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Free fermion representation of a boundary conformal field theory

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The theory of a massless two-dimensional scalar field with a periodic boundary interaction is considered. At a critical value of the period this system defines a conformal field theory and can be reexpressed in terms of free fermions, which provide a simple realization of a hidden SU(2) symmetry of the original theory. The partition function and the boundary S matrix can be computed exactly as a function of the strength of the boundary interaction. We first consider open strings with one interacting and one Dirichlet boundary, and then with two interacting boundaries. The latter corresponds to motion in a periodic tachyon background, and the spectrum exhibits an interesting band structure which interpolates between free propagation and tight binding as the interaction strength is varied.

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I. INTRODUCTION

Boundary conformal field theories [1] find applications in different branches of physics ranging from condensed matter systems [2–6] to particle physics [7,8] and string theory [9,10]. A simple but interesting example is provided by a scalar field in two spacetime dimensions which is free except for a periodic interaction at a boundary. This model arises in the Caldeira-Leggett [2] description of the dissipative quantum mechanics of a particle which is moving in a periodic potential. In this context, a perturbative renormalization group analysis, carried out by Fisher and Zwerger [3] and by Guinea *et al.* [4], showed that there is a critical value of the potential period where the system undergoes a localization transition. The same model comes up in the study of open string theory in background fields [10]. Here the critical potential corresponds to a particular solution of the equations of motion of open string field theory. From the string theory point of view it is natural to expect the critical theory to exhibit not only scale invariance but in fact an infinite dimensional symmetry generated by arbitrary reparametrizations of the boundary.

In a recent paper this problem was reexamined by Callan and Klebanov [11] who phrased it in the language of two-dimensional conformal field theory with boundary interactions. They obtained partial results for the partition function and scattering amplitudes for a boundary potential with the critical period and made a strong case for the exact conformal invariance of the model. In the present paper we analyze the model further and show how it can be rewritten as a theory of free fermions. The periodic potential translates into a fermion mass term, localized at the boundary, which twists the fermion boundary conditions by a coupling-dependent phase. The exact partition function is easily calculated in the fermion language and scattering amplitudes can be obtained in a straightforward manner. The fermions form a representation of a level one SU(2) current algebra. This symmetry is not immediately apparent in the original boson theory although it was clearly emerging in the work of Callan and Klebanov [11].

In Secs. II and III we consider open strings with one interacting and one Dirichlet boundary. In Sec. IV we briefly discuss the boundary S matrix. In Sec. V we consider the case of two interacting boundaries and find the band structure in the spectrum. Various subtleties, including cocycles, the precise mapping between bosonic and fermionic Hilbert spaces, and operator ordering, arise and are dealt with. After obtaining many of these results we received a paper by Callan *et al.* [12] who have also obtained an exact solution of the theory by using the underlying SU(2) algebra.

II. THE SYSTEM

We are interested in the physics of a massless scalar field living on a two-dimensional spacetime with a boundary, where it is subject to a periodic potential. The Lagrangian, in units where $\alpha' = 2$, is

$$L = \frac{1}{8\pi}\int_0^l d\sigma(\partial_\mu X)^2 - \frac{1}{2}(g e^{iX(0)/\sqrt{2}} + \bar{g} e^{-iX(0)/\sqrt{2}}), \qquad (1)$$

where g is a complex parameter which dials the strength of the boundary interaction. Following Callan and Klebanov [11] we include a second boundary at $\sigma = l$ and impose Dirichlet boundary conditions there to control infrared behavior. The period in (1) is chosen such that the potential has dimension one under boundary scaling and thus defines a marginal perturbation on the free theory [3,4].

