

Fractal basins and chaotic trajectories in multi-black-hole spacetimes

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We investigate the phase space for trajectories in multi-black-hole spacetimes. We find that complete, chaotic geodesics are well described by Lyapunov exponents, and that the attractor basin boundary scales as a fractal in a diffeomorphism-invariant manner.

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Chaos is an aspect of general relativity which has only recently been explored. Its existence is expected as general relativity contains generalizations of well-known chaotic systems from Newtonian physics such as the three-body problem. Moreover, the nonlinearity of Einstein's equations may give rise to chaos in systems whose Newtonian analogue is nonchaotic. The characterization of chaos in general relativity is complicated by the dynamical nature of time, which necessitates a careful generalization of the parameters which quantify chaos, such as Lyapunov exponents.

Relativistic systems in which chaos is known include charged particles in a magnetic field interacting with gravitational waves [1], particles near a magnetized black hole [2], and particles near a black hole interacting with gravitational waves [3], as well as particles in Majumdar-Papapetrou (MP) geometries, described below. There is also the general, sufficient but not necessary, criterion that chaotic geodesics will occur if the phase space is compact and the Ricci scalar is negative [4]. In MP spacetimes the Ricci scalar is identically zero, so this criterion is inapplicable.

Cosmological models such as the mixmaster universe [5-7] and Robertson-Walker models containing dynamical fields [8] have been found to show some of the characteristics of chaotic systems; however, there is difficulty in defining coordinate invariant measures of chaos for these models [9].

MP geometries [10,11] are the static solution of the Einstein-Maxwell equations with metric and electrostatic potential given by

$$ds^2 = -U^{-2}dt^2 + U^2(dx^2 + dy^2 + dz^2), \quad (1)$$

$$A_t = U^{-1}, \quad (2)$$

and the spatial components of A_μ are zero, with U a function of the spatial coordinates satisfying Laplace's equation

$$\nabla^2 U(x, y, z) \equiv U_{,xx} + U_{,yy} + U_{,zz} = 0. \quad (3)$$

Hartle and Hawking [12] showed that if U is of the form

$$U = 1 + \sum_{i=1}^N \frac{M_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}}, \quad (4)$$

the MP metric corresponds to a system of extreme Reissner-Nordström black holes with equal charge and mass $M_i > 0$ and horizons at (x_i, y_i, z_i) . Note that the MP coordinates are singular at these points, mapping a horizon of finite proper area to a single point. They also showed that if U takes certain, specific, forms different from the above expression, the spacetime contains naked singularities.

Chandrasekhar [13] and Contopoulos [14] have investigated the timelike and null geodesics of the two-black-hole system from the point of view of the periodic orbits and the weak field limit. For particles with elliptic (bound) energies the trajectories fall into several categories. There are stable periodic and quasiperiodic orbits, chaotic orbits trapped between periodic orbits, trajectories which fall into one or other of the black holes, and chaotic trajectories which lie on the boundary of these regions. This structure, together with the fact that the weak field limit is integrable, makes the MP geodesics problem a particularly interesting example of chaos in general relativity.

Working from the equations of motion, Contopoulos showed that the weak field limit of the two-black-hole system was integrable for uncharged test particles with zero angular momentum about the axis of symmetry. Using Hamilton-Jacobi methods [15], we have generalized this result to include test particles of arbitrary charge, energy, and angular momentum. In the background spacetime generated by two black holes with masses M_1, M_2 centered at $(0, 0, 0)$ and $(0, \pi, 0)$ in prolate spheroidal coordinates, (ψ, θ, ϕ) [14], we find the super-Hamiltonian for a test particle of charge e and mass m is given by

$$\mathcal{H} = \frac{\partial S}{\partial \lambda} = \frac{1}{2} U^2 \left(\frac{\partial S}{\partial t} - \frac{e}{U} \right)^2 - \frac{1}{2U^2 Q} \left(\frac{\partial S}{\partial \psi} \right)^2 - \frac{1}{2U^2 Q} \left(\frac{\partial S}{\partial \theta} \right)^2 - \frac{1}{2(U \sinh \psi \sin \theta)^2} \left(\frac{\partial S}{\partial \phi} \right)^2, \quad (5)$$

where

$$U = 1 + W/Q, \quad Q = \sinh^2 \psi + \sin^2 \theta,$$

$$W = (M_1 + M_2) \cosh \psi + (M_1 - M_2) \cos \theta.$$

Since t and ϕ are cyclic coordinates, the canonically conjugate momenta $\pi_t = -(E - e)$ and $\pi_\phi = L_z$ are constants of

the motion. E and L_z are the particle's energy and angular momentum at infinity. Invariance of the particle's rest mass along the trajectory affinely parameterized by λ gives rise to a third constant of motion m^2 . We see that the system is separable, and hence integrable, if

$$S = -\frac{1}{2} m^2 \lambda - (E - e)t + L_z \phi + S_\psi(\psi) + S_\theta(\theta). \quad (6)$$

