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Consistency of the chiral pion-pion scattering amplitudes with axiomatic constraints

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The pion-pion scattering amplitudes provided by second-order chiral perturbation theory are confronted with known rigorous constraints derived from the axioms of quantum field theory. We mainly test constraints restricting the π^0-π^0 S- and D-wave amplitudes in the unphysical interval $0 \leq s \leq 4m_\pi^2$. These constraints impose significant lower bounds for a linear combination of coupling constants specifying the second-order chiral Lagrangian. The accepted value of this combination is consistent with these bounds. The π^0-π^0 S and D waves are strongly correlated by a set of constraints.

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I. INTRODUCTION

During the 1960s and 1970s much effort was invested in deriving properties of scattering amplitudes which are exact consequences of the general principles of quantum field theory [1]. This program was pioneered by Martin and was very successful in the pion-pion case, the scattering of the lightest hadrons. Analyticity properties, i.e., the fact that scattering amplitudes are boundary values of analytic functions of two complex variables, constitute one of the main tools provided by axiomatic field theory. It implies the validity of dispersion relations with a number of subtractions restricted by the Froissart bound. The other tools are unitarity and crossing symmetry. Their interplay leads to a wealth of constraints on the low energy pion-pion scattering [2]. We call them "axiomatic constraints" as they follow from the axioms of quantum field theory.

A characteristic feature of the axiomatic constraints is that they do not depend on any specification of the interactions going beyond the requirement of crossing symmetry. Nowadays it is well established that the pion is a quasi-Goldstone boson associated with the breaking of chiral symmetry and the pion-pion scattering amplitudes should reflect the specificities of the dynamics of such particles. This has been worked out in chiral perturbation theory (CPT), which provides an approximate form of the low energy pion-pion amplitudes. Our aim is to check whether this chiral ansatz, i.e., the second-order one-loop chiral amplitudes first obtained by Gasser and Leutwyler [3], satisfies a representative set of the axiomatic conditions. Surprisingly, this has not been done until now, at least to the best of our knowledge.

The constraints we shall test restrict the shape of the pion-pion amplitudes in a triangle Δ of the Mandelstram plane:

$$\Delta = \{s,t,u | 0 \leq s \leq 4m_\pi^2, 0 \leq t \leq 4m_\pi^2, 0 \leq u \leq 4m_\pi^2\} , \tag{1.1}$$

where s, t, and u are the standard Mandelstam variables, $s+t+u = 4m_\pi^2$, m_π =pion mass. As the chiral ansatz is meant to provide a reliable approximation of the pion-pion amplitudes at low values of s, t, and u, it should verify our conditions with a good precision. In fact, the chiral amplitudes are analytic functions, they are exactly crossing symmetric and have positive absorptive parts. The axiomatic conditions being consequences of these properties, they might be expected to be satisfied automatically. This is not the case because the chiral amplitudes grow asymptotically as s^2 and violate the Froissart bound which is another ingredient of the axiomatic constraints. Since the chiral ansatz represents the first terms of a low energy expansion, bad asymptotic behavior can be expected. Our purpose is to determine whether the chiral ansatz, when restricted to its domain of validity, is compatible with the low energy implications of the Froissart bound. The latter being the mark of a local quantum field, we are asking if quasi-Goldstone bosons can be described by such a field.

Our constraints are inequalities which are linear in the amplitudes: they enforce bounds on combinations of parameters appearing in the chiral Lagrangian. This implies that these quantities cannot be chosen at will if compatibility with general field theoretical principles is required. In order to get first insight into the nature of these restrictions we apply the constraints to the standard chiral perturbation theory defined by Gasser and Leutwyler [3]. This may as well be done for other versions of this theory, for instance, the "generalized chiral perturbation theory" proposed by Stern and his collaborators [4]. In the case of the standard theory we find that a combination $\bar{l}$ of second-order coupling constants defined in (2.7) has to be larger than a lower bound equal to 6 in order of magnitude. Fits to experimental data give $\bar{l} \approx 21$ [5]. The bound is respected and the order of magnitude of the bound is not disproportionate to the actual value of $\bar{l}$. This is of importance because it proves that the restrictions imposed by the constraints are relevant, a fact which could not have been asserted beforehand. A sum rule requiring a phenomenological input accounts for the difference between the experimental value of $\bar{l}$ and its axiomatic lower bound.

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Since the axiomatic constraints are a rather old topic we find it worthwhile to start with an outline of their sources. This is done in Sec. II. The standard chiral ansatz for the π^0-π^0 amplitude is also displayed in this section and first constraints are checked. Section III deals with conditions on S and P waves and bounds for $\bar{l}$ are obtained, taking into account the uncertainties due to unknown third-order corrections. The sum rule which has just been mentioned is also discussed in this section. Section IV is devoted to constraints which strongly correlated the π^0-π^0 S and D waves. Our conclusions are presented in Sec. V.

