RAPID COMMUNICATIONS

PHYSICAL REVIEW D VOLUME 50, NUMBER 12 15 DECEMBER 1994

Multiwormholes and multi-black-holes in three dimensions

Gérard Clément*

Laboratoire de Gravitation et Cosmologie Relativistes, Université Pierre et Marie Curie, CNRS/URA769, Tour 22-12, Boîte 142, 4, place Jussieu, 75252 Paris Cedex 05, France

(Received 10 February 1994; revised manuscript received 3 August 1994)

We construct time-dependent multicenter solutions to three-dimensional general relativity with a zero or negative cosmological constant. These solutions correspond to dynamical systems of freely falling wormholes or black holes and conical singularities, with a multiply connected spacetime topology. Stationary multi-black-hole solutions are possible only in the extreme black-hole case.

PACS number(s): 04.60.Kz, 04.20.Gz, 04.20.Jb, 04.70.Bw

In a now well-known paper [1], Bañados *et al.* gave a black hole solution to the three-dimensional Einstein equations with a negative cosmological constant $\Lambda=-l^{-2}$, and studied its properties. This regular solution, which has inspired a number of recent papers [2], is given by

$$ds^2=\nu^2\left(\frac{r^2}{l^2}-M+\frac{J^2}{4r^2}\right)dt^2-r^2\left(d\theta-\frac{\nu J}{2r^2}\,dt\right)^2-\frac{l^2dr^2}{r^2-Ml^2+\frac{J^2l^2}{4r^2}}, \tag{1}$$

where θ is periodic with a period 2π, the two parameters M and J (with $M\geqslant 0$, $|J|\leqslant ML$) are interpreted [1] as the mass and angular momentum of the black hole, and the constant ν [$\equiv N(\infty)$ in [1]] sets the scale of time.

In the present work we wish to outline the construction of exact dynamical multi-black-hole solutions to three-dimensional cosmological gravity. Some time ago, conformal techniques were used to construct static [3] and stationary [4] multicenter solutions to pure gravity ($\Lambda=0$) associated with configurations of massive and spinning point particles, as well as a class of static multiparticle solutions to cosmological gravity ($\Lambda\neq 0$) [5]. Using similar methods, we shall construct what at first sight appears to be stationary multi-black-hole solutions. However, these stationary solutions turn out to be inconsistent, being generically plagued by extra conical singularities, which are unphysical in the sense that their world lines are not geodesics of the multi-black-hole space-time [5]. As we shall show, by taking the positions of the black-hole centers to be no longer constant but time dependent, one can derive intrinsically dynamical solutions corresponding to systems of freely falling black holes together with, now physical, auxiliary conical singularities.

We first consider for simplicity the case of pure gravity which, along with the well-known point particle solution [3], also admits a wormhole solution which, under certain circumstances, may behave as a black-hole solution. This may be obtained from the $J=0$ black hole of Eq. (1) by putting $\nu=\gamma l^2/c$, $M=c^2l^{-2}$, $r=c/\cos(l^{-1}X)$, $\theta=c^{-1}Y$, leading to

$$ds^2=\gamma^2l^2\tan^2(l^{-1}X)dt^2-\frac{1}{\cos^2(l^{-1}X)}(dX^2+dY^2), \tag{2}$$

and taking the limit $l\to\infty$, which yields

$$ds^2=\gamma^2X^2dt^2-dX^2-dY^2. \tag{3}$$

We recognize in (3) the well-known two-dimensional Rindler space-time [6] with an extra compact spatial dimension Y. As discussed in [6], the transformation $\bar{t}=X\sinh(\gamma t)$, $\bar{x}=X\cosh(\gamma t)$, $\bar{y}=Y$ maps the metric (3) into the two disjoint regions I ($\bar{x}^2>\bar{t}^2$) of the Minkowski cylinder $ds^2=d\bar{t}^2-d\bar{x}^2-d\bar{y}^2$ (with $\bar{y}$ periodic). The remaining two regions II ($\bar{t}^2>\bar{x}^2$) of the Minkowski cylinder may be obtained by extending the metric (3) through the horizon $X^2=0$ (of perimeter $2\pi c$) to $X^2=-\hat{X}^2<0$, and making the transformation $\bar{t}=\hat{X}\cosh(\gamma t)$, $\bar{x}=\hat{X}\sinh(\gamma t)$, $\bar{y}=Y$. The resulting Penrose diagram is shown in Fig. 1.