We find it convenient to map the theory (1), which is defined on a strip of width l into a chiral theory which lives on circle of circumstance $2l$. To see how this comes about let us first consider the $g=0$ theory where the boundary interaction has been turned off and the scalar field satisfies a Neumann condition at one boundary and a Dirichlet condition at the other. Away from the boundaries a free field can be written as a sum of left- and right-moving components: $X(\sigma,t) = X_L(\sigma+t) + X_R(\sigma-t)$. The Neumann boundary condition at $\sigma=0$ determines the right movers in terms of the left movers:

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$$X_R(0-t)=X_L(0+t).$$

Thus we can work with left movers on the circle. The Dirichlet condition on $X(\sigma,t)$ at $\sigma=l$ then implies that

$$X_R(l-t)=-X_L(l+t),$$

with opposite sign from the Neumann condition. The combination of the two boundary conditions implies that

$$X_L(\sigma+2l)=-X_L(\sigma),$$

so the left mover is antiperiodic on the circle of circumference $2l$.

The boundary interaction is easily expressed in terms of the left-moving field alone:

$$L_g=-\tfrac{1}{2}(ge^{i\sqrt{2}X_L(\sigma^+)}+\bar{g}e^{-i\sqrt{2}X_L(\sigma^+)})|_{\sigma=0}. \quad (2)$$

Note that because $X(0,t)=2X_L(\sigma^+)|_{\sigma=0}$ the coefficient in the exponential has changed.[1] The operators $e^{\pm i\sqrt{2}X_L(\sigma^+)}$ have conformal dimension (1,0) and along with $(i/\sqrt{2})\partial_+X_L(\sigma^+)$ they form the currents of a left-moving SU(2) algebra. It is this symmetry which enables the exact solution of the model.

III. EXACT SOLUTION IN TERMS OF FERMIONS

We now show how to express the interaction in terms of fermions. Since the scaling dimension of the boundary potential is (1,0) we will need two independent left-moving fermions, each of dimension $(\frac{1}{2},0)$. The first step is to introduce an auxiliary antiperiodic left-moving boson $Y_L(\sigma^+)$. We then define a pair Ψ of left-moving fermions:

$$\psi_1\sim e^{i(Y_L-X_L)/\sqrt{2}}\equiv e^{i\phi_{L1}}, \quad \psi_2\sim e^{i(Y_L+X_L)/\sqrt{2}}\equiv e^{i\phi_{L2}}. \quad (3)$$

The extra boson Y_L does not appear in the interaction and so decouples, but is needed for the fermionic representation.[2] Actually, the fermionization (3) is not quite right because ψ_1 and ψ_2 commute, being constructed from orthogonal linear combinations. A cocycle is needed. In familiar examples (such as the Neumann-Neumann case to be considered in Sec. V), this is constructed from the bosonic zero modes, but here the bosons have no zero modes. We instead add a two-state system S to the bosonic theory, and then

$$\psi_1=\boldsymbol{\sigma}^1e^{i\phi_{L1}}, \quad \psi_2=\boldsymbol{\sigma}^2e^{i\phi_{L2}}, \quad (4)$$

where the cocycle is written in terms of the σ matrices acting in S. Boldface is used to distinguish these operators in the space S from ordinary SU(2) matrices as will appear below.

The bosonic theory plus S is equivalent to the fermionic, as we now verify by comparing partition functions. The antiperiodic boundary condition on the bosons translates into conjugation on the fermions:

$$\psi_i(\sigma^++2l)=\psi_i^\dagger(\sigma^+). \quad (5)$$

If we work instead with real fermions χ_i, where $\sqrt{2}\psi_1=\chi_1+i\chi_2$ and $\sqrt{2}\psi_2=\chi_3+i\chi_4$, then these boundary conditions dictate that we have two periodic fermions and two antiperiodic fermions in the free theory. Writing $q=e^{-\pi\beta/l}$, the corresponding free partition function is

$$Z(q)=2q^{1/24}\prod_{n=1}^{\infty}(1+q^n)^2(1+q^{n-1/2})^2$$

$$=2q^{1/24}\prod_{n=1}^{\infty}(1-q^{n-1/2})^{-2}, \quad (6)$$

equaling that of two antiperiodic bosons plus S. The factor of 2 in the first line of (6) is from the fermionic zero modes, and the one in the second line is from S.