II. AXIOMATIC PROPERTIES OF PION-PION SCATTERING

We first recall the basis of the rigorous properties of the pion-pion amplitudes we shall exploit [1]. These properties hold in the triangle Δ defined in (1.1) where the amplitudes are real and satisfy the crossing conditions

$$\begin{aligned} T^I(s,t,u) &= \sum_{I'} C_{st}^{II'} T^{I'}(t,s,u) \\ &= \sum_{I'} C_{su}^{II'} T^{I'}(u,t,s) \\ &= \sum_{I'} C_{tu}^{II'} T^{I'}(s,u,t) \ . \end{aligned} \tag{2.1}$$

$T^I(s,t,u)$ is the s-channel isospin I amplitude ($I = 0,1,2$) and C_{st} C_{su}, C_{tu} are crossing matrices [2]. At each point of Δ, $T^I(s,t,u)$ is given by three dispersion relations evaluated either at fixed s, fixed t, or fixed u. These relations result from exact analyticity properties: the Froissart bound ensures that only two subtractions are needed. The fixed-t dispersion relations have their simplest form if written for the isospin I t-channel amplitudes $T^{(I)}(s,t,u)$ [$= T^I(t,s,u)$]:

$$T^{(I)}(s,t,u) = \mu_I(t) + \nu_I(t)(s-u) + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dx \frac{1}{x^2}\left(\frac{s^2}{x-s} + (-1)^I \frac{u^2}{x-u}\right)\sum_{I'} C_{st}^{II'} A^{I'}(x,t) \ , \tag{2.2}$$

where $\mu_I(t)$ and $\nu_I(t)$ are unknown t-dependent subtraction constants ($\mu_1 = 0$, $\nu_0 = \nu_2 = 0$). A^I is the absorptive part of the s-channel amplitude T^I:

$$A^I(s,t) = \mathrm{Im} T^I(s + i\epsilon, t, u), \quad s \geq 4m_\pi^2 \ . \tag{2.3}$$

If $(s,t,u) \in \Delta$, $T(s,t,u)$ is also given by a fixed-s or a fixed-t dispersion relation obtained from (2.2) by suitable substitutions. An important property of the absorptive parts is that they are positive for $0 \leq t < 4m_\pi^2$. This follows from their partial wave expansion:

$$A^I(s,t) = \sum_{l=0}^{\infty}(2l+1)\mathrm{Im} f_l^I(s+i\epsilon) P_l\left(1 + \frac{2t}{s-4m_\pi^2}\right) \ . \tag{2.4}$$

Each term on the right-hand side is positive because the partial waves $f_l^I(s)$ have positive imaginary parts in the physical domain $s \geq 4m_\pi^2$, and $P_l(z) \geq 1$ for $z \geq 1$. Our constraints are consequences of the twice subtracted dispersion relations, the positivity of the absorptive parts and crossing symmetry. They restrict the shape of the amplitudes T^I in the triangle Δ and the partial waves f_l^I in the unphysical interval $0 \leq s \leq 4m_\pi^2$.

Most of the constraints we shall use concern the totally symmetric π^0-π^0 amplitude $T = \frac{1}{3}(T^0 + 2T^2)$. The second-order chiral ansatz of this amplitude is [3,6]

$$T_\chi(s,t,u) = \frac{\lambda}{2} + \frac{\lambda^2}{4\pi}\{[(2s^2 - 4s + 3)I(s) + (s \to t) + (s \to u)] + \tfrac{1}{3}[\alpha(s^2 + t^2 + u^2) + \beta]\} \ , \tag{2.5}$$

with $\lambda = m_\pi^2/(16\pi F_\pi^2)$, where F_π is the pion decay constant. We treat λ as an expansion parameter: $\lambda = 0.0448$ if one takes $m_\pi = 140$ MeV and $F_\pi = 93.2$ MeV. In (2.5) and in the following the variables s,t,u are measured in units of m_π^2. Our normalization of scattering amplitudes differs from the one commonly used in CPT by a factor $(1/32\pi)$. The constant first-order term is the π^0-π^0 Weinberg amplitude. The first set of square brackets gives the finite part of the one-loop contributions with

$$I(s) = 2\left[\sqrt{(4/s) - 1}\left(\arctan\sqrt{(4/s)-1} - \frac{\pi}{2}\right) + 1\right] \ . \tag{2.6}$$

This form is adapted to the unphysical values $0 \leq s \leq 4$. The second set of square brackets in (2.5) comes from the tadpoles and second-order trees. The constants α and β are linear combinations of the scale independent coupling constants $\bar{l}_i$ of the second-order chiral Lagrangian

$$\begin{aligned} &\alpha = \bar{l} - 6, \quad \bar{l} = 2\bar{l}_1 + 4\bar{l}_2 \ , \\ &\beta = \bar{l}' + 21 \ , \\ &\bar{l}' = -8\bar{l}_1 - 16\bar{l}_2 - 9\bar{l}_3 + 12\bar{l}_4 \ . \end{aligned} \tag{2.7}$$

The exact π^0-π^0 amplitude $T(s,t,u)$ satisfies a simple version of the dispersion relation (2.2):

$$T(s,t,u) = \mu(t) + \frac{1}{\pi}\int_4^\infty dx\,\frac{1}{x^2}\left(\frac{s^2}{x-s}+\frac{u^2}{x-u}\right)A(x,t)\ , \tag{2.8}$$

where $A(s,t)$ is the s-channel π^0-π^0 absorptive part: it is positive for $s \geq 4$, $0 \leq t < 4$.

