PHYSICAL REVIEW D

PARTICLES, FIELDS, GRAVITATION, AND COSMOLOGY

THIRD SERIES, VOLUME 50, NUMBER 8 15 OCTOBER 1994

RAPID COMMUNICATIONS

Rapid Communications are intended for important new results which deserve accelerated publication, and are therefore given priority in editorial processing and production. A Rapid Communication in **Physical Review D** *should be no longer than five printed pages and must be accompanied by an abstract. Page proofs are sent to authors, but because of the accelerated schedule, publication is generally not delayed for receipt of corrections unless requested by the author.*

Black hole solutions in generalized two-dimensional dilaton-gravity theories

S. Mignemi*

Dipartimento di Scienze Fisiche, Università di Cagliari and INFN, Sezione di Cagliari, via Ada Negri 18, 09127 Cagliari, Italy

(Received 21 April 1994)

We consider two-dimensional dilaton-gravity theories with a generic exponential potential for the dilaton, and obtain the most general black hole solutions. We also show the relation of these models with higher-derivative theories and extended Poincaré gauge theories in two dimensions.

PACS number(s): 04.60.Kz, 04.50.+h, 04.70.Dy

Two-dimensional theories of gravity have been extensively studied in recent years, since they provide a large variety of toy models whose study may help to understand some still obscure aspects of four-dimensional gravity [1–8]. In particular, one could gain some insight on which properties of gravity, such as, for example, black hole thermodynamics or no-hair theorems, are generic to all geometric models and which are specific to Einstein theory. Especially relevant is also the possibility of studying in detail the quantization of two-dimensional (2D) models either in the context of string and conformal field theories [3], or in the context of gauge theories of the Poincaré group [7]. It is hoped that investigation of 2D models may shed some light on the quantization of the 4D theory.

For these reasons, it is interesting to study in some generality the various models which can be constructed in two dimensions [6,9,10]. A common feature of all the two-dimensional gravity theories is the presence of a scalar field (identified with the dilaton of string theory), nonminimally coupled to gravity. The actual form of this coupling, in turns, determines the physical properties of the theory. These can be most clearly revealed by the study of the black hole solutions [2–5,9–13], since, as is well known, black holes are the most fruitful laboratory for investigating the relations between gravity, quantum theory, and thermodynamics.

* Electronic address: mignemi@cagliari.infn.it

In this paper we study a dilaton-gravity model with a generic exponential potential for the dilaton. This form of the potential is especially interesting, for example, because it can be obtained from theories containing higher powers of the Ricci scalar in the action [13], which may arise from quantum corrections. We give here the most general black hole solutions for these models. We also show the relation between our action and the extended Poincaré group gauge formalism [7,8].

The 2D action we consider is

$$S=\int d^2x\sqrt{g}(\eta R+\Lambda\eta^h). \tag{1}$$

Defining $e^{-2\phi}=\eta$ and rescaling $\hat{g}_{\mu\nu}=e^{2\phi}g_{\mu\nu}$, the action can also be written as

$$S=\int d^2x\sqrt{\hat{g}}e^{-2\phi}[\hat{R}+4(\nabla\phi)^2+\Lambda e^{-2h\phi}]. \tag{2}$$

We shall call the metric $\hat{g}_{\mu\nu}$ the "string" metric (in contrast with the "standard" metric $g_{\mu\nu}$) since it is the one to which the string couples in 2D models. One can easily check that the limit $h=0$ of (2) corresponds to the string-inspired action [3], while $h=1$ is the Jackiw-Teitelboim (JT) theory [1].

An action of the form (1), or equivalently (2), can be derived from higher derivative 2D theories of the kind proposed in [13]. Consider, for example, the action

$$S=\int d^2x\sqrt{g}R^k. \tag{3}$$

For $k=1$, S is a total derivative, while for $k\neq 1$, it is easily seen [14] that, defining a scalar field $\eta\equiv dR^k/dR=kR^{k-1}$, the action (3) is equivalent to:

$$S=\int d^2x\sqrt{g}(\eta R+\Lambda\,\eta^{k/(k-1)}) \tag{4}$$

where $\Lambda=-(k-1)k^{-k/(k-1)}$, which is of the form (1) with $h=k/(k-1)$.

