pi/(2L_1+1)}\sqrt{4\pi/(2L_3+1)}\left(\frac{4\pi}{2L_2+1}\right)\frac{3(2J+1)}{4(4\pi)^3}[\epsilon_{L_3}\alpha_{L_1}\beta_d^{L_1 L_2}\gamma_{d'}^{L_3 L_2}] . \tag{40}$$

So one can measure the coefficients $\beta_d^{L_1 L_2}$ and $\gamma_{d'}^{L_3 L_2}$ for all possible values of L_1, L_2, L_3, d, and d'. From these we can determine the relative magnitudes as well as the relative phases of the angular momentum helicity amplitudes A^J_ν and E^J_ν $(J = 0, 1, 2;\ \nu = 0 \to J)$. We also get the relative magnitude of B_0 and B_1 as well as that of C_0 and C_1. For example, from Eq. (22) one finds that

$$\int d\Omega'' W_J = \frac{3(2J+1)}{4(4\pi)^2} \sum_{L_1}^{0,2} \alpha_{L_1} \sum_{L_2}^{0\to 2J} \sum_{d}^{0\to dm} \beta_d^{L_1 L_2} \gamma_0^{0L_2} \mathcal{Y}_{d0}^{L_1 0 L_2} \tag{41}$$

and

$$\int d\Omega W_J = \frac{3(2J+1)}{4(4\pi)^2} \sum_{L_3}^{0,2} \epsilon_{L_3} \sum_{L_2}^{0\to 2J} \sum_{d'}^{0\to d'm} \beta_0^{0L_2} \gamma_{d'}^{L_3 L_2} \mathcal{Y}_{d0'}^{0 L_3 L_2} . \tag{42}$$

So by measuring the $(L_1 L_2 d)$ coefficients of $\mathcal{Y}_{d0}^{L_1 0 L_2}$ in $\int d\Omega'' W_J$ and the $(L_3 L_2 d')$ coefficients of $\mathcal{Y}_{0d'}^{0 L_3 L_2}$ in $\int d\Omega W_J$ one makes the following determinations.

The $J = 0$ case is trivial. For the $J = 1$ case the measurements of the $(L_1 L_2 d)$ coefficients yield the following: Measuring (220) gives γ_0^{02}. Then (000) together with (020) yield $|A_0|^2$ and $|A_1|^2$. Next (200) determines α_2. Finally (221) gives $R_e(A_1 A_2^*)$. A similar procedure for the $(L_3 L_2 d')$ coefficients in Eq. (42) gives the corresponding values of the E amplitudes and ϵ_{L_3}. Measuring the coefficients of $\mathcal{Y}_{11}^{221}$ in Eq. (22) gives $\mathrm{Im}(A_1 A_0^*)\mathrm{Im}(E_1 E_0^*)$.

For the $J = 2$ case the measurement of the (L_1, L_2, d) coefficients in Eq. (41) give the following results. Measuring the (000), (200), (020), (220), (040), and (240) coefficients of $\mathcal{Y}_{d0}^{L_1 0 L_2}$ gives γ_0^{02}, γ_0^{04}, α_2, $|A_0|^2$, $|A_1|^2$, and $|A_2|^2$. Next the (221) and the (241) coefficients determine $R_e(A_1 A_0^*)$ and $R_e(A_2 A_1^*)$. Finally the (222) and the (242) coefficients would each determine $R_e(A_2 A_0^*)$. Measuring the $(L_3 L_2 d'')$ coefficients in Eq. (42) gives the corresponding values of ϵ_{L_3} and the E amplitudes. The (L_1, L_3, L_2, d, d') coefficients of $\mathcal{Y}_{dd'}^{L_1 L_3 L_2}$ in Eq. (22) determine the sine of the relative phases. Measuring any four of the coefficients (22111), (22311), (22312), (22321), and (22322) will determine the sines of the four relative phases between A_0 and A_1, A_1 and A_2, E_0 and E_1, and finally E_1 and E_2.

The integral of the angular distribution function W_J with respect to θ'' and ϕ'' gives the angular distribution function of the two γ photons γ_1 and γ_2 discussed in [1].

Our results are interesting. Previous studies [5] have shown that by studying the angular distribution of γ and of e^- in the cascade process $\bar{p}p \to \chi_J \to \psi + \gamma \to (e^+e^-) + \gamma$, when $\bar{p}$ and $\bar{p}$ are unpolarized, we can only get the relative magnitudes of the angular momentum helicity amplitudes and the cosines of the relative phases between the amplitudes in the radiative decay $\chi_J \to \psi + \gamma$. One cannot specify the relative phase unambiguously since the sine of the relative phase cannot be determined. Only by studying the angular distribution [7] of the decay products of χ_J formed by polarized $\bar{p}p$ collisions can one determine the relative phases unambiguously. Here we have shown that even with unpolarized $\bar{p}p$ collisions we can, in principle, determine the magnitudes as well as the phases of the angular momentum helicity amplitudes in the radiative decays $\psi' \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ if we can measure the combined angular distribution of γ_1, γ_2, and e^- in the cascade process

$$\bar{p}p \to \psi' \to \chi_J + \gamma_1 \to (\psi\gamma_2) + \gamma_1 \to (e^+e^-) + \gamma_2 + \gamma_1 .$$

Since our angular distribution function is given in terms of angles measured in the $\bar{p}p$ c.m. frame, it will be especially useful to the experimentalists for direct comparison with measurements.

Most of this work was done while the authors were visiting the University Park Campus of Pennsylvania State University. They would like to thank Professor Howard Grotch, Head of the Physics Department, for his kind hospitality.

[1] F. L. Ridener, Jr. and K. J. Sebastian, Phys. Rev. D **49**, 4617 (1994).

[2] K. J. Sebastian, H. Grotch, and F. L. Ridener, Jr., Phys. Rev. D **45**, 3163 (1992).

[3] K. J. Sebastian, Phys. Rev. D **49**, 3450 (1994).

[4] A. D. Martin and T. D. Spearman, *Elementary Particle Theory* (American Elsevier, NY, 1970).

[5] F. L. Ridener, Jr., K. J. Sebastian, and H. Grotch, Phys. Rev. D **45**, 3173 (1992).

[6] A. D. Martin, M. G. Olsson, and W. J. Stirling, Phys. Lett. **147B**, 203 (1984).

[7] F. L. Ridener, Jr. and K. J. Sebastian, Phys. Rev. D **49**, 5830 (1994).