With one exception, at the end of Sec. III, the constraints we shall discuss concern π^0-π^0 scattering and we want to know the conditions under which they are obeyed by the chiral amplitude T_χ. More precisely, we assume that the true amplitude T differs from T_χ by $O(\lambda^3)$ terms in the triangle Δ:

$$T(s,t,u) = T_\chi(s,t,u) + O(\lambda^3),\quad (s,t,u) \in \Delta\ . \tag{2.9}$$

If the amplitude T satisfies an axiomatic condition it can be violated by T_χ to an order of magnitude fixed by Eq. (2.9). Clearly, we have no control of the third-order corrections within the present context. The actual size of the $O(\lambda^3)$ term in (2.9) could well be λ^3 times a relatively large factor.

The axiomatic constraints we shall use follow from analyticity, positivity of absorptive parts, and crossing. These conditions being linear convex and homogeneous, they constrain only the shape of the amplitudes and not their size. However, by including nonlinear aspects of unitarity, it has been possible to derive absolute axiomatic bounds for the π^0-π^0 amplitude [7]. These bounds are remarkable in that they have to hold independently of the details of the dynamics, whenever the $I = 0$ and $I = 2$ mass spectra start at $2m_\pi$ (absence of two pions bound states). The most stringent bounds are

$$T(3,2,-1) > -1.30,\quad T(\tfrac{4}{3},\tfrac{4}{3},\tfrac{4}{3}) < 2.70\ . \tag{2.10}$$

With $\lambda = 0.0448$ and reasonable values of α and β, T_χ is of the order of a percent of the bounds. This means that the chiral ansatz describes a pion-pion interaction which is weak at the scale defined by the axiomatic bounds (2.10). It is precisely one of the achievements of current algebra and CPT to explain the smallness of the pion-pion interaction.

From now on we consider only constraints derived from the linear and homogeneous conditions of analyticity, positivity, and crossing. They leave the first-order Weinberg amplitudes completely free. This comes from the fact that these amplitudes are linear in s, t, and u and that the arbitrariness in the subtraction constants in (2.2) leaves crossing symmetric linear terms of the full amplitudes T^I undetermined. Consequently, all conditions we shall examine test exclusively the second-order chiral ansatz.

As already mentioned in the Introduction, the constraints lead to bounds for linear combinations of the second-order coupling constants $\bar{l}_i$, the bounds being determined by the loop and tadpole terms. To establish the relevance of these bounds we look at the simplest axiomatic constraint on the π^0-π^0 amplitude $T(s,t,u)$. It tells us that the symmetry point $s = t = u = \frac{4}{3}$ is an absolute minimum of T [8]. In fact T_χ has an extremum at this point because of crossing symmetry. Inspection of (2.5) shows that it will be a maximum, instead of a minimum, if α is large and negative. Therefore α has to be larger than some lower bound. At the symmetry point,

$$\frac{\partial^2 T_\chi}{\partial s^2} = \frac{\partial^2 T_\chi}{\partial t^2} = \frac{1}{2}\frac{\partial^2 T_\chi}{\partial s \partial t} = \frac{\lambda^2}{4\pi}\left[\frac{2}{3}\alpha + 1.729\right]\ . \tag{2.11}$$

Insofar as the second-order derivatives of the $O(\lambda^3)$ term in (2.9) are also $O(\lambda^3)$, the symmetry point will be a minimum of T if

$$\bar{l} > 3.4 + 6\pi O(\lambda) = 3.4 + O(0.8)\ . \tag{2.12}$$

This condition shows that $\bar{l}_i = 0$ is excluded: second-order trees have to be included in the chiral Lagrangian in order to get an ansatz which is compatible with the axiomatic constraints.

The actual value of $\bar{l}$ quoted in the Introduction, $\bar{l} = 21 \pm 4$, comes from $\bar{l}_1 = -1.7 \pm 1.0$ and $\bar{l}_2 = 6.1 \pm 0.5$ [5]. It is compatible with (2.12) and, as announced, the order of magnitude of the bound is comparable with this accepted value.

III. S AND P WAVES, MAINLY π^0-π^0 S WAVE

We investigate two sets of axiomatic constraints on the π^0-π^0 S-wave $f_0(s)$. The first set restricts the shape of f_0 on the interval $[0,4]$ through the signs of its derivatives. The second set consists of inequalities relating the values of f_0 at two points of $[0,4]$. For $0 \leq s \leq 4$, $f_0(s)$ is given by

$$f_0(s) = \frac{2}{4-s}\int_0^{(4-s)/2} dt\, T(s,t,u)\ . \tag{3.1}$$

The constraints on the derivatives follow directly from properties of $T(s,t,u)$ implied by the dispersion relation (2.8) and the positivity of $A(x,t)$. One finds [8–10]

$$\frac{df_0(s)}{ds} < 0 \quad \text{for } 0 < s < 1.217\ , \tag{3.2}$$

$$\frac{df_0(s)}{ds} > 0 \quad \text{for } 1.697 < s < 4\ , \tag{3.3}$$

$$\frac{d^2 f_0(s)}{ds^2} > 0 \quad \text{for } 0 < s < 1.7\ . \tag{3.4}$$

These conditions show that f_0 has a minimum in the interval (1.217, 1.697): this minimum is clearly a reflection of the minimum of the full amplitude at the symmetry point.