We notice that in this formalism, the string action corresponds to the limit $k\to 0$ [8], and the JT theory to $k\to\infty$. Actions of the form (2) can also be obtained from dimensional reduction of higher dimensional theories.

The field equations stemming from (1) are

$$R=-\Lambda h\,\eta^{h-1}, \tag{5a}$$

$$(\nabla_\mu\nabla_\nu-g_{\mu\nu}\nabla^2)\eta+\frac{\Lambda}{2}g_{\mu\nu}\eta^h=0. \tag{5b}$$

Separating the trace and traceless parts, (5b) can be written as

$$\nabla^2\eta=\Lambda\,\eta^h,\quad (\nabla_\mu\nabla_\nu-g_{\mu\nu}\nabla^2)\eta=0. \tag{5c}$$

For our purposes, it is, however, more convenient to consider the equivalent field equations arising from (2):

$$4\nabla^2\phi-4(\nabla\phi)^2+(1+h)\Lambda e^{-2h\phi}+\hat{R}=0, \tag{6a}$$

$$-2\nabla_\mu\nabla_\nu\phi-\hat{g}_{\mu\nu}\left[-2\nabla^2\phi+2(\nabla\phi)^2-\frac{\Lambda}{2}e^{-2h\phi}\right]=0. \tag{6b}$$

The equations (6) can be rearranged to read

$$2\nabla^2\phi-4(\nabla\phi)^2=-\Lambda e^{-2h\phi},\quad \hat{R}=-2\nabla^2\phi-\Lambda he^{-2h\phi}. \tag{6c}$$

We seek for a solution depending only on the spatial coordinate. It is convenient to adopt the conformal gauge $d\hat{s}^2=e^{2\rho(x)}(-dt^2+dx^2)$, for which

$$\rho''-\phi''=\frac{\Lambda}{2}he^{2(\rho-h\phi)}, \tag{7}$$

$$\phi''-2\phi'^2=-\frac{\Lambda}{2}e^{2(\rho-h\phi)}, \tag{8}$$

$$2\rho'\phi'-2\phi'^2=-\frac{\Lambda}{2}e^{2(\rho-h\phi)}. \tag{9}$$

Equations (8) and (9) yield

$$\phi''=2\rho'\phi', \tag{10}$$

which can be immediately integrated to give

$$e^{2\rho}=A\phi' \tag{11}$$

with A an integration constant. Substituting in (8),

$$\phi''-2\phi'^2=-\frac{\Lambda}{2}Ae^{-2h\phi}\phi' \tag{12}$$

whose first integral is

$$\phi'e^{-2\phi}=\frac{\Lambda A}{4(h+1)}(e^{-2(h+1)\phi}-C). \tag{13}$$

Finally, from (10)–(12), one gets

$$\rho'=\phi'-\frac{\Lambda e^{2(\rho-h\phi)}}{4\phi'}=\phi'-\frac{\Lambda}{4}Ae^{-2h\phi}, \tag{14}$$

which, derived, recovers (7) (Bianchi identity).

If $C=0$ the equations can be immediately integrated:

$$e^{2h\phi}=\frac{\Lambda Ah}{2(h+1)}(x+B),\quad e^{2\rho}=\frac{A}{2h}(x+B)^{-1}. \tag{15}$$

The metric function (15) describes an asymptotically flat spacetime singular at $x=-B$. This result has also been obtained in [13] in the context of higher derivative theories.

