

# PHYSICAL REVIEW D

## PARTICLES, FIELDS, GRAVITATION, AND COSMOLOGY

THIRD SERIES, VOLUME 50, NUMBER 12

15 DECEMBER 1994

### RAPID COMMUNICATIONS

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#### Filtering post-Newtonian gravitational waves from coalescing binaries

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(Received 28 June 1994)

Gravitational waves from inspiralling binaries are expected to be detected using a data analysis technique known as *matched filtering*. This technique is applicable whenever the form of the signal is known accurately. Though we know the form of the signal precisely, we will not know *a priori* its parameters. Hence it is essential to filter the raw output through a host of search templates each corresponding to different values of the parameters. The number of search templates needed in detecting the Newtonian waveform characterized by three independent parameters is itself several thousands. With the inclusion of post-Newtonian corrections the inspiral waveform will have four independent parameters and this, it was thought, would lead to an increase in the number of filters by several orders of magnitude—an unfavorable feature since it would drastically slow down data analysis. In this Rapid Communication I show that by a judicious choice of signal parameters we can work, even when the first post-Newtonian corrections are included, with as many number of parameters as in the Newtonian case. In other words I demonstrate that the effective dimensionality of the signal parameter space does not change when first post-Newtonian corrections are taken into account.

PACS number(s): 04.80.Nn, 04.25.Nx, 95.55.Ym

Coalescing systems of compact binaries are the most promising sources of gravitational radiation for the planned laser interferometric gravitational wave detectors. As a binary system of stars inspirals, due to radiation reaction, the gravitational wave sweeps up in amplitude and frequency. The resulting inspiral waveform is often called the chirp waveform. As the wave sweeps up in frequency secular post-Newtonian corrections to the phase of the waveform become substantial [1]. When such corrections are incorporated in the templates that are used in detecting the gravitational wave signal it may be possible to glean useful astrophysical information such as the masses of the component stars [1,2], their equation of state, etc. In addition, observation of several such coalescence events can facilitate an accurate determination of cosmological parameters, such as the Hubble constant and after a few hundred events the density parameter [3,4]. In order to extract information of this kind, it is absolutely essential that the parameters of the waveform be determined very accurately.

Optimal Weiner filtering or matched filtering is a data analysis technique that allows a very precise determination of signal parameters [5]. The method consists in correlating the raw output of a detector with a waveform, variously known as a template or a filter, whose Fourier transform is the Fourier transform of the signal divided by the noise power spectrum. A decision about the presence or the absence of a signal is made by looking at the maximum of the correlation. A template whose parameters are exactly matched with those of the signal enhances the signal-to-noise ratio (SNR) in proportion to the square root of the number of cycles that the signal spends in the detector output, as opposed to the case when the shape of the waveform is not known *a priori* and all that can be done is to prebandpass filter the detector output to the frequency band where the signal is assumed to lie, and then look at the SNR for each data point in the time domain individually [6]. For an interferometric detector, such as the Laser Interferometric Gravitational Wave Observatory (LIGO) or the VIRGO, operating with a lower frequency cutoff  $\sim 40$  Hz and an upper cutoff  $\sim 1$  kHz, this means an amplification in the SNR  $\sim 30$ – $40$  for

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the inspiral waveform from a typical binary. This enhancement in the SNR not only increases the number of detectable events but, more importantly, it also allows a more accurate determination of signal parameters—the error in the estimation of a parameter being inversely proportional to the SNR. In order to take full advantage of matched filtering it is essential that the inspiralling binary waveform, and in particular the evolution of its phase, be known to a very high degree of accuracy [1]. A mismatch in the phases of the template and the signal can drastically reduce the SNR. Though we are confident about the event rate, and hence know the probability of the signal being present in the data train, we will not know *a priori* what the signal parameters are. Consequently, the detector output needs to be filtered through a number of templates each corresponding to a particular set of “test” parameters. The number of search templates needed to cover the astrophysically relevant range of the parameter space depends primarily on the dimensionality of the parameter space.