The chiral ansatz for the π^0-π^0 S wave is

$$f_0^\chi(s) = \frac{\lambda}{2} + \frac{\lambda^2}{4\pi}\left\{\left[(2s^2-4s+3)I(s) + \frac{2}{4-s}\int_0^{4-s} dt(2t^2-4t+3)I(t)\right] + \frac{1}{3}\left[\frac{\alpha}{3}(5s^2-16s+32)+\beta\right]\right\}\ . \tag{3.5}$$

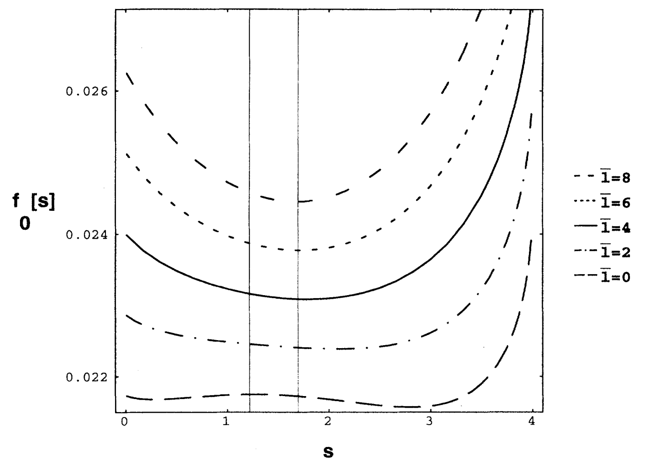

FIG. 1. The chiral π^0-π^0 S wave as given in (3.5) with $\bar{l}' = 0$ ($\beta = 21$), for $s \in [0,4]$ and various values of $\bar{l} = \alpha + 6$ with the two vertical lines delimiting the position of the axiomatic minimum following from (3.2)–(3.4).

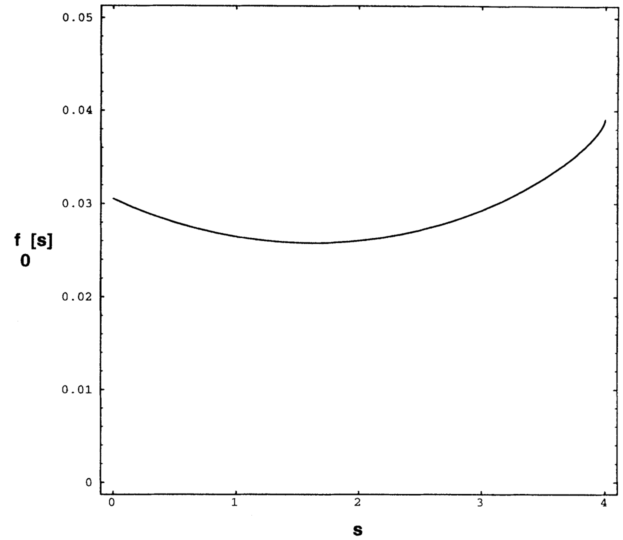

FIG. 2. The chiral π^0-π^0 S wave on the interval [0,4] for central values of the parameters $\bar{l}$ and $\bar{l}'$ defined in (2.7): $\bar{l} = 21$ [5], $\bar{l}' = -58.5$ [5,6].

We ask if this ansatz satisfies the conditions (3.2)–(3.4) as it stands, ignoring the $O(\lambda^3)$ corrections in (2.9). As in the case of the full chiral amplitude, f_0^χ will have a maximum instead of a minimum if α is too negative. The minimum value of α is determined by the shape of the loop contribution, given by the first set of square brackets in (3.5). This contribution is found to satisfy conditions (3.2) and (3.4) but it marginally violates condition (3.3) because its minimum is slightly above $s = 1.697$, at $s = 1.701$. The polynomial in the second set of square brackets, coming from the tadpoles and second-order trees, has its own extremum at $s = 1.6$. Thus α has to be slightly positive in order to bring the minimum of f_0^χ into the allowed interval: this corresponds to

$$\bar{l} > 6.63 \ . \tag{3.6}$$

The situation is illustrated in Fig. 1 which shows the evolution of the shape of f_0^χ as $\bar{l}$ varies between 0 and 8. Figure 2 shows f_0^χ for the central phenomenological value $\bar{l} = 21$.

Our findings about the loop term are instructive because the loop contribution to the full amplitude verifies all the required exact properties except the Froissart bound and its S-wave projection violates conditions (3.2)–(3.4) only weakly. This shows that the axiomatic constraints are not necessarily very sensitive to wrong asymptotic behavior. In the present case, the practical effect of these constraints is to impose correct behavior of the tree and tadpole contributions. Note that the third-order corrections introduce an uncertainty ~ 1.7 into (3.6).

The constraints relating the values of $f_0(s)$ at two points in [0,4] are obtained by eliminating the subtraction constant $\mu(t)$ in (2.8). Projecting this dispersion relation onto the t-channel S-wave $f_0(t)$ gives an equation relating $\mu(t)$ and $f_0(t)$. Using this equation, $\mu(t)$ can be eliminated from (2.8) in favor of $f_0(t)$, giving

$$T(s,t,u) = f_0(t) + \frac{1}{\pi}\int_4^\infty dx\, A(x,t) F(x,s,t) \ , \tag{3.7}$$

with

$$F(x,s,t) = \frac{1}{x-s} + \frac{1}{x-4+s+t} + \frac{2}{4-t}\ln\left(\frac{x-4+t}{x}\right) \ . \tag{3.8}$$

As $T(s,t,u) = T(t,s,u)$, (3.7) implies that

$$f_0(s) - f_0(t) = \frac{1}{\pi}\int_4^\infty dx[A(x,t)F(x,s,t) - A(x,s)F(x,t,s)] \ . \tag{3.9}$$

