Combinatorics of boundaries in string theory

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We investigate the possibility that stringy nonperturbative effects appear as holes in the world sheet. We focus on the case of Dirichlet string theory, which we argue should be formulated rather differently than in previous work, and we find that the effects of boundaries are naturally weighted by $e^{-O(1/g_{st})}$.

I. INTRODUCTION

It is essential to develop a nonperturbative understanding of string theory. The discovery of unusually large nonperturbative effects, of order $e^{-O(1/g_{st})}$, is likely to be an important clue. These effects are found in matrix models and are expected more generally from the large-order behavior of string perturbation theory [1], but they have no field-theory analogue and their nature is not in general known. In this paper we suggest that the leading stringy nonperturbative effects might make their appearance through holes (boundaries) in the string world sheet, and study in detail the case of Dirichlet boundary conditions. We show in particular that holes naturally have weight $e^{-O(1/g_{st})}$.

Dirichlet boundary conditions in string theory have been a subject of frequent study. Boundaries with Dirichlet conditions on all coordinates have been considered as a source of partonic behavior in string theory [2–4] and as external probes on the theory [5,6]. Boundaries with Dirichlet conditions on some coordinates and Neumann on others have been suggested to represent a form of compactification [7] or an extended object in spacetime [8]. The interpretation here is new—that they represent an example of the sought after $e^{-O(1/g_{st})}$ stringy nonperturbative effects. Our claim hinges on a treatment of the combinatorics of the boundaries which differs from that previously assumed in the fully Dirichlet case, though it is closely related to the work [8] on extended objects. We will argue that our combinatorics is that which arises naturally from duality, and that it is necessary for the Fischler-Susskind treatment of string divergences. However, while we will at times contrast our approach with others, we are not necessarily asserting that the latter are incorrect. These represent distinct theories, with different physics, and the consistency of each (including our own) is still an open issue.

In the next section we review the interpretation of mixed Dirichlet-Neumann boundary conditions in terms of extended objects, with emphasis on the combinatorics. We then consider the fully Dirichlet case and show how the $e^{-O(1/g_{st})}$ weight arises. We discuss divergences and conformal anomalies, which are always important constraints in string theory, and show that the Fischler-Susskind mechanism operates. We conclude with a brief discussion of the implications, and suggest that the Dirichlet boundary is only one example of a much larger class.

II. D-BRANES

We begin by reviewing the critical bosonic string with Dirichlet conditions on one coordinate X^{25} and Neumann conditions on the remaining $X^{\mu} \equiv X^0, \ldots, X^{24}$. This can be obtained by duality from the more familiar fully Neumann theory by compactifying the Neumann coordinate $\tilde{X}^{25}$ and taking the radius to zero [8,9]. To see this, consider some world-sheet path C which begins at a boundary point p_1 and ends at any other boundary point p_2. The Neumann coordinate $\tilde{X}^{25}$ and the dual coordinate X^{25} are related by

$$\tilde{X}^{25}(z,\bar{z}) = X_L^{25}(z) - X_R^{25}(\bar{z}), \quad X^{25}(z,\bar{z}) = X_L^{25}(z) + X_R^{25}(\bar{z}).$$

(1)

Then

$$X^{25}(p_2) - X^{25}(p_1) = \int_C dz(\partial_z X^{25} + \partial_{\bar{z}} X^{25})$$

$$= \int_C dz(\partial_z \tilde{X}^{25} - \partial_{\bar{z}} \tilde{X}^{25}) = 0.$$

(2)

The last equality holds because the line integral is the total $\tilde{p}^{25}$ momentum crossing C, and this is zero for all states surviving in the $R \to 0$ limit. Thus, the two end points of any open string are at the same value of X^{25}. Moreover, for two different open strings, if we consider an interaction (such as gravitation) between them, there will be world-sheet paths connecting the end points of the different strings. So the end points of *all* open strings must be at the same X^{25}. In particular, if the world sheet has several disconnected boundary components, the field X^{25} satisfies a Dirichlet boundary condition with *the same* value on each component. Let us for now define the zero mode of X^{25} such that the value is zero.