Substituting this expression into (5) and using the weak field expansion

$$QU^n = Q + nW + O(W^2/Q), \quad (7)$$

we find

$$S_\psi = \int \left((E^2 + m^2) \sinh^2 \psi + 2(M_1 + M_2) \right. \\ \left. \times (2E^2 + m^2 - Ee) \cosh \psi - \frac{L_z^2}{\sinh^2 \psi} + \alpha \right)^{1/2} d\psi, \quad (8)$$

$$S_\theta = \int \left((E^2 + m^2) \sin^2 \theta + 2(M_1 - M_2)(2E^2 + m^2 - Ee) \cos \theta \right. \\ \left. - \frac{L_z^2}{\sin^2 \theta} - \alpha \right)^{1/2} d\theta. \quad (9)$$

The separation constant α is a fourth constant of the motion, which in turn guarantees the system is integrable.

The next order in the weak field expansion destroys separability in prolate spheroidal coordinates. While this does not prove the system is nonintegrable, it does show that if chaos is present, it is due to relativistic effects. In this case, the chaos is related to the relativistic perihelion advance of elliptic trajectories, which invalidates Bonnet's theorem for motion with two centers [13].

For MP geometries with three or more black holes, even the Newtonian limit is nonintegrable. The one exception to this occurs in the Newtonian regime for "extremal" test particles with $e = m$, since then the gravitational and electrostatic forces cancel exactly. This cancellation breaks down in general relativity where gravity couples to all forms of energy, including kinetic energy. The scattering of an extremal test particle by an extremal black hole of mass M illustrates this effect since we find the scattering angle $\Delta\phi$ differs from the Newtonian value ($\Delta\phi = 0$) by $\Delta\phi = -(M/b)[3 + u^2/4 + O(u^4, M/b)]$. The impact parameter b is related to L_z and the asymptotic velocity u via $L_z = mbu/\sqrt{1-u^2}$. This result leads us to expect that the trajectories of extremal test particles in MP geometries will be chaotic beyond the Newtonian limit, thus providing another example where chaos arises due to relativistic effects.

For rapid numerical calculations we use the original MP coordinates (t, x, y, z) , because the equations of motion do not contain transcendental functions. Writing the upper components of the four-velocity in an orthonormal basis as (v_0, \mathbf{v}) , the equations of motion are

$$\dot{\mathbf{v}} = U^{-2} [(v_0^2 + v^2 - ev_0/m) \nabla U - \mathbf{v} \cdot \nabla U], \quad (10)$$

$$\dot{x} = U^{-1} \mathbf{v}, \quad (11)$$

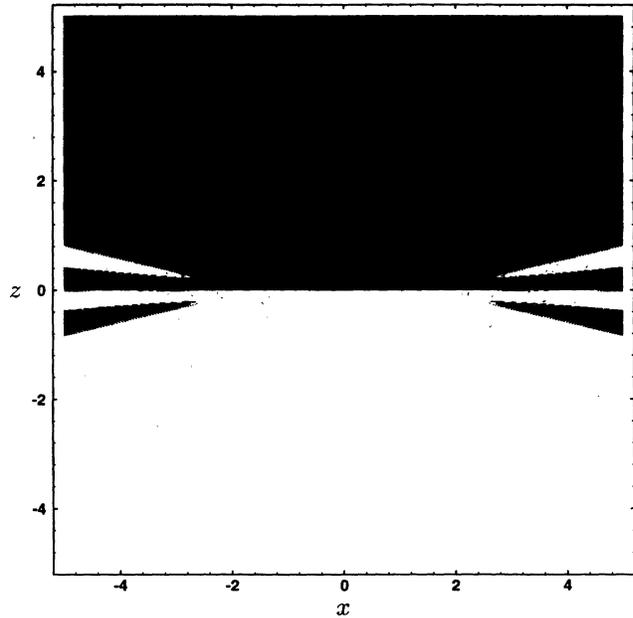


FIG. 1. A two-dimensional section of phase space determined by $\mathbf{v} = \mathbf{0}$, $y = 0$. The black regions are trajectories which fall into the black hole of mass $1/3$ at $(0, 1)$. The white regions are those which fall into the black hole of mass $1/3$ at $(0, -1)$.

$$\dot{t} = Uv_0, \quad (12)$$

$$v_0 = \sqrt{1 + v^2}, \quad (13)$$

where an overdot indicates the derivative with respect to proper time. The conserved energy is given by

$$E = U^{-1}(mv_0 - e). \quad (14)$$

We have integrated the equations using a fourth-order Runge-Kutta technique with adaptive stepsize [16]. The results are shown in Figs. 1–3 for the case $e = 0$. These figures show two-dimensional sections of the phase space given by the initial conditions $\mathbf{v} = \mathbf{0}$, $y = 0$. All of the points shown fall into one of the black holes and are color coded accordingly. The boundary between the basins of attraction clearly has a complicated structure, which may be quantified using the concept of fractal dimension.