We see that the inequality

$$f_0(s) - f_0(t) > 0 \tag{3.10}$$

holds for every pair (s,t) such that

$$F(x,s,t) > 0 \text{ and } (F(x,s,t) - F(x,t,s)) > 0 \quad \text{for } x \geq 4 \text{ if } t > s \ ,$$
$$F(x,t,s) < 0 \text{ and } (F(x,s,t) - F(x,t,s)) > 0 \quad \text{for } x \geq 4 \text{ if } s > t \ . \tag{3.11}$$

The inequality $A(x,s) > A(x,t)$, valid if $4 > s > t \geq 0$, has been taken into account: it results from (2.4). The first known inequalities (3.10) [9] have been obtained from the more restrictive condition

$$F(x,s,t) > 0 \text{ and } F(x,t,s) < 0 \text{ for } x \geq 4 \; . \qquad (3.12)$$

We have computed anew the domain of the (s,t) plane defined by condition (3.12) and determined the significantly larger domain defined by condition (3.11). The result is displayed in Fig. 3.

The assumption (2.9) implies that the inequality (3.10) imposes the following constraint on the second-order chiral S wave:

$$f_0^\chi(s) - f_0^\chi(t) > O(\lambda^3) \; . \qquad (3.13)$$

Using expression (3.5) of f_0^χ we see that (3.13) is equivalent to an inequality of the form

$$a(s,t)\bar{l} - b(s,t) > O(\lambda) \; , \qquad (3.14)$$

where $a(s,t) > 0$ and we consider those pairs for which $b(s,t) > 0$. These give lower bounds for $\bar{l}$. It turns out that the loop term in (3.5) satisfies all the inequalities arising from the strong condition (3.12). This means that the lower bounds produced by these inequalities are smaller than 6. The pair $s = 2.30$, $t = 1.08$, is an example verifying (3.11) and not (3.12) and leading to an inequality (3.10) slightly violated by the loop term. With $a = 0.0097$, $b = 0.0588$ and $\lambda \sim 0.05$ this gives

$$\bar{l} > 6.1 + O(5) \; . \qquad (3.15)$$

The large uncertainty in this lower bound comes from the fact that, for reasonable values of $\bar{l}$, the difference $[f_0^\chi(s) - f_0^\chi(t)]$ is small compared with λ^2 and is, in fact, $O(\lambda^3)$. Therefore a is small and the ratio λ/a is large. All inequalities leading to interesting ratios b/a follow this trend. Furthermore the precise numbers appearing in bounds like (3.15) depend on the fine details of the chiral ansatz. There are pairs which lead to slightly larger bounds but s and t are very close in these pairs and it would be unrealistic to assume that the correction to $[f_0^\chi(s) - f_0^\chi(t)]$ is effectively of order λ^3. Assuming that the derivative of the correction to f_0^χ is also $O(\lambda^3)$ we may safely suppose that the correction to the difference is $O((s-t)\lambda^3)$. If we adopt this procedure, the pair $s = 1.720$, $t = 1.675$ with $a = 0.000\,388$ and $b = 0.002\,55$ gives the largest lower bound combined with a relatively small uncertainty:

$$\bar{l} > 6.6 + O(5) \; . \qquad (3.16)$$

Although the bounds (3.15) and (3.16) are not disportionate to the phenomenological value of $\bar{l}$, the difference between this value and the bound nevertheless is relatively large and has to be explained. Comparing the equality (3.9) and the inequality (3.10) we see that this difference is determined by the integral on the right-hand side of (3.9). This integral cannot be evaluated with the help of the chiral ansatz alone and the explanation we are looking for has to be based in part on the phenomenological analysis of pion-pion scattering. In other words we are no longer confronting the chiral ansatz with rigorous constraints alone but are using (3.9) as a sum rule. To estimate the right-hand side of (3.9) we cut off the integral at an energy of 1400 MeV ($x = 100$) and replace the

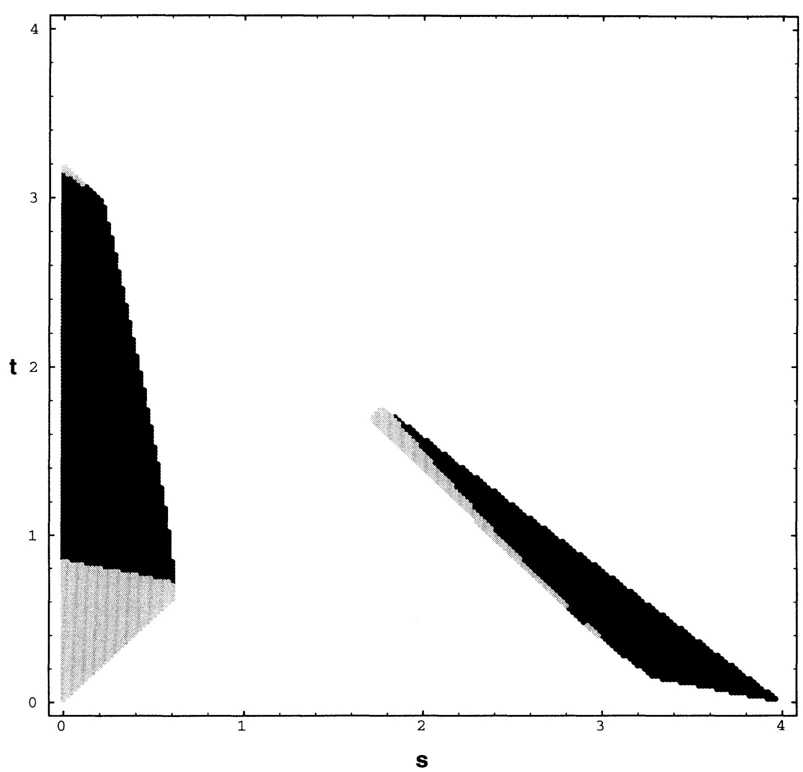

FIG. 3. The domains of pairs (s,t) for which an inequality (3.10) holds true. The black domain contains the pairs satisfying condition (3.12). The domains in grey are the extensions obtained by replacing (3.12) by (3.11).