Of course, this maximally extended Rindler cylinder is indistinguishable from the Minkowski cylinder. The distinction comes about if for instance the metric (3) arises as an interior solution generated by a one-dimensional ring of exotic matter [7]. This ring and its mirror image under the symmetry $X\to -X$ separate space-time into three regions (Fig. 2): two exterior regions where the metric

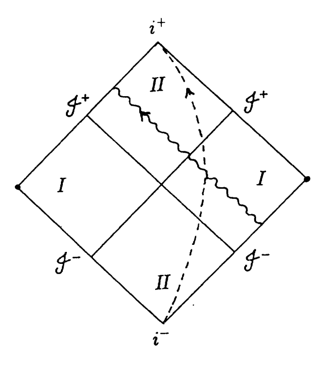

FIG. 1. Penrose diagram for the $\Lambda=0$ wormhole (Rindler cylinder), with a timelike geodesic (dashed line) and a radial lightlike geodesic (wavy line).

*Electronic address: GECL@CCR.JUSSIEU.FR

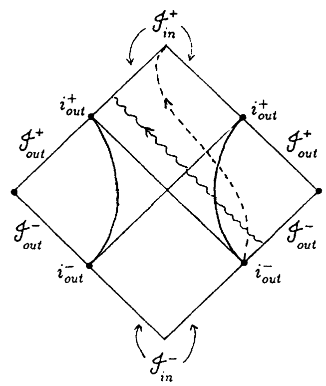

FIG. 2. Penrose diagram for the black-hole space-time generated by two mirror-symmetric rings (heavy world lines). The exterior regions are truncated conical space-times, while the interior region is a truncated Rindler cylinder.

$$ds^2_{\rm out}=dt^2-dX^2-\left(\mu X\pm\frac{\gamma-\mu}{\gamma}\right)^2 dY^2 \tag{4}$$

appears to be generated by a fictitious point particle hidden behind the ring $X=\pm\gamma^{-1}$, and an interior region where the metric is (3) ($\mu<0$ and $\gamma>0$ are the energy density and the stress of the ring). An observer falling inside the ring may send lightlike signals to the outside until he crosses the horizon $X=0$. Thereafter the observer continues falling towards interior timelike infinity, while his signals ($\bar{x}-\bar{t}=$negative constant), which can never catch up with the receding ring ($\bar{x}^2-\bar{t}^2=\gamma^{-2}$), end up at interior lightlike infinity $\mathscr{I}^+_{\rm in}$. The future Rindler region II is thus, for an external observer, a genuine black hole (this fact was not fully appreciated in [7]). Let us here mention that the four-dimensional Rindler cylinder, $ds^2=\gamma^2X^2dt^2-dX^2-dY^2-dZ^2$ with Y periodic, may similarly arise as an interior solution generated by an infinite cylinder of exotic matter, leading to a black cosmic string [8].

Restricting ourselves to the sourceless case, let us now construct from (3) multiwormhole solutions to pure three-dimensional gravity; we shall then generalize this construction to that of multi-black-hole solutions to three-dimensional cosmological gravity. We first recall that the conformal map $X+iY=Z(z)$ (with $z=x+iy$) generates from (3) the family of stationary flat metrics [4]