There are a number of different fractal dimensions defined in the literature [17]. The box-counting dimension of a set embedded in an E -dimensional Euclidean space is obtained by finding the minimum number $N(\epsilon)$ of E -dimensional cubes needed to cover the set, and calculating

$$d_B = -\lim_{\epsilon \rightarrow 0} \frac{\ln N}{\ln \epsilon}, \quad (15)$$

assuming that this limit exists. The generalization to Riemannian or pseudo-Riemannian spaces is direct, since d_B is invariant under diffeomorphisms, so all choices of coordinates give the same value. If the basin is defined on some spatial hypersurface, as is the case here, the dimension may depend on the chosen slicing of spacetime.

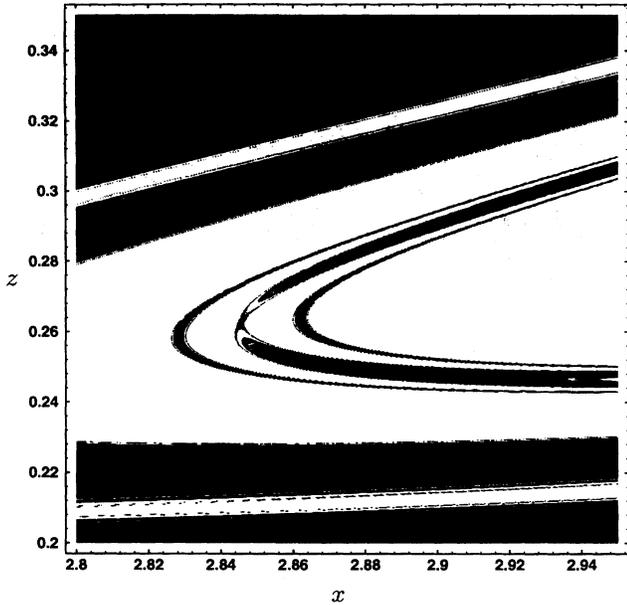


FIG. 2. A small region of Fig. 1 showing the fractal structure.

The value of d_B of the fractal boundary for a region of phase space near that shown in Fig. 2 was evaluated using the above equation. The region, containing 2520^2 points, was covered by a grid. Each square of size ϵ^2 (ϵ being a factor of 2520) was counted if it contained trajectories leading to both black holes. Figure 4 shows a plot of $\ln N$ vs $\ln \epsilon$. The straight line is a fit to all but the three largest and smallest values of ϵ , and gives a dimension of 1.464... This result clearly demonstrates that the 8-dimensional

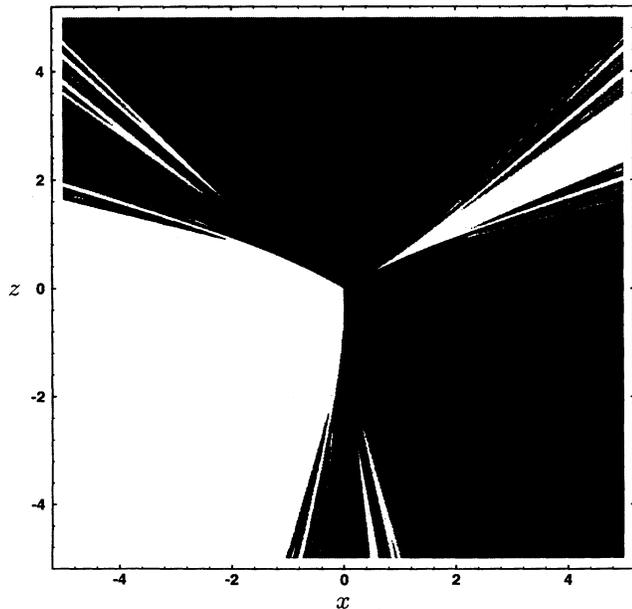


FIG. 3. The $v=0, y=0$ section of phase space for the 3-hole problem. The black holes of mass $1/3$ are situated at $(0,1)$, $(\sqrt{3}/2, -1/2)$, and $(-\sqrt{3}/2, -1/2)$, corresponding to gray, black, and white, respectively.