What is the physical interpretation? Open strings end points lie on the hyperplane $X^{25} = 0$, while in the rest of the dual spacetime only closed strings are found. The physics is that of a closed string theory in interaction with a 24-dimensional extended object, the "D(irichlet)-brane." In a relativistically invariant theory one does not expect to find a rigid object, and indeed, the shape of the D-brane is dynamical. The vertex operator

$$\oint_B dsA(X^{\mu})\partial_n X^{25}$$

(3)

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$(n = \text{normal}, t = \text{tangential})$ corresponds to fluctuations of the shape of the D-brane. This is evident from consideration of the boundary state: the normal derivative $\partial_n X^{25}$ is the canonical momentum for X^{25}, so the vertex operator introduces an X^μ-dependent shift of X^{25}. The leading action for these fluctuations is the world volume swept out by the D-brane, as developed in Ref. [8]. The vertex operator (3) is dual to the gauge field vertex operator $\oint_B ds A(X^\mu) \partial_t \bar{X}^{25}$ of the Neumann theory.

From cluster decomposition, one would expect that there should also exist configurations with two or more D-branes. Indeed, these arise from duality when Chan-Paton factors are included. Introduce a Chan-Paton degree of freedom with states $a = 1, \ldots, N$. The diagonal vertex operators

$$\lambda_{aa} \oint_B ds A_a(X^\mu) \partial_n X^{25} \tag{4}$$

(no sum on a) produce an a-dependent shift of X^{25}, so the boundary shape depends on the state a of the boundary: each of the N states corresponds to a *different* D-brane.[1] The path integral thus includes

$$\sum_{N=0}^{\infty} \left\{ \left(\prod_{a=1}^{N} \int [dX_a^{25}] \right) \sum_{n=0}^{\infty} \sum_{a_1, \ldots, a_n = 1}^{N} \right\}. \tag{5}$$

That is, for each N sum over the number n of world-sheet boundaries and sum each of the n Chan-Paton degrees of freedom from 1 to N; then, integrate over the configurations of the N D-branes, and sum over the number N of D-branes. Summing over n and the Chan-Paton factors is equivalent to summing, for each value of the Chan-Paton factor (each D-brane), over the number of world-sheet boundaries lying in the given D-brane. Thus, the sum (5) is equivalent to

$$\sum_{N=0}^{\infty} \prod_{a=1}^{N} \left\{ \int [dX_a^{25}] \sum_{n_a=0}^{\infty} \right\}. \tag{6}$$

Note that cluster decomposition in the dual theory requires a sum over the number N of Chan-Paton degrees of freedom. This suggests that string theories with different Chan-Paton groups should be regarded as different states of a single theory, not a surprising result given the increasing evidence for the unity of string theory. Incidently, we have been limited thus far to D-branes with the topology $\mathbf{R}^{24}$. One would expect other topologies, such as S^{24}, but these do not seem to have any simple dual description.[2]

III. D-INSTANTONS

For the fully Dirichlet case, the boundary is at a single spacetime point and so corresponds to an event, a D-instanton. The sum (6) becomes

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[1]The off-diagonal vertex operators correspond to open strings with end points on different D-branes, so these are all massive when the D-branes are separated.

[2]Even D-branes of topology $\mathbf{R}^{24}$ are problematic in the dual theory if they become folded over so that $X^{25}(X^\mu)$ is multivalued.

$$\sum_{N=0}^{\infty} \prod_{a=1}^{N} \left\{ \int d^{26}X_a \sum_{n_a=0}^{\infty} \right\}. \tag{7}$$

That is, the functional integral over the configuration reduces to a 26-dimensional integral over the position of each D-instanton. The sum (7) differs from that considered in the partonic Dirichlet theory [2–4] where the spacetime position of each world-sheet boundary is integrated independently— equivalently, each n_a is fixed at 1 for $a = 1, \ldots, N$. Roughly speaking, with the combinatorics (7), the D-instanton is an extrinsic event in spacetime, to which any number of world-sheet boundaries may attach, while with $n_a \equiv 1$ each D-instanton is associated with exactly one world-sheet boundary.