absorptive parts by their S-wave contributions leading to the integral

$$D(s,t) = \frac{1}{\pi}\int_4^{100} dx[F(x,s,t) - F(x,t,s)]\frac{1}{3}\left(\frac{x}{x-4}\right)^{1/2} \times[\sin^2\delta_0^0(x) + 2\sin^2\delta_0^2(x)] \ . \tag{3.17}$$

We expect $D(s,t)$ to be a good approximation of the right-hand side integral of (3.9). In fact, if (s,t) verifies (3.11), $D(s,t)$ is smaller than this integral and we have an improved version of (3.14):

$$a(s,t)\bar{l} - b(s,t) > \frac{1}{\lambda^2}D(s,t) + O(\lambda) \ . \tag{3.18}$$

To evaluate $D(s,t)$ we take the phase shifts obtained from the chiral ansatz between threshold and 600 MeV according to the procedure described in [6], for the central values of the coupling constants [5,6]. Above 600 MeV and up to 1400 MeV the chiral amplitudes can no longer be trusted and we take the $I = 0$ phase shift adopted by the Particle Data Group [11] and the $I = 2$ phase shift given by Martin *et al.* [12]. We find $D(2.30, 1.08) = 2.31 \times 10^{-4}$ and $D(1.720, 1.675) = 8.69 \times 10^{-6}$. These results lead to the following lower bounds which, according to (3.18) replace (3.15) and (3.16):

$$\bar{l} > 17.9 + O(5), \quad \bar{l} > 17.7 + O(5) \ . \tag{3.19}$$

The coincidence of these two bounds obtained from quite different (s,t) pairs is striking. It supports the assumption that $D(s,t)$ is a good approximation of the integral in (3.9). This allows us to turn the inequalities (3.19) into approximate equalities, that is into approximate versions of the sum rules (3.9). Dropping the third-order corrections we notice that the second-order chiral ansatz is not quite consistent with the sum rules: the right-hand side has been evaluated assuming $\bar{l} = 21$. We shall not enter into a detailed analysis of this issue taking into account the uncertainties on the $\bar{l}_i$. We limit ourselves only to the conclusion that matters here: the size of $D(s,t)$ which forces $\bar{l}$ to be substantially larger than the rigorous lower bounds (3.15) and (3.16).

A refinement of the technique leading to the π^0-π^0 constraints (3.10) produces inequalities involving the $I = 0$, 1 and 2 S and P waves [13]. We discuss only one of these constraints:

$$f_0^0(s) - f_0^2(s) - 1.693\,269 f_1^1(s) - \frac{1}{3}[f_0^0(t) + 2f_0^2(t)] + 4.751\,676 f_1^1(t) > 0 \ , \tag{3.20}$$

with $s = 2.6097$, $t = 1.0873$. A high accuracy in the coefficients of f_1^1 is mandatory in order to ensure a vanishing contribution of the first-order linear amplitudes. After insertion of the chiral ansatz of the S and P waves, the left-hand side of (3.20) becomes a linear function of $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_4$. Remarkably, it depends practically only on the combination $\bar{l} = 2\bar{l}_1 + 4\bar{l}_2$ which appears in the π^0-π^0 amplitudes. Eliminating $\bar{l}_2$, (3.20) becomes

$$0.0338\bar{l} + 1.3 \times 10^{-3}\bar{l}_1 - 6 \times 10^{-7}\bar{l}_4 - 0.0764 > O(\lambda) \ . \tag{3.21}$$

This inequality effectively constrains only $\bar{l}$:

$$\bar{l} > 2.3 + O(1.3) \ . \tag{3.22}$$

This bound is weaker than the preceding ones. It may be that the uncertainty is underestimated because it is a combination of the λ^3 corrections of the six amplitudes appearing in the constraint.

IV. π^0-π^0 S AND D WAVES

The partial waves $l \geq 2$ do not have the same status as the S and P waves. The reason is that the twice-subtracted fixed-s dispersion relations lead to a Froissart-Gribov representation for these higher partial waves, which involves only absorptive parts and no subtraction constants. For the π^0-π^0 D wave and $0 \leq s < 4$,

$$f_2(s) = \frac{4}{4-s}\frac{1}{\pi}\int_4^{\infty} dx\, A(x,s) Q_2\left(\frac{2x}{4-s} - 1\right) \ . \tag{4.1}$$

On the other hand, the chiral ansatz gives

$$f_2^{\chi}(s) = \frac{\lambda^2}{4\pi}\left[\frac{2}{4-s}\int_0^{4-s} dt(2t^2 - 4t + 3)P_2\left(1 - \frac{2t}{4-s}\right) I(t) + \frac{1}{45}\alpha(4-s)^2\right] \ . \tag{4.2}$$

Positivity immediately implies that $f_2(s)$ as given by (4.1) is positive on [0,4]. Moreover, it has been shown that $df_2(s)/ds$ is negative for $1.435 \leq s < 4$ [14].