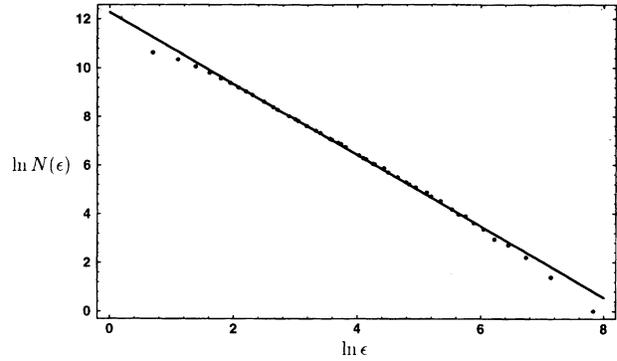


FIG. 4. The dimension d_B of a region of the fractal boundary near that shown in Fig. 2 is equal to 1.464... .

boundary as a whole is a fractal, thus giving us a diffeomorphism invariant measure of chaos.

For complete geodesics, a useful characteristic of chaotic systems is the presence of positive Lyapunov exponents, which demonstrate that the system is nonintegrable [18]. These are defined in flat spacetime by choosing a point x in phase space, at the center of a ball of radius $\epsilon \ll 1$. After a time t the ball evolves to an ellipsoid with semiaxes $\epsilon_k(t)$, where k ranges from one to the dimension of the phase space. The Lyapunov exponents are

$$\lambda_k(x) = \lim_{t \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{1}{t} \ln \frac{\epsilon_k(t)}{\epsilon}, \quad (16)$$

again assuming the limits exist. The λ_k are constant along a trajectory, and are often constant over larger regions of phase space, such as the basin of an attractor. Numerically these are calculated by integrating the linearized equations and periodically performing a Gram-Schmidt orthonormalization to ensure that the exponentially growing solutions do not cause overflow errors [19].

There are a number of subtleties associated with the definition and calculation of Lyapunov exponents in curved spaces, and in particular, in the MP spacetimes. The definition of the λ_k requires a metric on phase space to calculate distances, but the λ_k are independent of this metric if the phase space is compact, since the trajectory passes arbitrarily close to at least one point an infinite number of times at unbounded values of t , causing the metric terms appearing in the ratio of radii to effectively cancel out.

In relativity, there is no unambiguously defined global time in general. If this is the case, the only reasonable option is to use the particle's proper time as t in Eq. (16). This approach emphasizes the quasilocal nature of Lyapunov exponents, that is, their relation to particular trajectories, and not (necessarily) the system as a whole. If the system has a timelike Killing vector, and hence a global time t , then this may be used in the definition of the exponents. The results of this approach correspond to measurements of stationary observers. We choose the latter approach, following Ref. [2].

In the MP case, there is an additional difficulty in that many of the geodesics are incomplete. Particles may cross the horizon of a black hole after finite proper time and encounter a singularity, rendering the limit in Eq. (16) unde-

finer. Thus, we attempt to calculate Lyapunov exponents only for those trajectories which survive for an infinite proper time. This is not a problem for the regions of phase space which are nearly integrable, but for the basin boundary special techniques are required. It is impossible to find a point which is exactly on the boundary in general, and a trajectory which begins near the boundary will eventually move away and fall into a black hole. One solution is to periodically check that the trajectory is sufficiently close to the boundary, using a criterion based on whether the particle subsequently survives for a given proper time. If the criterion is not met, the trajectory is adjusted by making small random shifts until it is. We have checked that this procedure does not cause the exponents or energy to drift significantly over the integration times used. We have calculated the Lyapunov exponents for the two-black-hole system using the 4-dimensional submanifold of phase space (x, z, v_x, v_z) , defined by $(L_z=0, m=\text{const})$, for the initial values $(3.33467, 0.23509, 0, 0)$, and found them to be $\lambda_k = (0.03609, 0.00006, -0.00006, -0.03609)$. The statistical uncertainty is about 0.00009. Note that the Lyapunov exponents are measured in the eigendirections of the linearized evolution operator, and do not correspond to the above coordinate system.

As in nonrelativistic mechanics, the symmetries of the system impose constraints on the Lyapunov exponents [20]. Liouville's theorem implies that they sum to zero; time reversal symmetry implies that they come in $+/-$ pairs; the presence of a constant of the motion requires that one pair of the exponents be zero. Since the phase space for our system is eight dimensional, we require four constants of motion to ensure the vanishing of all four pairs of Lyapunov exponents. For geometries with three or more black holes, there will generally only be two constants of motion, necessitating the calculation of two pairs of Lyapunov exponents. These results will be presented in a future study. The two-black-hole geometry furnishes three constants of motion, leaving one pair of nonzero Lyapunov exponents as given above.