The fermionic action

$$L_F=\frac{i}{2\pi}\int_{-l}^{l}d\sigma\Psi^\dagger[\partial_t-\partial_\sigma-iM\delta(\sigma)]\Psi, \quad (7)$$

with $\Psi=[\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}]$, $M=\pi(g_1\sigma^2+g_2\sigma^1)$, and $g=g_1+ig_2$, is then equivalent to the X_L-Y_L-S system with the interaction

$$L_g'=-\frac{\boldsymbol{\sigma}^3}{2}(ge^{i\sqrt{2}X_L(\sigma^+)}+\bar{g}e^{-i\sqrt{2}X_L(\sigma^+)})|_{\sigma=0}. \quad (8)$$

We thus have a sum of two copies of the interacting theory, with opposite signs for g. Since the fermions (4) flip the sign of $\boldsymbol{\sigma}^3$, the desired single copy can be obtained by projection on fermion number mod 2.

The normal modes of frequency ν,

$$\Psi_\nu(t,\sigma)=e^{-i\nu t}\tilde{\Psi}_\nu(\sigma)+e^{i\nu t}\tilde{\Psi}_{-\nu}(\sigma), \quad (9)$$

satisfy

$$\tilde{\Psi}_\nu'=-i[\nu+M\delta(\sigma)]\tilde{\Psi}_\nu. \quad (10)$$

This is ill-defined because the δ function multiplies $\tilde{\Psi}$, which is discontinuous at $\sigma=0$. This is where the issue of regularization comes up in the fermion language. We deal with it by smearing out the δ function symmetrically around $\sigma=0$ by a smooth function which satisfies $f(\sigma)=f(-\sigma)$ and $\int_{-l}^{l}d\sigma f(\sigma)=1$. This prescription produces an unambiguous answer for the partition function. In terms of the original bosonic theory, this is equivalent (in Euclidean time) to extending the interaction analytically in $\sigma+i\tau$ from $\sigma=0$ and then smearing the integration contour. It is thus the same as the principle value prescription adopted by Callan *et al.* [12]. The integrated result is then

$$\tilde{\Psi}_\nu(l)=e^{-2il\nu-iM}\tilde{\Psi}_\nu(-l). \quad (11)$$

The boundary condition $\tilde{\Psi}_\nu(l)=\tilde{\Psi}^*_{-\nu}(-l)$ then implies

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[1] More generally, $e^{ikX(0,t)}=e^{ik_LX_L(t)}$ with $k_L=2k$. Thus we must, and will, be careful to distinguish k_L and k.

[2] The representation of the boundary interaction in terms of free fermions is due to Guinea *et al.* [4]. The extra boson was implicit in their work, while for our purposes (the partition function and spectrum) we need to develop the Fermi-Bose equivalence in much more detail.

$$\tilde{\Psi}_\nu(l)=e^{-4il\nu}e^{-iM}e^{iM^*}\tilde{\Psi}_\nu(l), \quad (12)$$

thus determining the eigenvalues ν. A short calculation gives $e^{4il\nu}=e^{\pm 2i\Delta(g,\bar{g})}$, with

$$\Delta(g,\bar{g})=\arcsin\left[\frac{g+\bar{g}}{2|g|}\sin\pi|g|\right]. \quad (13)$$

The boundary interaction shifts the frequencies and lifts the degeneracy of the two pairs of real fermions.