Consider the one-D-instanton amplitude A_1. The sum (7) includes world-sheet components which have Dirichlet boundaries at X but are otherwise disconnected; the leading such contribution would be the disk amplitude with no vertex operators, denoted $\langle 1 \rangle_{D_2}$. The amplitude with ν such disks includes a symmetry factor $1/\nu!$, so the sum exponentiates,

$$A_1 = \exp(\langle 1 \rangle_{D_2} + \cdots) A_1^{\text{connected}}, \tag{8}$$

where the ellipsis represents higher-order disconnected topologies. A string world sheet of Euler number χ, with m vertex operators, is weighted by $g_{\text{st}}^{m-\chi}$. In particular, the disk with no vertex operators has weight g_{st}^{-1}, giving the advertised $e^{-O(1/g_{\text{st}})}$ (the sign in the exponent will be obtained below). We are thus interpreting $-\langle 1 \rangle_{D_2}$ as the D-instanton action. This $1/g_{\text{st}}$ behavior has previously been noted in the tension of the D-brane [8].

For m vertex operators, the leading contribution to the connected amplitude, order g_{st}^0, comes from m disks each with a single vertex operator,

$$A_1 = \exp(\langle 1 \rangle_{D_2} + \cdots) \int d^{26}X \prod_{i=1}^{m} \langle V_i \rangle_{D_2, X} + \cdots, \tag{9}$$

with the position X of the boundary noted by a subscript where relevant. Notice that momentum is not conserved on the individual world-sheet components; it flows through the boundary and is only conserved in the total process. This amplitude can be summarized in terms of an effective action

$$S_{\text{eff}} \sim \int d^{26}X \exp\left(\langle 1 \rangle_{D_2} + \sum_\alpha A_\alpha \langle V_\alpha \rangle_{D_2, X} \right). \tag{10}$$

The sum α runs over all vertex operator types (including an implicit momentum integration), and A_α is the corresponding creation or annihilation operator.

The actual coefficient of $1/g_{\text{st}}$ is of some interest. The results [10] are readily extended to the Dirichlet case, giving

$$\langle 1 \rangle_{D_2} = -\left(\frac{3}{16\pi} \right)^{3/2} \frac{2}{\pi^2} \cdot (2\pi)^{13/2} \cdot \left(\frac{16\pi}{3} \right)^{3/2} \frac{2^{-11/2}\pi^2}{\kappa_{26}}$$

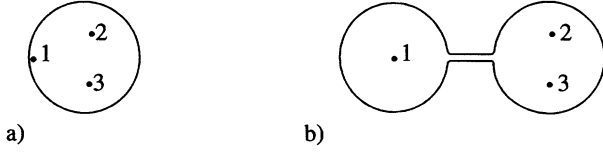$$\cdot (2\pi\alpha')^6 = -\frac{2^8 \pi^{25/2} \alpha'^6}{\kappa_{26}}. \tag{11}$$

FIG. 1. (a) Dirichlet disk amplitude with three closed-string vertex operators, with V_1 approaching the boundary. (b) Conformally equivalent picture, two disks joined by a degenerating strip.

In the first equality the four groups of terms, separated by dots, are from the conformal Killing volume, the relation of determinants on the disk to those on the sphere, the relation of the sphere amplitude to the gravitational coupling, and a dimensional factor restoring units. This is for the oriented case; the unoriented would have an additional $2^{-1/2}$. The negative value of the Möbius volume, which appeared to be a mere curiosity in Ref. [10], provides the sign needed in the exponent. Convert to an effective four-dimensional coupling,

$$\kappa_{26}=V^{1/2}\kappa_4=v^{1/2}(4\pi^2\alpha')^{11/2}\kappa_4, \tag{12}$$

where V is the compactification volume and v is the same in units of the minimum volume of toroidal compactification. Also, $\kappa_4=g_4\alpha'^{1/2}/2$, taking the example of a level one $SU(n)$ group. In all,

$$\langle 1\rangle_{D_2}=-\frac{\pi^{3/2}}{4v^{1/2}g_4}. \tag{13}$$

In terms of the four-dimensional gauge coupling, the coefecient is of order 1 for moderate values of v. This is comparable to the value estimated by Banks and Dine by very different means [11], and so is consistent with their proposal that the stringy nonperturbative effects can be large even while field-theoretic nonperturbative effects remain quite small.