Cutler *et al.* [1] pointed out that for the purpose of constructing a set of matched filters to detect gravitational waves from inspiralling binaries it is both necessary and sufficient to work with the so-called *restricted* post-Newtonian gravitational waveform. In this approximation one incorporates the post-Newtonian corrections only to the phase of the waveform working always with just the Newtonian amplitude. Going beyond the restricted post-Newtonian approximation is not expected to change appreciably the magnitude of the statistical errors in the parameter extraction, so the restricted post-Newtonian approximation can be used to estimate statistical errors. However, in the postdetection analysis it is necessary to employ more accurate templates since the use of just the restricted post-Newtonian waveform would give rise to some systematic errors. In the restricted post-Newtonian approximation the gravitational waves from a binary system of stars, modeled as point masses orbiting about each other in a circular orbit, induce a strain  $h(t)$  at the detector given by

$$h(t) = A[\pi f(t)]^{2/3} \cos[\varphi(t)], \quad (1)$$

where  $f(t)$ , the instantaneous gravitational wave frequency, is equal to twice the orbital frequency of the binary; the constant  $A$  involves the distance to the binary, its reduced and total mass, and the antenna pattern of the detector. The detailed form of  $A$  will not be of any concern in this Rapid Communication. The phase of the waveform can be schematically written as

$$\varphi(t) = \varphi_N(t) + \varphi_{P1N} + \varphi_{P1.5N} + \dots \quad (2)$$

Here  $\varphi_N(t)$  is the dominant Newtonian part of the phase and  $\varphi_{PnN}$  represents the  $n$ th order post-Newtonian correction to it. In the quadrupole approximation we have only the Newtonian part of the phase given by [5]

$$\varphi(t) = \varphi_N(t) = \frac{16\pi f_a \tau_N}{5} \left[ 1 - \left( \frac{f}{f_a} \right)^{-5/3} \right] + \Phi. \quad (3)$$

Here  $f(t)$  is the instantaneous Newtonian gravitational wave frequency given implicitly by

$$t - t_a = \tau_N \left[ 1 - \left( \frac{f}{f_a} \right)^{-8/3} \right], \quad (4)$$

$\tau_N$  is a constant having dimensions of time given by

$$\tau_N = \frac{5}{256} \mathcal{M}^{-5/3} (\pi f_a)^{-8/3}, \quad (5)$$

and  $f_a$  and  $\Phi$  are the instantaneous gravitational wave frequency and the phase of the signal, respectively, at  $t = t_a$ . Even though it is possible to invert  $f$  in terms of  $t$  we shall continue to work with (4) since it allows a straightforward interpretation of the parameter  $\tau_N$  and later of a similar post-Newtonian parameter. We shall refer to the time elapsed starting from an epoch when the gravitational wave frequency is  $f_a$  till the epoch when it becomes infinite, at which time the two stars would theoretically coalesce, as the *chirp time* of the signal. In the quadrupole approximation  $\tau_N$  is the chirp time. The Newtonian part of the phase, namely, Eq. (3), is essentially characterized by three parameters: (i) the *time of arrival*  $t_a$  when the signal first becomes *visible* in the detector, (ii) the *phase*  $\Phi$  of the signal at the time of arrival, and (iii) the *chirp mass*  $\mathcal{M} = (\mu^3 M^2)^{1/5}$ , where  $\mu$  and  $M$  are the reduced and the total mass of the binary, respectively. Note that at this level of approximation the phase (as also the amplitude) depends on the masses of the two stars only through the above combination of the individual masses. Consequently, two coalescing binary signals of the same chirp mass but of different sets of individual masses would be degenerate and thus exhibit exactly the same time evolution. This degeneracy, as we shall see below, will be removed when post-Newtonian corrections are included in the phase of the waveform.

How many search templates are needed to cover an interesting range of the parameter space if we restrict ourselves to the Newtonian waveform? Sathyaprakash and Dhurandhar [7] have made a detailed analysis of this question and a typical number of filters they quote is about a thousand. They have also pointed out that the present-day computer technology is well equipped to filter the detector output on-line. However, as pointed out earlier it is not enough to consider just the Newtonian waveform. Inclusion of post-Newtonian corrections serves a dual purpose: On the one hand unless the secular post-Newtonian corrections are included in the phase of the search templates there would be a severe drop in the SNR. On the other hand, and more importantly, in order to do interesting astrophysics with gravitational waves it is essential to remove the degeneracy in the wave forms by including post-Newtonian corrections.