If $C\neq 0$, one has, instead,

$$x+B=-\frac{2(h+1)}{A\Lambda}\int\frac{dy}{y^{h+1}-C}\quad\text{where } y=e^{-2\phi}. \tag{16}$$

For positive C, the solution can be written in terms of the hypergeometric function F:

$$x+B=\frac{2(h+1)e^{-2\phi}}{\Lambda AC^{h/(h+1)}}F\left(\frac{1}{h+1},1,\frac{h+2}{h+1};\frac{e^{-2(h+1)\phi}}{C}\right). \tag{17}$$

(One may take $A=1$, $B=0$ without loss of generality.)

By inverting (17), one gets ϕ as a function of x. In general it is not possible to obtain it in analytical form. Nevertheless, Eq. (11) permits us to write, in terms of ϕ the metric function,

$$e^{2\rho}=\frac{\Lambda A^2}{4(h+1)}(e^{-2h\phi}-Ce^{2\phi}) \tag{18}$$

and, with the help of (7) and (13), also the curvature

$$\hat{R}=-\frac{\Lambda}{h+1}(h^2e^{-2h\phi}-Ce^{2\phi}). \tag{19}$$

One can thus discuss in general the properties of the solutions: the behavior of $e^{-2\phi}$ depends on the sign of C and h. For $C<0$, $e^{-2\phi}$ vanishes for $x\to 0$ and diverges for a finite value $x=x_0$ if h is positive, or for $x\to\infty$, if $h<0$. For positive C, instead, one has in general two branches for the solutions: if $h>0$, $e^{-2\phi}\to C^{1/(h+1)}$ for $x\to-\infty$ and goes to zero or infinity for $x\to 0$, depending on the branch considered. If $h<0$, for one branch $e^{-2\phi}\to 0$ for $x\to 0$ and $e^{-2\phi}\to C^{1/(h+1)}$ for $x\to-\infty$. For the other $e^{-2\phi}\to\infty$ for $x\to\infty$ and $e^{-2\phi}\to C^{1/(h+1)}$ for $x\to-\infty$. The point where $e^{-2\phi}=C^{1/(h+1)}$ in general corresponds, as we shall see, to the event horizon of the metric. In conformal coordinates, this point is always at infinity. This fact renders a bit obscure the description of the metric.

Let us discuss the properties of the solution (18): for positive C, if $h>0$, a horizon is located at $x=-\infty$, but both branches of the solution diverge at $x\rightarrow\infty$. At these points the curvature is also divergent.

For $h\leqslant 0$, a horizon is present at $x=-\infty$. One of the two branches is regular also at $x=\infty$, while the other diverges at finite x. In the first case one has a regular asymptotically flat black hole. The case $h=0$, in particular, yields the Mandal-Sengupta-Wadia metric for the string [3,12], or its dual [15]:

$$ds^2=(1\pm Ce^{-\Lambda x})^{-1}(-dt^2+dx^2). \tag{20}$$

For negative C, if $h>0$, the metric and the curvature are singular at the origin and at a point x_0 at finite distance, while for $h\leqslant 0$ a naked singularity is placed at the origin, the metric being regular at infinity. None of the $C<0$ solutions is therefore physically relevant.

From a physical point of view is perhaps more interesting to consider the "standard" metric $g_{\mu\nu}=e^{2\sigma}\delta_{\mu\nu}\equiv e^{2(\rho-\phi)}\delta_{\mu\nu}$, which is obtained by rescaling the "string" metric $\hat{g}_{\mu\nu}$. The "standard" metric is in fact the relevant one both for the Jackiw-Teitelboim theory and for the higher derivative models. It turns out that the solutions have interesting properties also in this case.

For $C=0$, $e^{2\sigma}=(x+B)^{-1-1/h}$ and $R\propto(x+B)^{-1+1/h}$. These metrics are singular at the origin except for $0<h<1$, but in this case are singular at infinity.