$$ds^2=\gamma^2X^2(z,\bar{z})dt^2-|Z'(z)|^2dzd\bar{z} \tag{5}$$

($Z'\equiv dZ/dz$). Consider the multicenter map

$$Z=\sum_{i=1}^{n} c_i\ln(z-a_i)+d \tag{6}$$

(c_i and d real, a_i complex) of the region $X(z,\bar{z})>0$ of the Euclidean (x,y) plane into the spatial sections of the three-dimensional Rindler space-time; this map preserves spatial infinity ($X\rightarrow+\infty\Leftrightarrow|z|\rightarrow+\infty$) if all the c_i are positive. For $n=1$ and $a_1=0$, $c_1=c>0$, we recover the Rindler cylinder with $X=c\ln r+d$, $Y=c\theta$, where $z=re^{i\theta}$, $r>e^{-d/c}$. For $n>1$, we obtain what appears to be a system of p wormholes, $p\leqslant n$ being the number of connected components of the horizon $X(x,y)=0$, the total horizon perimeter being $2\pi\Sigma_{i=1}^{n}c_i$. This solution may be maximally extended by taking two identical copies of the multiply connected $X>0$ region, which generalize the two regions I of the Rindler space-time, and connecting the corresponding p horizon components via p two-sided bridges made of two copies (past and future) of a region of type II.

However, a serious problem with the above construction is that the metric (5) has $n-1$ conical singularities associated with the zeros of $Z'(z)$. As conical singularities correspond to point particles, we must require for consistency [5] that these follow geodesics of the multiwormhole space-time. Now a point particle at rest in the geometry (3) feels a static gravitational field

$$-\Gamma^X_{00}=-\gamma^2X \tag{7}$$

which vanishes only on the horizon $X=0$, and the configurations such that the zeros of $Z'(z)$ sit on the horizon are obviously rather special. Of course, the problem is avoided for those zeros of $Z'(z)$ which lie behind the horizon and do not belong to the multiwormhole space-time (the regions $X<0$ are cut out and replaced by connecting bridges). However, for all the zeros of $Z'(z)$ to lie behind the horizon this must be simply connected ($p=1$), in which case the space-time geometry reduces to that of the original Rindler cylinder. The conclusion is that the previously discussed static multiwormhole solution is inconsistent. However, the preceding analysis hints strongly towards a dynamical solution. Consider for instance the map

$$Z=c\ln(z^2-a^2)+d \tag{8}$$

($c>0$); in the case $c\ln|a|^2+d>0$ this leads to a two-wormhole "solution" with an unphysical conical singularity located at the "center of mass" $z=0$. The gravitational field (7) acting on this singularity, which tends to reduce $X(0)$ and thus the separation $2|a|$ between the two centers, pulls the two wormholes together until they merge in a single wormhole for $c\ln|a|^2+d=0$.

To translate this picture into an exact solution, we must introduce a time dependence in the multiwormhole solution. This can be done by generalizing the conformal map $Z=Z(z)$ to the time-dependent map

$$Z=Z(z,t) \tag{9}$$

which leads from the static flat metric (3) to a dynamical flat metric. As we want to describe a system of moving wormholes, we shall assume $Z(z,t)$ to be given by (6), where the positions a_i of the centers are now time dependent, leading to the metric

$$ds^2=(\gamma^2X^2-|A|^2)dt^2+(\bar{A}Z'dz+A\bar{Z}'d\bar{z})dt-|Z'^2|dzd\bar{z}, \tag{10}$$

where

$$A(z,t)=\sum_{i=1}^{n}\frac{c_i\partial_0a_i(t)}{z-a_i(t)} \tag{11}$$

(such a transformation was previously used by Letelier and Gal'tsov [9] to construct multiple moving cosmic strings). The metric (10) has again a horizon at $X(z,t)=0$, and

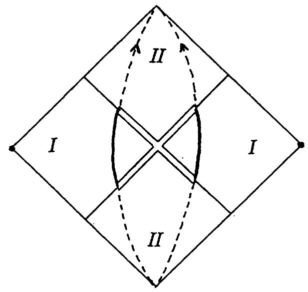

FIG. 3. Penrose diagram for the timelike two-wormhole space-time with $Y=c\pi$ $(\Lambda=0)$. The mirror-symmetric conical singularities are shown as heavy world lines in the two regions I (their dashed analytic extensions into the regions II are not associated with conical singularities). The double lines result from the superposition, induced by the map $z^2=\zeta$, of the two disjoint horizon components.