Summing over eigenfrequencies $\nu=(n\pi\pm\Delta)/2l$, the fermion partition function is

$$Z_F^\eta(q,\alpha)=\mathrm{Tr}[\eta^F q^{H\pi/l}]=q^{\alpha^2-(1/2)\alpha+1/24}(1+\eta q^\alpha)\times\prod_{n=1}^{\infty}(1+\eta q^{n/2+\alpha})(1+\eta q^{n/2-\alpha}), \quad (14)$$

where the shift $\alpha=\Delta(g,\bar{g})/2\pi$ is a coupling-dependent constant. The required projection which selects out states of odd fermion number is

$$Z_F^{\mathrm{odd}}=\tfrac{1}{2}[Z_F^{(+)}-Z_F^{(-)}], \quad (15)$$

with $Z_F^{(\pm)}(q,\alpha)=q^{\alpha^2-(1/2)\alpha+1/24}(1\pm q^\alpha)\Pi_{n=1}^{\infty}(1\pm q^{n/2+\alpha})\times(1\pm q^{n/2-\alpha})$. Changing the sign of the second term in (15) selects states of even fermion number and leads to the same projected partition function up to an overall sign change of both g and $\bar{g}$. The desired sign (15) is most easily determined by comparing with lowest order perturbation theory.

By using Jacobi's triple product formula, the projected partition function can be represented as a sum:

$$Z_F^{\mathrm{odd}}(q,\alpha)=q^{-1/48}\prod_{m=1}^{\infty}(1-q^{m/2})^{-1}\times\sum_{n=-\infty}^{\infty}q^{(1/4)[n+1/2+2\alpha]^2}\left[\frac{1+(-1)^n}{2}\right]. \quad (16)$$

Finally, to obtain the exact partition function of the original interacting boson theory we have to divide (16) by the partition function, $Z_0(q)=q^{1/48}\Pi_{n=1}^{\infty}(1-q^{n-1/2})^{-1}$, of the free antiperiodic boson we added to the system:

$$Z_g(q)=\frac{Z_F^{\mathrm{odd}}(q,\alpha)}{Z_0(q)}=q^{-1/24}\prod_{m=1}^{\infty}(1-q^m)^{-1}\times\sum_{n=-\infty}^{\infty}q^{(1/4)\{2n+1/2+[\Delta(g,\bar{g})/\pi]\}^2}. \quad (17)$$

The terms in the sum can be rearranged to get it into the form of a sum over Virasoro modules:

$$Z_g(q)=q^{-1/24}\prod_{m=1}^{\infty}(1-q^m)^{-1}\times\sum_{k=0}^{\infty}q^{(1/4)\{k+1/2+(-1)^k[\Delta(g,\bar{g})/\pi]\}^2}. \quad (18)$$

This result was obtained independently by Callan *et al.* [12] using a different approach.

IV. BOUNDARY S MATRIX

It is also straightforward to calculate amplitudes for scattering off the interacting boundary in this model. In this case one wants to discuss asymptotic incoming and outgoing states so we take the Dirichlet boundary off to $l\to\infty$. The map to a chiral theory is very useful for amplitude calculations and we also find it convenient to rotate to Euclidean signature. Incident left movers then have $\mathrm{Re}(z)>0$ and the outgoing states are also left moving but have $\mathrm{Re}(z)<0$. The boundary interaction is localized on the contour $\mathrm{Re}(z)=0$. Away from the boundary, left-moving bosons are created by the insertion of $\partial_z X(z)$ operators into the Euclidean path integral.