For nonvanishing C, one has, instead,

$$e^{2\sigma}=\frac{\Lambda A^2}{4(h+1)}(e^{-2(h+1)\phi}-C) \tag{21}$$

and

$$R=-2\sigma''e^{-2\sigma}=-\Lambda he^{-2(h-1)\phi}. \tag{22}$$

For positive C all the solutions have a horizon at $x=-\infty$. For $h>1$, both branches are asymptotically flat, while for $0<h<1$ both are singular at $x=+\infty$. For $h<0$, instead, one of the branches is asymptotically flat, while the other is singular at a finite value of x. Special examples of these classes of solutions are the anti–de Sitter and pro–de Sitter[1] solutions of the JT theory $h=1$, given by $e^{2\sigma}=(e^x\pm e^{-x})^{-2}$, respectively. Our general solutions $h>1$ have essentially the same properties, except constant curvature.

Finally, for $C<0$, the solution either have naked singularity at the origin if $h<1$, or are singular at infinity, if $h>1$.

As an illustration of the method, we consider the special cases $h=-1$. This case corresponds to the nonlinear action $S=\int d^2x\sqrt{g R}$, and is distinguished because (13) is no more valid. The solution of (12) is, in fact,

$$\frac{\Lambda}{2}x=\mathrm{Ei}(-2\phi)=\mathrm{li}(e^{-2\phi}). \tag{23}$$

The dilaton is then given by the inverse function of the exponential integral Ei and goes to zero for $x\rightarrow-\infty$, whereas it diverges for $x\rightarrow\infty$. Furthermore, from (11) and (12) follows that $e^{2\rho}=-(\Lambda/2)A^2\phi e^{2\phi}$. Thus the metric function $e^{2\phi}$, as well as the curvature $\hat{R}=(2/A^2)e^{2\rho}$, vanish at both $x=\pm\infty$. The solution is hence a solitonlike regular spacetime, with a horizon at both ends. In the "standard" gauge $g_{\mu\nu}$, the metric is instead given by $e^{2\sigma}=-(\Lambda/2)A^2\phi$ and is therefore proportional to the dilaton. The curvature turns out to be $R=-(\Lambda/2)e^{4\phi}$. The structure of the solution is analogous to that of the "string" metric.

We conclude this Rapid Communication by showing that our model can be written as a gauge theory of the extended Poincaré group [8], where the symmetry is broken to U(1)×U(1) by imposing some constraints on the fields. This formulation could be useful for the quantization of the theory.

Consider the two-dimensional extended Poincaré algebra [8]:

$$[P_a,J]=\epsilon_a^bP_b,\quad [P_a,P_b]=\frac{\Lambda}{2}\epsilon_{ab}I,\quad [P_a,I]=[J,I]=0$$

and the corresponding gauge field

$$A=e^aP_a+\omega J+aI \tag{24}$$

with the field strength

$$F=dA+A^2=P_aT^a+Jd\omega+I\left(\frac{\Lambda}{4}\epsilon_{ab}e^ae^b+da\right), \tag{25}$$

where T^a is the torsion $T^a=de^a+\epsilon^a{}_b\omega e^b$. According to [8], the fields transform under the gauge transformations generated by $\Theta=\theta^aP_a+\alpha J+\beta I$ as

$$e^a\rightarrow(\mathscr{M}^{-1})^a{}_b(e^b+\epsilon^b{}_c\theta^c+d\theta^b),$$

$$\omega\rightarrow\omega+d\alpha, \tag{26}$$

$$a\rightarrow a-\theta^a\epsilon_{ab}e^b-\frac{1}{2}\theta^2\omega+d\beta+\frac{1}{2}d\theta^a\epsilon_{ab}\theta^b,$$

where $\mathscr{M}^a{}_b=\delta^a{}_b\cosh\alpha+\epsilon^a{}_b\sinh\alpha$.