$n-1$ conical singularities following the world lines $z_\alpha(t)$ which solve the equation $Z'(z_\alpha,t)=0$. For consistency, these world lines must obey the geodesic equations

$$\ddot{x}^\mu_\alpha+\Gamma^\mu_{\nu\rho}(x_\alpha)\dot{x}^\nu_\alpha\dot{x}^\rho_\alpha=0, \tag{12}$$

where $x^0_\alpha\equiv t$ for all α, and an overdot denotes $d/d\sigma_\alpha$, σ_α being the affine parameters on the αth geodesic. Eliminating the σ_α in favor of coordinate time t, we are left with a system of $2(n-1)$ second-order differential equations for the $2n$ unknowns $[a_i(t),\bar{a}_i(t)]$. The remaining twofold arbitrariness is of course due to the possibility of arbitrary global time-dependent translations $z\to z+w(t)$; if we choose for instance the origin of the complex z plane to coincide with the "center of mass" of the multiwormhole system, then the relative dynamics of the system are fully determined by integrating the consistency equations (12) with appropriate initial conditions.

We consider in more detail the symmetrical two-wormhole system (8) with fixed conical singularity $z=0$. The two-body problem may be reduced to that of the motion of one wormhole relative to the fixed point $\zeta=0$ by the transformation $z^2=\zeta$, which transforms (8) to

$$Z=c\ln[\zeta-\alpha(t)]+d \tag{13}$$

with $\alpha=a^2$. Then, this last motion may be transformed, by the global time-dependent translation $\zeta=\psi+\alpha(t)$, into that

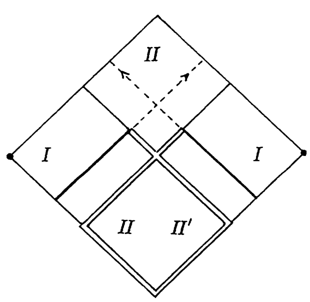

FIG. 4. Penrose diagram for the lightlike two-wormhole space-time with $Y=c\pi$ $(\Lambda=0)$.

of a point particle following the world line $\psi+\alpha(t)=0$ relative to the fixed wormhole (the background Rindler space-time):

$$Z=c\ln\psi+d. \tag{14}$$

These successive coordinate transformations mapping the geodesic $z=0$ into a geodesic, it follows that the motion of the two wormhole centers is given by $\pm a=(-\psi)^{1/2}$, where $\psi=\psi(t)$ is a geodesic of the Rindler cylinder metric (14). These geodesics may easily be derived from those of Minkowski space-time by the Rindler transformation. A typical timelike and a radial null geodesic are shown in Fig. 1. In the timelike (or generic lightlike) case, as the geodesic crosses the horizon from region I into region II, a single wormhole bifurcates into two wormholes which separate to a finite distance and merge again (the conical singularity falls back on the horizon) after an infinite coordinate time. The global structure of the maximal extension of this space-time is schematized in Fig. 3; because of its multiply connected topology, this dynamical solution is clearly not equivalent to a stationary solution with point singularities. In the radial lightlike case (Fig. 4) the two wormholes, infinitely separated at $t=-\infty$, fall upon each other and merge, again after an infinite coordinate time; the time-reversed evolution is equally possible. In all cases, the total horizon perimeter $4\pi c$ is a constant of the motion.