Finally, we note that recently the MP solutions have been generalized to include a positive cosmological constant [21,22]. These solutions may describe coalescing extremal black holes in a de Sitter-type universe. It would be interesting to investigate this system, and how its dynamical nature and negative Ricci curvature might affect the chaotic structure of the geodesics.

A long follow-up paper to this work will appear elsewhere [23].

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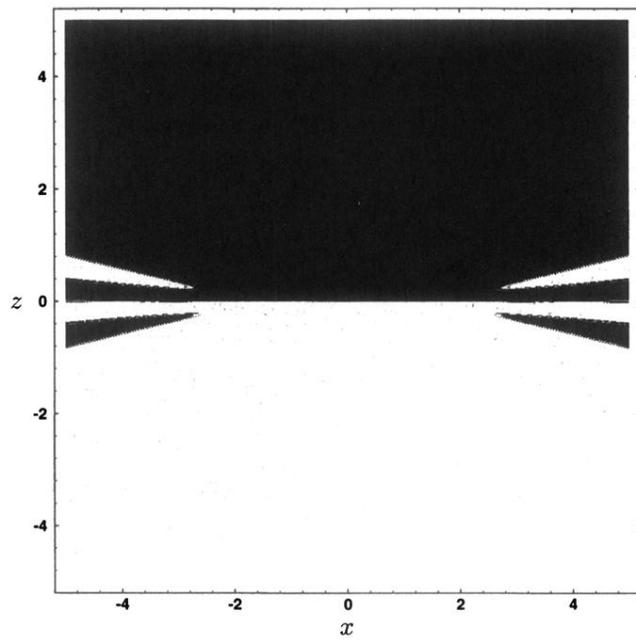


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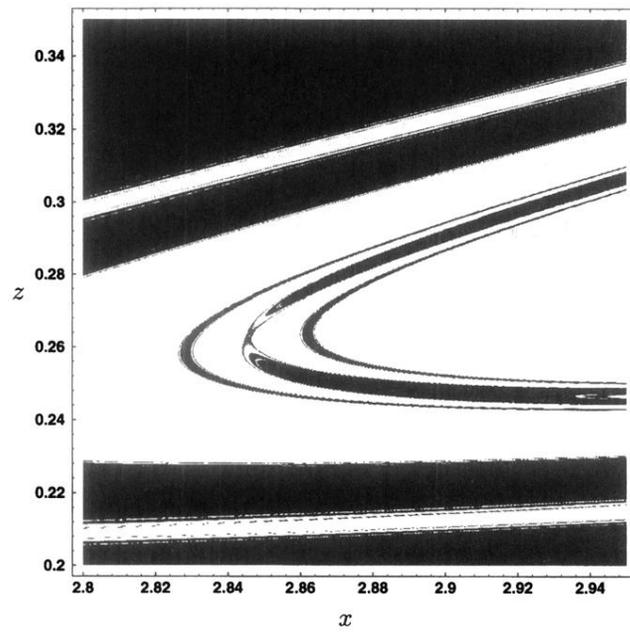


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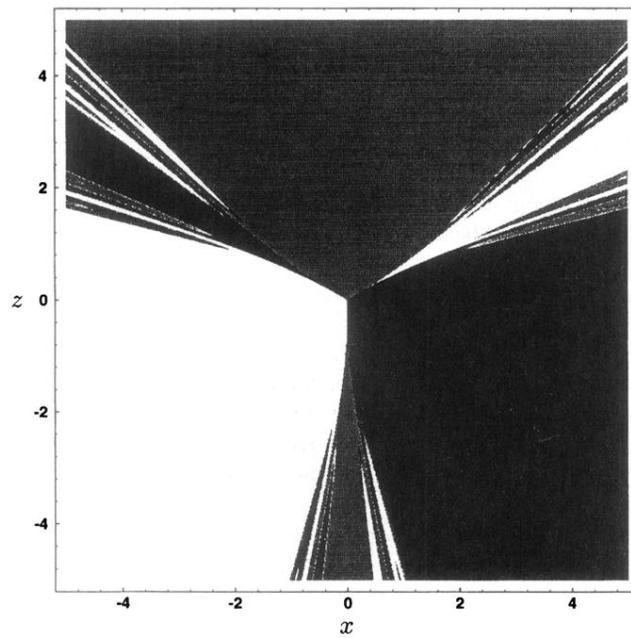


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