When post-Newtonian corrections are included the parameter space of waveform acquires an extra dimension. It was feared that this would mean a severe burden on data analysis: an extra dimension of the parameter space implies that it would be necessary to construct for each of the thousand odd Newtonian filters a similar number of filters corresponding to the post-Newtonian parameter. Several authors have therefore analyzed the effectiveness of a Newtonian template with “wrong” parameters to detect a post-Newtonian signal [8,9]. However, these authors conclude that even after allowing for a mismatch in the parameters of a Newtonian template and a post-Newtonian signal the SNR

would reduce by about 10–20%. Such a drop in the SNR is unfavorable considering the low event rate of these sources. In this Rapid Communication I will show that when the first post-Newtonian corrections are included in the phase of the waveform it is possible to make a judicious choice of the parameters so that the parameter space essentially remains only three dimensional. While such a strategy is suitable for an unambiguous and easy detection it does by no means guarantee a precise estimation of all of the binary's parameters. The bottom line of this Rapid Communication is that the algorithm presented here enables an enhancement in the SNR by including post-Newtonian corrections in the search templates without at the same time causing any extra burden on data analysis. It should, however, be noted that further off-line analysis would be necessary to extract useful astrophysical information.

With the inclusion of first post-Newtonian correction the phase of the waveform becomes [10,11]

$$\varphi(t) = \varphi_N(t) + \varphi_{P1N}(t), \quad (6)$$

where  $\varphi_N(t)$  is given by (3) and

$$\varphi_{P1N}(t) = 4\pi f_a \tau_{P1N} \left[ 1 - \left( \frac{f}{f_a} \right)^{-1} \right]. \quad (7)$$

Now  $f(t)$  is the instantaneous post-Newtonian frequency given implicitly by

$$t - t_a = \tau_N \left[ 1 - \left( \frac{f}{f_a} \right)^{-8/3} \right] + \tau_{P1N} \left[ 1 - \left( \frac{f}{f_a} \right)^{-2} \right], \quad (8)$$

$\tau_N$  is given by (5) and  $\tau_{P1N}$  is a constant having dimensions of time given by

$$\tau_{P1N} = \frac{5(743 + 924\eta)}{64512\mu(\pi f_a)^2}. \quad (9)$$

The phase (7) now contains the reduced mass  $\mu$  and the parameter  $\eta = \mu/M$ , in addition to the chirp mass  $\mathcal{M}$ . Taking  $(\mathcal{M}, \eta)$  to be the post-Newtonian mass parameters the total mass and the reduced mass are given by  $M = \mathcal{M} \eta^{-3/5}$ ,  $\mu = \mathcal{M} \eta^{2/5}$ . Note that the total chirp time  $\tau$  of the signal has a Newtonian contribution  $\tau_N$  and a post-Newtonian contribution  $\tau_{P1N}$ . The time left starting from an epoch when the gravitational wave frequency is  $f_a$  until an epoch when the frequency becomes infinite is  $\tau = \tau_N + \tau_{P1N}$ . We shall refer to  $\tau_N$  as the Newtonian chirp time and to  $\tau_{P1N}$  as the post-Newtonian chirp time. Note that instead of working with the parameters  $\mathcal{M}$  and  $\eta$  we can equivalently work with the parameters  $\tau_N$  and  $\tau_{P1N}$ .

Thus the post-Newtonian filter is characterized by four parameters:  $\lambda_k = \{t_a, \Phi, \tau_N, \tau_{P1N}\}$  where we have used the symbol  $\lambda_k$ ,  $k = 1, \dots, 4$ , to collectively denote the four parameters. Note that  $t_a$  and  $\Phi$  are *kinematical* parameters that fix the origin of the measurement of time and phase, respectively, whereas the Newtonian and the post-Newtonian chirp times are *dynamical* parameters in the sense that they decide the evolution of the phase and the amplitude of the signal. We shall now set out to see if it is possible, for the purpose of filtering, to reduce the dimensionality of the parameter space from four to three.