We now sketch how the above construction may be generalized to the case of $\Lambda<0$ cosmological gravity (fuller details shall be given elsewhere). In the case $J=0$, multi-black-holes may similarly be obtained from the one black hole (2) by the time-dependent conformal map $X+iY=Z(z,t)$, where now X varies between $ml\pi$ (the horizon) and $(m+1/2)l\pi$ (the line at spatial infinity, which may also be multiply connected) for a given integer m, and the functions $a_i(t)$ in (6) are determined by the condition that the zeros of $Z'(z,t)$ follow geodesics. In the general case $J\neq 0$ $(J^2\leq M^2l^2)$, we can write the one-black-hole solution (1) as

$$ds^2=h^2\left(\nu dt+\frac{J}{2ch^2}\,dY\right)^2-\frac{l^2}{c^2}\left(h^2+M+\frac{J^2}{4l^2h^2}\right)\times(dX^2+dY^2), \tag{15}$$

where $h^2=r^2/l^2-M$ is related to X by

$$\frac{dh}{dX}=c^{-1}\left[h^2+M+\frac{J^2}{4l^2h^2}\right], \tag{16}$$

and $Y=c\theta$. The construction then proceeds as before, except that the solution $Z(z,t)$ must be analytically continued beyond $h^2=0$ to the largest, negative root h^2_+ of the right-hand side of (16) (the event horizon).

The evolution of the symmetrical two-black-hole system with a conical singularity[1] at $z=0$ may again be inferred

[1]The conclusion that static multi-black-hole systems necessarily contain conical singularities was independently obtained by Coussaert and Henneaux [10].

from the geodesic motion of a point particle in the one-black-hole metric. While the Penrose diagram for the maximally extended one-black-hole space-time contains an infinite sequence of regions I, II, and III [1], we are only concerned with geodesic motion of the conical singularity in a given region I, which is qualitatively similar to that shown in Fig. 3, the Rindler horizons being replaced by event horizons (the conical singularity is born on the past event horizon, and dies on the future event horizon). It follows that, given an initial two-black-hole (plus conical singularity) configuration at time $t=0$, a distant observer sees the two black holes falling toward each other, eventually merging (and absorbing the conical singularity) after an infinite coordinate time.

Particularly interesting is the extreme case $J^2=M^2l^2$, in which we would expect that the gravitational attraction and the centrifugal repulsion may balance, resulting in stationary solutions. Indeed, it is easy to show that the lines $r=a_0$, $d\theta=(\nu/l)dt$ are lightlike geodesics for arbitrary a_0, corresponding to stationary systems of two black holes orbiting around the conical singularity at the constant angular velocity $\nu/2l$.

We have studied the classical dynamics of wormholes and black holes in three-dimensional cosmological gravity. Limiting cases of special interest are pure gravity ($\Lambda=0$) where, despite the fact that space-time is (almost everywhere) flat, we have obtained dynamical systems of freely falling wormholes and conical singularities with nontrivial topology, and extreme black holes ($J^2=-M^2/\Lambda$), which may interact together with conical singularities to form stationary planetary systems.

I wish to acknowledge stimulating discussions with Bernard Linet, Gary Horowitz, Marc Henneaux, and Olivier Coussaert.

[1] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).

[2] G. T. Horowitz and D. L. Welch, Phys. Rev. Lett. **71**, 328 (1993); A. Achúcarro and M. E. Ortiz, Phys. Rev. D **48**, 3600 (1993); N. Kaloper, *ibid.* **48**, 2598 (1993); D. Cangemi, M. Leblanc, and R. B. Mann, *ibid.* **48**, 3606 (1993); J. Gamboa and A. J. Seguí-Santonja, Class. Quantum Grav. **9**, L111 (1992).

[3] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).

[4] G. Clément, Int. J. Theor. Phys. **24**, 267 (1985).

[5] S. Deser and R. Jackiw, Ann. Phys. (N.Y.) **153**, 405 (1984).

[6] W. Rindler, *Essential Relativity* (Springer-Verlag, New York, 1977).

[7] G. Clément, Ann. Phys. (N.Y.) **201**, 241 (1990).

[8] G. Clément and I. Zouzou, this issue, Phys. Rev. D **50**, 7271 (1994).

[9] P. S. Letelier and D. V. Gal'tsov, Class. Quantum Grav. **10**, L101 (1993).

[10] O. Coussaert and M. Henneaux, Phys. Rev. Lett. **72**, 183 (1994).