What is the effect of the D-instanton amplitude (9)? The main previous interest in Dirichlet boundaries has been their partonic scattering behavior. Indeed, there is no Regge suppression in the one-point amplitude on the disk, so the D-instanton gives rise to hard scattering amplitudes of order $e^{-O(1/g_{st})}$. In fact the effective action (10) is essentially a local exponential in spacetime, which suggests that the theory may have severe problems in the ultraviolet. This may not be fatal, as will argue in the next section that the divergences in the D-instanton amplitude cancel. In any case, we again note that we are more interested in the combinatoric properties of the boundaries than in the particular case of Dirichlet boundary conditions.

IV. DIVERGENCES AND ANOMALIES

Dirichlet boundaries lead to a string divergence of a rather severe sort, arising when a vertex operator or group of vertex operators approaches the boundary. An example is shown in Fig. 1(a), and an equivalent representation in terms of a degenerating strip in Fig. 1(b). The divergence comes from the $L_0=1$ state $\alpha^\mu|0\rangle$ in the Hilbert space of the open string with end points fixed at identical positions, and is of the form
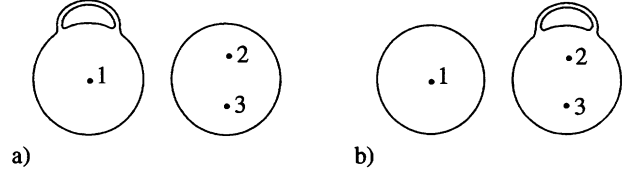

FIG. 2. (a,b) Amplitudes which cancel the divergence and anomaly of Fig 1. In each amplitude, there are three boundaries at a common position X^μ.

$$\langle V_1 B^\mu\rangle_{D_2,X}\langle V_2 V_3 B_\mu\rangle_{D_2,X}\int_0^\cdot\frac{dt}{t} \tag{14}$$

where

$$B^\mu=\oint_B ds\,\partial_n X^\mu \tag{15}$$

is the vertex operator for this state and $t\to 0$ is the limit of moduli space. There is also a tachyon divergence $\int_0 dt/t^3$, which can as usual be defined by analytic continuation. More generally, this divergence appears whenever a strip with coincident end points degenerates; this includes the short-distance divergence noted above.

To deal with this divergence we recall the Fischler-Susskind principle [12], that physically sensible quantities are free of divergences. Note that the vertex operator (15) is just the collective coordinate (3) for the position of the D-instanton. In the original Fischler-Susskind situation [12], to cancel divergences it was necessary to expand around the correct background configuration. The present case is different, because there is no momentum dependence in the divergence, and because there is no "correct" value of the collective coordinate—rather, we must integrate over it. Note now that the divergence (14) can be put in the form

$$\frac{\partial}{\partial X_\mu}\langle V_1\rangle_{D_2,X}\frac{\partial}{\partial X^\mu}\langle V_2 V_3\rangle_{D_2,X}\int_0^\cdot\frac{dt}{t}. \tag{16}$$

This is not a total derivative as it stands, but there are two other divergent amplitudes, shown in Figs. 2(a) and 2(b), which combine with the divergence (17) to give

$$\frac{1}{2}\frac{\partial}{\partial X_\mu}\frac{\partial}{\partial X^\mu}\langle V_1\rangle_{D_2,X}\langle V_2 V_3\rangle_{D_2,X}\int_0^\cdot\frac{dt}{t}, \tag{17}$$

which now vanishes upon integration (the factor of $\frac{1}{2}$ correctly accounts for the symmetry of the annulus). This generalizes directly to all other divergences involving this intermediate state. As in the original Fischler-Susskind mechanism, summing over topologies has produced a finite result. Again, this mechanism depends in a essential way on

the inclusion of multiple boundaries at the same D-instanton; the cancellation is between $n_a = 1$ in Fig. 1 and $n_a = 3$ in Fig. 2.[3]