We begin by defining the scalar product of waveforms which plays a crucial role in deciding the filters that are required to span the range of parameters and hence to assess the effective dimensionality of the parameter space. Given two waveforms  $g(t)$  and  $h(t)$  their scalar product is defined by

$$\langle g, h \rangle \equiv \int_{-\infty}^{\infty} \frac{\tilde{g}(f) \tilde{h}^*(f)}{S_n(f)} df, \quad (10)$$

where  $S_n(f)$  is the *two-sided* detector noise power spectral density and  $\tilde{g}(f) = \int_{-\infty}^{\infty} g(t) \exp(2\pi i f t) dt$  and  $\tilde{h}(f) = \int_{-\infty}^{\infty} h(t) \exp(2\pi i f t) dt$ , are the Fourier transforms of the waveforms  $g(t)$  and  $h(t)$ , respectively. The SNR  $\rho$  obtained for a signal  $h(t)$  using an optimal Weiner filter is simply the norm of the signal computed using the above definition of the scalar product:

$$\rho = \langle h, h \rangle. \quad (11)$$

A waveform is said to be *normalized* if its norm is equal to unity. Let us consider the behavior of the scalar product of two chirp waveforms  $g(t; \lambda_k)$  and  $h(t; \lambda_k^0)$  which differ in all their parameter values, i.e.,  $\lambda_k$  being in general different from  $\lambda_k^0$ , and are normalized, i.e.,  $\langle h, h \rangle = \langle g, g \rangle = 1$ . Their scalar product  $C(\lambda_k, \lambda_k^0)$  is given by

$$C(\lambda_k, \lambda_k^0) = \langle g(\lambda_k), h(\lambda_k^0) \rangle. \quad (12)$$

Here  $\lambda_k$  can be thought of as the parameters of a signal while  $\lambda_k^0$  those of a template. Then  $C(\lambda_k, \lambda_k^0)$  is the SNR obtained using a template that is not necessarily matched on to the signal. Since the waveforms are of unit norm  $C(\lambda_k, \lambda_k^0) = 1$ , if  $\lambda_k = \lambda_k^0$  and  $C(\lambda_k, \lambda_k^0) < 1$ , if  $\lambda_k \neq \lambda_k^0$ . In general, as indicated by its arguments  $C(\lambda_k, \lambda_k^0)$  depends on the individual values of the parameters both of the signal and the template. In what follows I will first show that the SNR (12) depends only on the difference in the parameter values  $\lambda_k - \lambda_k^0$ . Second, I will show that a template of a given total chirp time obtains roughly the same SNR for all waveforms of the same total chirp time though their Newtonian and post-Newtonian chirp times may be different from that of the template (see, however, the discussion at the end of the paper). The former of these two results implies uniformity in the spacing of filters [7,12] while the latter result facilitates a massive reduction in the number of templates required in spanning the parameter space since instead of constructing filters separately for each of the Newtonian and post-Newtonian chirp times we can construct filters simply for the total chirp time.

In the stationary phase approximation the Fourier transform of the restricted first-post-Newtonian chirp waveform for positive frequencies is given by [5,7,13]

$$\tilde{h}(f) = \tilde{A} f^{-7/6} \exp \left[ i \sum_{k=1}^4 \psi_k(f) \lambda_k - i \frac{\pi}{4} \right], \quad (13a)$$

where  $\tilde{A}$  is a constant and

$$\psi_1 = 2\pi f, \quad (13b)$$

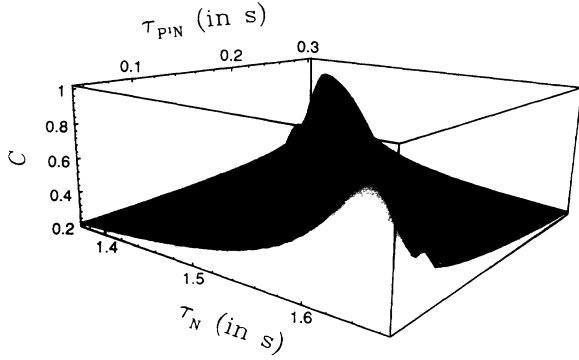


FIG. 1. Surface showing the maximum, over  $t_a$  and  $\Phi$ , of the SNR  $C(\delta\lambda_k)$ .

$$\psi_2 = 1, \quad (13c)$$

$$\psi_3 = 2\pi f - \frac{16\pi f a}{5} + \frac{6\pi f a}{5} \left(\frac{f}{f_a}\right)^{-5/3}, \quad (13d)$$

$$\psi_4 = 2\pi f - \pi f a + 2\pi f a \left(\frac{f}{f_a}\right)^{-1}. \quad (13e)$$