One can now define the gauge multiplet of scalar fields $\eta_A=(\eta_a,\eta_2,\eta_3)$, which permits us to construct the topological Lagrangian

$$\mathscr{L}=\sum\eta_AF^A=\eta_aT^a+\eta_2d\omega+\eta_3\left(\frac{\Lambda}{4}\epsilon_{ab}e^ae^b+da\right) \tag{27}$$

invariant under the extended Poincaré group. By imposing the constraints (possibly enforced by means of Lagrange multipliers)

$$da=0,\qquad \eta_3=\eta_2^h, \tag{28}$$

which break the symmetry to U(1)×U(1) generated by J and I, the Lagrangian can be written

$$\mathscr{L}=\eta_aT^a+\eta_2d\omega+\frac{\Lambda}{4}\eta_2^h\epsilon_{ab}e^ae^b. \tag{29}$$

The ensuing field equation are

---

[1]We call "pro–de Sitter" the regular, constant curvature black hole solution discussed by many authors [10,16].

$$
\begin{aligned}
&T^a=0,\\
&R+h\Lambda\eta_2^{h-1}=0,\\
&d\eta_a+\epsilon^b{}_a\omega\eta_b+\Lambda\eta_2^h\epsilon_{ab}e^b=0,\\
&d\eta_2+\eta_a\epsilon^a{}_be^b=0.
\end{aligned}
\tag{30}
$$

The first equation implies the vanishing of the torsion and hence the usual relation between the spin connection and the zweibein, while the second coincides with (5a). Finally, the last two equations, combined, yield (5b).

This way of writing the theory should lead to a straightforward quantization, on the lines of [8]. It would, however, be interesting to find a mechanism of spontaneous symmetry breaking of the extended Poincaré invariance instead of introducing explicitly symmetry breaking constraints. Another interesting point which deserves further investigations are the thermodynamical properties of the black hole solutions we have derived. It is in fact well known that 2D models can lead to several different kinds of thermodynamics [10,17]. All these possibilities are presently actively investigated.

I wish to thank M. Cadoni for discussions.

[1] C. Teitelboim, in *Quantum Theory of Gravity*, edited by S. M. Christensen (Hilger, Bristol, 1984); R. Jackiw, *ibid.*
[2] R. B. Mann, A. Shiekh, and I. Tarasov, Nucl. Phys. **B341**, 134 (1990); R. B. Mann, Gen. Relativ. Gravit. **24**, 433 (1992).
[3] E. Witten, Phys. Rev. D **44**, 314 (1991).
[4] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D **45**, R1005 (1992).
[5] V. P. Frolov, Phys. Rev. D **46**, 5383 (1992).
[6] J. G. Russo and A. A. Tseytlin, Nucl. Phys. **B382**, 259 (1992).
[7] K. Isler and C. A. Trugenberger, Phys. Rev. Lett. **63**, 834 (1989); A. H. Chamseddine and D. Wyler, Phys. Lett. B **228**, 75 (1989).
[8] D. Cangemi and R. Jackiw, Phys. Rev. Lett. **69**, 233 (1992); R. Jackiw, Lectures given at the International Colloquium on Group Theoretical Methods in Physics, Salamanca, June, 1992.
[9] J. P. L. Lemos and P. M. Sá, Phys. Rev. D **49**, 2897 (1994).
[10] M. Cadoni and S. Mignemi, Report No. INFN-CA-20-93 and No. INFNCA-TH-94-4 (unpublished).
[11] J. D. Brown, M. Henneaux, and C. Teitelboim, Phys. Rev. D **33**, 319 (1986).
[12] G. Mandal, A. M. Sengupta, and S. R. Wadia, Mod. Phys. Lett. A **6**, 1685 (1991).
[13] H. J. Schmidt, J. Math. Phys. **32**, 1562 (1991).
[14] G. Magnano, M. Ferrais, and M. Francaviglia, Gen. Relativ. Gravit. **19**, 465 (1987).
[15] A. Giveon, Mod. Phys. Lett. A **6**, 2843 (1991).
[16] D. Christensen and R. B. Mann, Class. Quantum Grav. **9**, 1769 (1992); J. P. L. Lemos and P. M. Sá, Report No. DF/IST-8.93 (unpublished).
[17] P. T. Landsberg and R. B. Mann, Class. Quantum Grav. **10**, 2373 (1993).