The divergence is accompanied by a conformal anomaly in the individual graphs, because any cutoff on the integration over t will rescale under a conformal transformation. Again this cancels between the various topologies. This has a curious consequence. Comparison with matrix model results requires a linear dilaton background. It would appear that Dirichlet conditions are inconsistent with a gradient of the dilaton Φ, because the world-sheet field $X^\mu(\sigma)$ is not Weyl invariant:

$$\delta X^\mu(\sigma) = \delta\phi(\sigma)\partial^\mu\Phi(X(\sigma)). \tag{18}$$

However, one is free to take the vertex operator V_1 to be a dilaton with nonzero momentum, and the tree-level conformal anomaly from Fig. 1 is canceled by the amplitudes of Fig. 2. So the Dirichlet boundary condition is evidently consistent with a linear dilaton background even though it does *not* give rise to a conformal field theory, by cancellation in the string loop expansion.

V. DISCUSSION

We first summarize our results for the specific case of Dirichlet boundaries, before turning to broader and more speculative issues. We have argued on several grounds— duality, the analogy with the D-brane, and the Fischler-Susskind mechanism—that the nature of the theory with Dirichlet boundaries is rather different from what has been considered previously. A consequence is that the D-instanton carries a weight $e^{-O(1/g_{st})}$, so that the Dirichlet theory is identical to the usual closed string theory to all orders of perturbation theory. The difference is of the same order as the indeterminacy in the perturbation series itself [1].

Are we to interpret this as a nonperturbative ambiguity in the theory, that the usual closed string and the Dirichlet string theory are two different theories? Our speculation is that there is not a large nonperturbative ambiguity in string theory.[4] Rather, given an exact nonperturbative formulation

of string theory, the leading nonperturbative effects would be given as a sum over specific boundary types. Note that the Dirichlet boundary condition is only one possible example. The condition for a boundary state $|B\rangle$ to define a conformal field theory is

$$(L_n - \bar{L}_{-n})|B\rangle = 0. \tag{19}$$

There are many such states, one for each primary field in the theory [6,15]. There are other consistency conditions as well [16]—for example, there must be a Hilbert space interpretation in the open string channel, in order that amplitudes properly factorize—and we do not know the general solution.

In $d = 2$ string theory [17], the stringy nonperturbative effects are understood in terms of tunneling of matrix model fermions.[5] This would correspond to effective interactions of the general form

$$e^{-O(1/g_{st})}e^{i\{S_{R,L}(q) - S_{R,L}(q')\}/\sqrt{2\pi}}, \tag{20}$$

where $S(q)$ is the canonically normalized tachyon field. This is of the general form (10), the exponent in the second term being of order g_{st}^0. It is therefore consistent with our more general speculation about boundaries. It does not appear to correspond specifically to a Dirichlet boundary condition, however, as we see no way to obtain the necessary i in the exponent from these. Also, the Dirichlet amplitudes have a very simple momentum dependence, while a more complicated structure, including leg poles, seems to arise in the matrix model.[6]

Dirichlet boundaries for the type II superstring were discussed in Ref. [20], but boundaries in the heterotic string are problematic. The condition (19) has no natural extension to a chiral algebra; perhaps it must simply be replaced by Becchi-Rouet-Stora-Tyutin invariance (together with the other conditions mentioned above).

To conclude let us emphasize a possible general lesson. This is the nonperturbative breakdown of the world sheet through the appearance of holes (which has been also discussed in a different context [21]) and in the extrinsic nature of the D-instanton.