For  $f < 0$  the Fourier transform is computed using the identity  $\tilde{h}(-f) = \tilde{h}^*(f)$  obeyed by real functions  $h(t)$ . With the above expression for the Fourier transform the SNR (12), using (10), takes the form

$$C(\delta\lambda_k) \propto \int_0^\infty \frac{f^{-7/3}}{S_n(f)} \cos\left[\sum_{k=1}^4 \psi_k(f) \delta\lambda_k\right] df, \quad (14)$$

where  $\delta\lambda_k = \lambda_k - \lambda_k^0$ . As in the Newtonian case the SNR is independent of the individual parameter values of the signal and the template: *For all signal-template pairs that have the same differences in times of arrival, phases, and chirp times one obtains the same SNR.* Consequently, constancy of the distance, measured using the scalar product (10), between two nearest neighbor filters, that is required in making a choice of filters, translates into the constancy of the distance, measured using the difference in their parameter values.

We now seek to analyze the behavior of the SNR (14) when the template's parameters are mismatched with those of the signal. While it is essential to do the analysis for noise power spectral density in real laser interferometers, such as the one discussed by Finn and Chernoff [13], the results obtained in that case are qualitatively the same as in the case of white noise [14]. In order not to divert attention from the main theme of discussion, here I will only quote the results for white noise.  $C(\delta\lambda_k)$  traces out a four-dimensional surface as we vary  $\delta\lambda_k$ . In what follows I consider the two-dimensional subspace obtained by maximizing  $C(\delta\lambda_k)$ , over  $t_a$  and  $\Phi$ , for every pair of  $\tau_N$  and  $\tau_{P1N}$  of the signal keeping the parameters of the template  $\tau_N^0$  and  $\tau_{P1N}^0$  constant. The surface so obtained is plotted for white noise [i.e.,  $S_n(f) = \text{const}$ ] in Fig. 1. Since the parameters of the template are constant I have shown on the  $x$  and  $y$  axis of Fig. 1 (and Fig. 2) the parameters of the signal  $\tau_N$  and  $\tau_{P1N}$  and not the difference  $\delta\tau_N$  and  $\delta\tau_{P1N}$ . The same surface is obtained

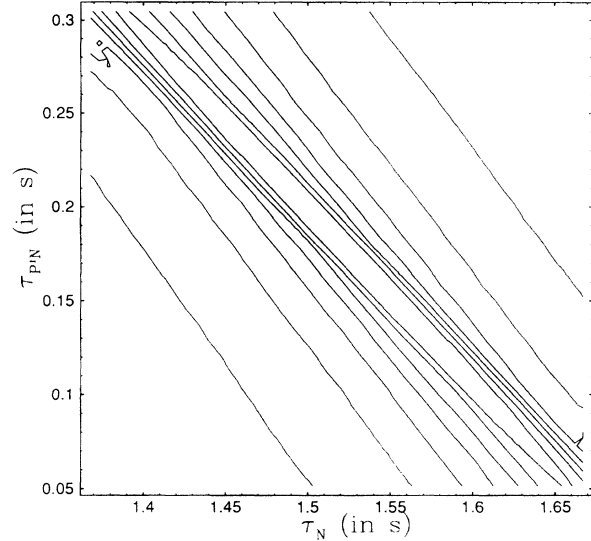


FIG. 2. Contours of the SNR surface shown in Fig. 1. The contours are approximately straight lines  $\tau_N + \tau_{P1N} = \text{const}$ .

irrespective of what values we choose for the parameters of the template provided we keep the range of the signal parameters the same. In this sense, the correlation surface in Fig. 1 only depends on the difference in the parameters of the signal and the template and not on their absolute values. For the astrophysically relevant range of the masses of the two stars (say,  $M_1, M_2 \in [1, 10]M_\odot$ ) and for  $f_a = 100$  Hz,  $\tau_N \in [4, 0.08]$  s and  $\tau_{P1N} \in [0.3, 0.03]$  s. In Fig. 1 the post-Newtonian chirp time is varied over the whole of its relevant range while the Newtonian chirp time is only varied over a portion of its relevant range. The contours of this surface shown in Fig. 2 are *almost* straight lines  $\tau = \tau_N + \tau_{P1N} = \text{const}$ , except for the innermost one or two contours. The value of  $C$  corresponding to the innermost contour is 0.9 and reduces by 0.1 with successive outer contours.