The loop contribution to the chiral D wave is marginally in conflict with these constraints, being slightly negative above $s = 2.71$. It is found that f_2^{χ} has the correct shape if $\bar{l} > 7.85$. Although this is our largest rigorous lower bound, it is not really useful because there is no reliable way of estimating the uncertainty coming from the third-order D-wave corrections. The shape of f_2^{χ} for various values of $\bar{l}$ is shown in Fig. 4.

Apart from the properties of the D wave alone, there are two known sets of inequalities involving both the π^0-π^0 S and D waves [15]. The inequalities of the first set are of the form

$$C(s,t)f_2(s) + C(t,s)f_2(t) \gtrless f_0(s) - f_0(t) \tag{4.3}$$

for appropriate pairs $(s,t), C(s,t)$ being a known func-

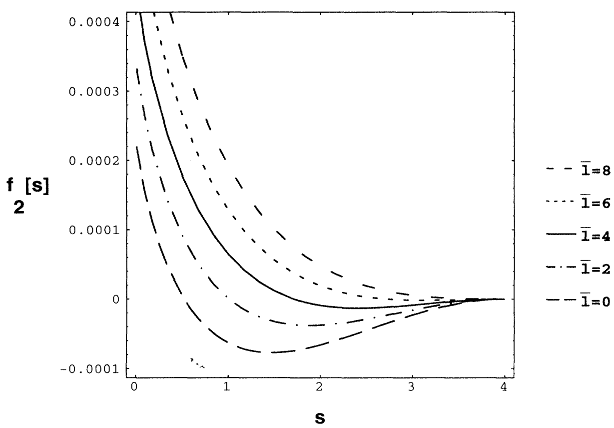

FIG. 4. The chiral π^0-π^0 D wave on the interval [0,4] for various values of $\bar{l}$.

tion. These inequalities again provide bounds for $\bar{l}$ but they do not improve our previous results.

The second set of constraints gives upper or lower bounds for the D wave at given points in terms of the difference of the S wave between two points:

$$f_2(s) \gtrless \left[5P_2\left(1-\frac{2t}{4-s}\right)\right]^{-1}[f_0(s)-f_0(t)]\,. \quad (4.4)$$

These inequalities impose lower bounds on $\bar{l}$ as well as upper bounds. The lower bounds are weaker than the previous ones and the upper bounds are too large ($\sim$1000) to be of any interest. The real strength of the inequalities (4.4) is that they strongly constrain the shape of the π^0-π^0 D wave once the S wave is given. This phenomenon has already been recognized by Martin in a general context [15]. In our case, with $\bar{l}=21$, we obtain an impressive picture, displayed in Fig. 5. The chiral S wave defines gates through which the D wave must pass. The gates located below $s=1$ are very narrow. The chiral D wave passes through all the gates, often close to the lower edge. Therefore the second-order chiral π^0-π^0 S and D waves obey all the constraints (4.4). This

TABLE I. The chiral π^0-π^0 D wave f_2^χ/λ^3 and its lower and upper bounds from (4.4) for $\bar{l}=21$.

s	(Lower bounds)/λ^3	f_2^χ/λ^3	(Upper bounds)/λ^3
0.0341		15.256 09	15.822 05
0.073	14.459 34	14.685 68	
0.304		11.996 60	12.124 49
0.325	11.749 86	11.788 77	
0.572	9.479 23	9.630 69	10.661 81
0.589	5.584 49	9.498 39	
0.747		8.348 95	8.363 08
0.803		7.973 07	8.330 16
0.826	7.815 48	7.823 04	
1.000	6.587 37	6.763 39	9.118 61
1.200	5.575 84	5.691 12	9.930 5
1.400	4.541 94	4.752 4	10.433 5
1.435	3.538 43	4.600 94	10.489 9
1.500	3.960 47	4.327 54	10.569 6
1.600	3.435 62	3.929 92	10.628 5
1.800	0.930 242	3.210 38	10.513 6
1.900	0.286 525	2.885 82	5.050 84
1.950	0.475 615	2.731 84	4.305 03
2.000	0.708 817	2.583 2	3.887 46
2.050	0.883 305	2.439 8	3.594 93
2.100	1.191 99	2.301 5	3.365 35
2.288	1.627 80	1.825 27	6.103 39
2.500	1.116 38	1.366 13	5.355 27
2.857	0.304 35	0.761 78	
3.000		0.541 21	2.827 7
3.102		0.457 97	6.596 16
3.106		0.453 71	1.009 08

picture remains qualitatively unchanged if the value of $\bar{l}$ is reduced. It is only for $\bar{l}=3.7$ that one of the bounds starts to be violated. The constraints (4.4) also impose correlations on the third-order corrections, due to the fact that the width of the narrow gates is a fraction of λ^3, as shown in Table I. Thus, one may expect third-order S-wave corrections to be larger than the widths of the narrow gates: they will be shifted by these corrections and the D wave must comply with these shifts.