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[3]In the $n_a \equiv 1$ Dirichlet theory there are therefore uncanceled divergences, about which Green has put forward a very interesting proposal [13]. He interprets the state (15) as a Lagrange multiplier, whose most notable effect is to remove the dilaton from the string spectrum. This is completely different from the role of this state in our approach, but we can point to no obvious inconsistency.

[4]For example, in the $d = 1$ matrix model, we have found that most of the nonperturbative ambiguity is removed by considerations of causality [14].

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[5]For a recent discussion see Ref. [18].

[6]For a recent discussion of leg poles, with references to earlier work, see Ref. [19].

[1] S. Shenker, in *Random Surfaces and Quantum Gravity*, edited by O. Alvarez, E. Marinari, and P. Windey (Plenum, New York, 1991).

[2] M. B. Green, Phys. Lett. **69B**, 89 (1977); Phys. Lett. B **201**, 42 (1988); **282**, 380 (1992).

[3] Z. Yang, "Asymptotic Freedom and Dirichlet String Theory," Report No. UR-1288, ER-40685-737, hep-th/9211092, 1992 (unpublished); M. Li, Nucl. Phys. **B420**, 339 (1994).

[4] M. B. Green and J. Polchinski, Phys. Lett. B **335**, 377 (1994).

[5] J. H. Schwarz, Nucl. Phys. **B65**, 131 (1973); E. F. Corrigan

and D. B. Fairlie, *ibid.* **B91**, 527 (1975); M. B. Green, *ibid.* **B103**, 333 (1976); A. Cohen, G. Moore, P. Nelson, and J. Polchinski, *ibid.* **B267**, 143 (1986); **B281**, 127 (1987).

[6] A. Cohen, G. Moore, P. Nelson, and J. Polchinski, in *Unified String Theories*, edited by M. Green and D. Gross (World Scientific, Singapore, 1986).

[7] W. Siegel, Nucl. Phys. **B109**, 244 (1976).

[8] J. Dai, R. Leigh, and J. Polchinski, Mod. Phys. Lett. A **4**, 2073 (1989); R. Leigh, *ibid.* **4**, 2767 (1989).

[9] M. B. Green, Phys. Lett. B **266**, 325 (1991).

[10] B. Grinstein and M. Wise, Phys. Rev. D **35**, 655 (1987); **35**, 3285(E) (1987); M. R. Douglas and B. Grinstein, Phys. Lett. B **183**, 52 (1987); **187**, 442(E) (1987); J. Liu and J. Polchinski, *ibid.* **203**, 39 (1988).

[11] T. Banks and M. Dine, Phys. Rev. D (to be published).

[12] W. Fischler and L. Susskind, Phys. Lett. B **171**, 383 (1986); **173**, 262 (1986).

[13] M. B. Green, Phys. Lett. B **302**, 29 (1993).

[14] J. Polchinski, "On the Nonperturbative Consistency of the $D=2$ String," Report No. NSF-ITP-94-96, hep-th 9409168, 1994 (unpublished).

[15] N. Ishibashi, Mod. Phys. Lett. A **4**, 251 (1989); J. Cardy, Nucl. Phys. **B324**, 581 (1989).

[16] D. Lewellen, Nucl. Phys. **B372**, 654 (1992).

[17] D. J. Gross and N. Miljković, Phys. Lett. B **238**, 217 (1990); E. Brézin, V. A. Kazakov, and A. B. Zamolodchikov, Nucl. Phys. **B333**, 673 (1990); P. Ginsparg and J. Zinn-Justin, Phys. Lett. B **240**, 333 (1990); G. Parisi, *ibid.* **238**, 209 (1990).

[18] R. Brustein, M. Faus, and B. A. Ovrut, "Instanton Effects in Matrix Models and String Effective Lagrangians," Report No. CERN-TH. 7301/94, UPR-608T, hep-th/9406179, 1994 (unpublished).

[19] M. Natsuume and J. Polchinski, Nucl. Phys. **B424**, 137 (1994)

[20] M. B. Green, Phys. Lett. B **329**, 435 (1994).

[21] J. Atick and E. Witten, Nucl. Phys. **B310**, 291 (1981).