A useful interpretation of the surface in Fig. 1 is the following: Imagine that we have a template with parameters corresponding to the center of the grid. It obtains an SNR of unity with a signal whose parameters are exactly matched onto it. The SNR that it obtains for other wave forms is in general less than unity but, as is evident from Fig. 1, the SNR is almost equal to unity for every waveform whose total chirp time is the same as its own total chirp time. Thus, if we choose our search templates along the curve perpendicular to the contours of Fig. 2 then we will in effect be covering the entire subspace of the signal. In other words *we can span the two-dimensional  $(\tau_N, \tau_{P1N})$  subspace of the four-dimensional parameter space with just one parameter.* This curve is an appropriate one of a family of straight lines  $\tau_N = \tau_{P1N} + \text{const}$ . Consequently, as far as the choice of filters is considered we need only work with three parameters, namely  $\{t_a, \Phi, \tau\}$ . This reduction in the effective number of parameters can be traced to the fact that there is a strong covariance between the parameters  $\tau_N$  and  $\tau_{P1N}$ . This result, together with the details of a Monte Carlo simulation demonstrating the effectiveness of the claim made in this paper and the results of including higher-order post-Newtonian corrections, and noise envisaged in real interferometers will be published elsewhere [14].

Let me conclude by making two cautionary remarks. The first one concerns the choice of parameters: It should not be thought that the effective dimensionality of the parameter space is three only when the set  $\{t_a, \Phi, \tau_N, \tau_{P1N}\}$  is employed in constructing a lattice of filters. After all the reduction of dimensionality is related to the property of the scalar product (10) which is reparametrization invariant. The advantage of the set used in this Rapid Communication is that it allows us to conclude about the effective dimensionality without recourse to complicated mathematical analysis. However, the final justification has to come from a more rigorous analysis which will be taken up in a future paper [14]. The second comment is about the scope of the reduced dimensionality of the parameter space itself. The parameter space would be truly three-dimensional provided the correlation function is a constant in the direction  $\tau_N + \tau_{P1N} = \text{const}$ . However, as can be inferred from Figs. 1 and 2, strictly speaking, this is not the case. The correlation func-

tion slowly decreases as we move away from the maximum of the correlation function in the direction  $\tau_N + \tau_{P1N} = \text{const}$ . This means two things: (i) The argument about the reduction in dimensionality is only valid as long as the correlation function has not dropped too much (ideally less than about 1%) along lines of constant  $\tau_N + \tau_{P1N}$  and (ii) a post-Newtonian filter of a given total chirp time *cannot* be replaced by a Newtonian filter of the same chirp time. In other words, the presence of the post-Newtonian term cannot be mimicked by a Newtonian filter alone. However, for the astrophysically relevant range of the parameter  $\tau_{P1N}$  it turns out that we need only use a three-dimensional lattice of filters or at worst two sets of a three-dimensional lattice.

It is a pleasure to thank the members of the gravitational wave group at IUCAA, especially R. Balasubramanian, Sanjeev Dhurandhar, and Kanti Jotania, for many useful discussions.

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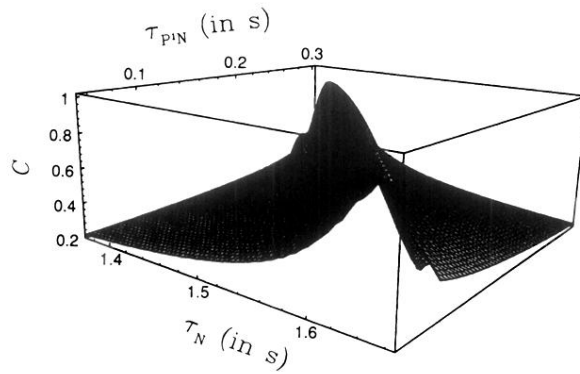


FIG. 1. Surface showing the maximum, over  $t_a$  and  $\Phi$ , of the SNR  $C(\delta\lambda_k)$ .