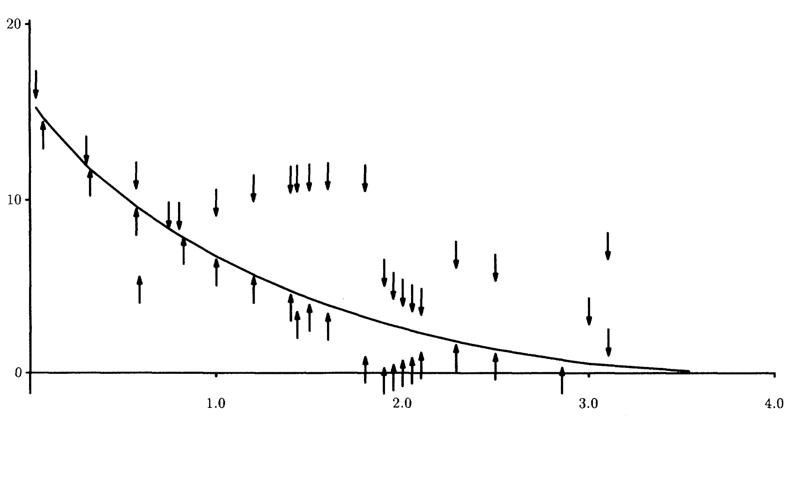

FIG. 5. The upper and lower bounds imposed on the π^0-π^0 D wave by the chiral π^0-π^0 S wave according to the inequalities (4.4) for $\bar{l}=21$. The orientation of the arrows distinguishes the upper from the lower bounds. The chiral D wave satisfies all constraints. Bounds and D wave are divided by λ^3.

V. CONCLUSIONS

We have checked whether the second-order chiral pion-pion amplitudes obey a set of known axiomatic constraints. We have found that this is the case as long as the value of the combination $\bar{l}$ of second-order coupling constants is larger than about 6. As $\bar{l} \approx 21$, the constraints are actually satisfied. It is remarkable that the best bounds (3.6) and (3.16) produced by different families of π^0-π^0 constraints are nearly equal, and slightly larger than 6. This arises from the fact that the loop terms either obey the constraints or violate them marginally. In the latter case, a small positive α removes the violation. Since $\alpha = \bar{l} - 6$ is the coefficient of polynomials in (2.5) and (3.5) coming from tadpoles and second-order trees, $\alpha \approx 0$ essentially means that the trees nearly cancel the tadpoles. A sum rule involving data at energies which are above the domain of validity of the chiral ansatz shows that $\bar{l}$ has to be substantially larger than its axiomatic lower bound. It is remarkable that constraint (3.20), involving all S and P waves, effectively restricts only that combination $\bar{l}$ which appears in the π^0-π^0 amplitudes.

Finally, we have discovered that the second-order chiral π^0-π^0 S wave practically fixes the D wave below $s = 1$ by means of a set of axiomatic constraints. Surprisingly, the second-order chiral D wave agrees completely with all the conditions imposed by the S wave if $\bar{l}$ is large enough. This implies strong correlations between the third-order S- and D-wave corrections.

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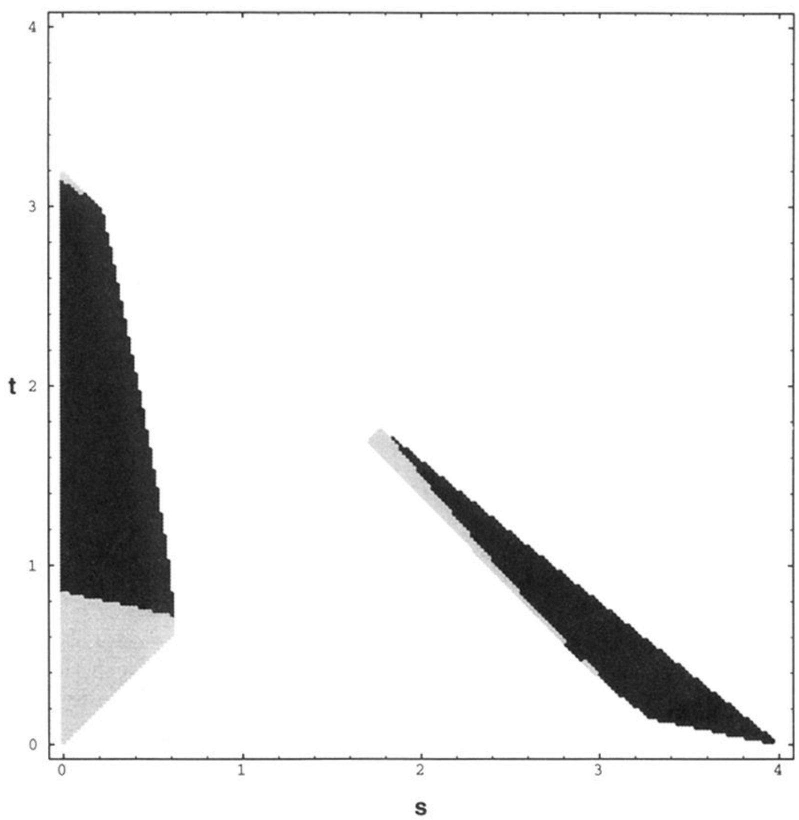

FIG. 3. The domains of pairs (s,t) for which an inequality (3.10) holds true. The black domain contains the pairs satisfying condition (3.12). The domains in grey are the extensions obtained by replacing (3.12) by (3.11).