The boundary interaction with critical period defines a dimension (1,0) operator in the chiral theory. It is integrated along $\mathrm{Re}(z)=0$ in the Euclidean action but we are free to deform the contour into the complex plane. For the calculation of any given amplitude the contour can be moved to the right until it surrounds each of the in-state vertex insertions (or, if preferred, it can be moved to the left to surround the out states). This generates a global SU(2) transformation $e^{i\pi(gJ_++\bar{g}J_-)}$, where $J_+=\oint(dz/2\pi i)\psi_1^\dagger\psi_2$ and $J_-=\oint(dz/2\pi i)\psi_2^\dagger\psi_1$, on each in state. The effect of the SU (2) rotation on each state can be computed by using the SU (2) current algebra, as shown by Callan *et al.* [12], or it can be obtained directly from the free fermion operator product expansion. Once the effect of the boundary interaction has been captured by the SU(2) rotations, the amplitude calculation reduces to an exercise in free field theory. Since the asymptotic states themselves are created by SU(2) generators, one can also utilize the current algebra for this computation. Callan *et al.* [12] work out some explicit examples using this procedure and we will not repeat those calculations here. As these authors point out, the SU(2) rotation generates the soliton operators $e^{i\sqrt{2}X(z)}$ and $e^{-i\sqrt{2}X(z)}$. These operators create nonperturbative kink states, where the X field shifts between adjacent minima of the boundary potential, and amplitudes involving such states must be included in order to obtain a unitary S matrix. In the fermion language both $\partial_z X(z)$ and the soliton operators are represented as fermion bilinears and thus they all enter on equal footing from the start. Since the fermion theory is manifestly a free theory it is clear that we have included all states necessary for unitarity and that soliton-antisoliton states, or states which shift the boson by more than one period, are not independent objects.

V. PERIODIC TACHYON BACKGROUNDS

The boundary conformal field theory can be regarded as an open string tachyon background. An open string propagating in such a background then has Neumann conditions with interactions at both ends. Each end-point interaction can be written in terms of the left mover X_L, but we need to be careful about the relative phase of the two interactions. The open string mode expansion for X is

$$X(\sigma,t)=x+\frac{4\pi}{l}pt+i\sum_{m\neq 0}\frac{\alpha_m}{m}(e^{-im(t+\sigma)\pi/l}+e^{-im(t-\sigma)\pi/l}), \tag{19}$$

while

$$X_L(\sigma+t)=\frac{x}{2}+\frac{2\pi}{l}p(t+\sigma)+i\sum_{m\neq 0}\frac{\alpha_m}{m}e^{-im(t+\sigma)\pi/l}. \tag{20}$$

The zero mode part of $e^{ikX(\sigma,t)}$ is $e^{ikx+4\pi ikpt/l}$, where the symmetric ordering of x and p is appropriate for a tensor in the $\sigma\pm t$ frame, while that of $e^{2ikX_L(\sigma+t)}$ is

$$e^{ikx+4\pi ikp(t+\sigma)/l}=e^{ikx+4\pi ikpt/l}e^{2\pi ik(2p+k)\sigma/l}. \tag{21}$$

The nonzero modes are the same in both operators, so for $k=1/\sqrt{2}$ we have

$$e^{i\sqrt{2}X_L(t)}=e^{iX(0,t)/\sqrt{2}},$$
$$e^{i\sqrt{2}X_L(l+t)}=-e^{iX(l,t)/\sqrt{2}}e^{2\sqrt{2}\pi ip}. \tag{22}$$

In terms of the left-moving boson the interaction is then

$$L_{\rm int}=-\frac{g}{2}(e^{i\sqrt{2}X_L(t)}-e^{i\sqrt{2}X_L(l+t)}e^{-2\sqrt{2}\pi ip})+\text{H.c.} \tag{23}$$

Again the spectrum is easily obtained either from current algebra or the free Fermi representation, and we work with the latter. The fermionization is

$$\psi_1=e^{i\phi_{L1}},\quad \psi_2=e^{i\phi_{L2}}e^{-2\sqrt{2}\pi ip_X}, \tag{24}$$

where we remind the reader of the notation $p_X=p_{LX}/2=(p_{L2}-p_{L1})/2\sqrt{2}$. Now the cocycle is constructed from the bosonic zero mode and the theories are equivalent without additional discrete degrees of freedom.[3] However, we must be careful about the correspondence between bosonic momentum and fermionic boundary conditions. A complex left-moving fermion with the boundary condition

$$\psi_i(l)=-e^{2\pi i\zeta_i}\psi_i(-l) \tag{25}$$

is equivalent to a left-moving boson with momentum $p_{Li}=\zeta_i$ mod $\mathbb{Z}$. This is evident from consideration of the vertex operators, for example $e^{i\zeta\phi_{Li}}$ for the ground state, or from the mode expansion (20) for ϕ_{Li}. For reasons soon to be explained, it is useful to restrict further to states of even fermion number relative to the ground state. This is equivalent to the space of bosonic states with $p_{Li}=\zeta_i$ mod $2\mathbb{Z}$. The respective partition functions are

$$Z_0(q,\zeta)=q^{(1/2)\zeta^2-1/24}\frac{1}{2}\sum_{\pm}\prod_{n=1}^{\infty}(1\pm q^{n+\zeta-1/2})\times(1\pm q^{n-\zeta-1/2})$$
$$=\sum_{m=-\infty}^{\infty}q^{(1/2)(\zeta+2m)^2-1/24}\prod_{n=1}^{\infty}(1-q^n)^{-1}. \tag{26}$$

We will focus on the case that the physical boson X_L is noncompact and take the extra boson Y_L also noncompact, and so must integrate $(2\pi)^{-2}\int\int_0^2 d\zeta_1 d\zeta_2$ to cover the full space.

In fermionic form the interaction is

$$L'_{\rm int}=-\frac{g}{2}[\psi_1^\dagger(t)\psi_2(t)e^{2\sqrt{2}\pi ip_X}-\psi_1^\dagger(l+t)\psi_2(l+t)]+\text{H.c.} \tag{27}$$

This is inconvenient because $e^{2\sqrt{2}\pi ip_X}$ anticommutes with the Fermi fields. However, we are free to consider instead

$$L''_{\rm int}=-\frac{g}{2}[\psi_1^\dagger(t)\psi_2(t)e^{2\sqrt{2}\pi ip_X}(-1)^F-\psi_1^\dagger(l+t)\psi_2(l+t)]+\text{H.c.}, \tag{28}$$

since we have expressed the bosonic space in terms of states of even fermion number. The combination $e^{2\sqrt{2}\pi ip_X}(-l)^F=w$ commutes with the Fermi fields, and is simply equal to $e^{i\pi(\zeta_2-\zeta_1)}$ in a given sector.

The fermionic action for the interacting theory is now

$$L_{BG,F}=\frac{i}{2\pi}\int_{-l}^{l}d\sigma\Psi^\dagger[\partial_t-\partial_\sigma+iN_1\delta(\sigma)-iN_2\delta(\sigma-l)]\Psi, \tag{29}$$

where

$$N_1=\pi\begin{bmatrix}0 & wg\\ \bar{w}\bar{g} & 0\end{bmatrix},\quad N_2=\pi\begin{bmatrix}0 & g\\ \bar{g} & 0\end{bmatrix}. \tag{30}$$

The normal modes now satisfy the periodicity

$$\tilde{\Psi}_\nu(l)=-e^{2\pi i(\zeta_+ +\sigma^3\zeta_-)}\tilde{\Psi}_\nu(-l), \tag{31}$$

where $\zeta_\pm=\frac{1}{2}(\zeta_1\pm\zeta_2)$. The equation of motion gives

$$\tilde{\Psi}_\nu(l)=e^{-2il\nu}e^{-iN_2}e^{iN_1}\tilde{\Psi}_\nu(-l). \tag{32}$$

Together these give the eigenvalue equation

$$-e^{-2il\nu}\tilde{\Psi}_\nu(-l)=e^{-iN_1}e^{iN_2}e^{2\pi i(\zeta_+ +\sigma^3\zeta_-)}\tilde{\Psi}_\nu(-l). \tag{33}$$

The solution is

[3]There is some freedom in the choice of cocycle, and we have chosen to construct it entirely from p_X so that Y will remain decoupled.

$$\frac{l\nu}{\pi}=n+\frac{1}{2}-\zeta_+\pm\lambda, \quad \sin\pi\lambda=\cos\pi|g|\sin\pi\zeta_-, \tag{34}$$

with the value of λ determined by continuity from $\lambda=\zeta_-$ at $g=0$.

The fermionic partition function is then the product of two copies of (26), with $\zeta=\zeta_+\pm\lambda$. In bosonic form this gives

$$Z=\int\int_0^2\frac{d\zeta_1}{2\pi}\frac{d\zeta_2}{2\pi}\sum_{m_1,m_2\in Z}q^{(\zeta_++2m_+)^2+(\lambda+2m_-)^2-1/12}\times\prod_{n=1}^{\infty}(1-q^n)^{-2}, \tag{35}$$

where $m_\pm=(m_1\pm m_2)/2$. We must now regroup in order to separate the contribution of the Y boson. Note first that the sum on (m_1,m_2) amounts to summing (m_+,m_-) over all integer points *and* all integer points plus $(\frac{1}{2},\frac{1}{2})$. The periodicity under $\zeta_1\to\zeta_1+2$ takes the form $m_\pm\to m_\pm+\frac{1}{2}$ and thus we can use this periodicity to enlarge the integration range to $0\leq\zeta_1\leq4$ while restricting (m_+,m_-) to the integer points. We can then use the manifest periodicity in ζ_+ and ζ_- mod $2\mathbb{Z}$ to shift the integration range to $0\leq\zeta_+\leq2$, $0\leq\zeta_-\leq2$ (this is most easily seen by drawing the two regions). Finally, ζ_+ and m_+ combine into a single variable $k_Y/\sqrt{2}=\zeta_++2m_+$ with range $-\infty$ to ∞. Thus,

$$Z=\int_{-\infty}^{\infty}\frac{dk_Y}{2\pi}\int_0^2\frac{d\zeta_-}{\sqrt{2}\pi}\sum_{m_-=-\infty}^{\infty}q^{(1/2)k_Y^2+(\lambda+2m_-)^2-1/12}\times\prod_{n=1}^{\infty}(1-q^n)^{-2}. \tag{36}$$

The desired partition function is

$$\frac{Z}{Z_Y}=\int_0^2\frac{d\zeta_-}{\sqrt{2}\pi}\sum_{m_-=-\infty}^{\infty}q^{\kappa^2-1/24}\prod_{n=1}^{\infty}(1-q^n)^{-1}, \tag{37}$$

where $\kappa=\lambda+2m_-$.

As expected for a particle in a periodic potential, the spectrum has split up into bands. As ζ_- runs through its range, λ runs through the values

$$0\leq\lambda\leq r, \quad 1-r\leq\lambda\leq1+r, \quad 2-r\leq\lambda\leq2,$$
$$2r=|1-2F(|g|)|, \tag{38}$$

with $F(|g|)$ denoting the fractional part of $|g|$. In all, the allowed values of κ consist of bands of width $2r$ centered at every integer, with gaps of width $1-2r$ between. When $|g|$ is any integer, $2r=1$ and the gaps disappear; this corresponds to free propagation, the particle not seeing the potential. When $|g|$ is integer plus $\frac{1}{2}$, the bands have zero width. This is the tight-binding limit, with no tunneling between the minima of the potential.[4] Thus we see the physics of strings moving in a tachyonic crystal. We would expect also to see a "phonon" mode; indeed, at $|\kappa|=1$, the wave number of the potential, there are two massless (weight one) states, representing the phase (translation) and the magnitude of the background.

We can extend readily to the theory compactified at radius $R=q\sqrt{2}$, where q must be an integer because the potential must respect the periodicity. The only effect of compactification of open strings is to restrict to momenta

$$\zeta_-=-k_X\sqrt{2}\in\frac{\mathbb{Z}}{q}. \tag{39}$$

At the self-dual radius $q=1$, ζ_- is an integer and so Eq. (34) implies that the spectrum is unaffected by the potential as found in Ref. [12].

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[4]This agrees with the observation [12] that the latter values correspond to the Dirichlet boundary state.

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