PHYSICAL REVIEW D VOLUME 50, NUMBER 4 15 AUGUST 1994

Information in black hole radiation for initial mixed states

Paul B. Slater*
Community and Organization Research Institute, University of California, Santa Barbara, California 93106-2150
(Received 8 April 1994)

The question of whether black holes deplete the Universe of information is usually addressed in terms of initial gravitationally collapsing pure states. Pure states are nongeneric, having null prior probability of occurrence. The more realistic case of initial mixed states is, thus, examined here. The average information in an m-dimensional system in a random mixed state is found to approach asymptotically, as $m\to\infty$, an upper limit of $\frac{1}{6}$ nats, that is, $\frac{1}{6}\ln 2$ or 0.240449 bits. Also, the average information in an m-dimensional subsystem A of an mn-dimensional system AB proves to be smaller if AB is in a random mixed state than in a random pure state. This finding reinforces the recently drawn conclusion of Page that information in black hole radiation may come out initially so slowly that it would never show up in an analysis perturbative in M_{Planck}/M.

PACS number(s): 04.70.Dy, 02.50.Sk, 03.65.−w, 05.30.Ch

The "cosmic information paradox" is a topic of much current interest [1–3]. It is typically posed in terms of a black hole forming from matter in a pure quantum state. However, in a certain generic or prior sense, the probability of a pure state occurring is null. "The wave-function language applies strictly only to pure states which seldom—some physicists would say *never*—occur in nature." [4] "In practice, nature usually presents us with states that are not pure, and it is the preparation of pure states that provides the greatest challenge" ([5], p. 156). It had been indicated by Hawking [6] that "in fact, the initial situation in general will also be described by a density matrix because of the hidden surface occurring at earlier times . . . In general, the initial situation will not be a pure quantum state either because of the evaporation of black holes at earlier times."

Here, we investigate the cosmic information paradox in the more realistic case of initial mixed states. In pursuing this line of analysis, heavy reliance is placed on the recent work of Page [7] and Foong and Kanno [8]. They were concerned with the average entropy ($S_{m,n}=\langle S_A\rangle$) of an m-dimensional subsystem A of an mn-dimensional system in a random pure state $|\psi\rangle\langle\psi|$. The average was defined with respect to the unitarily invariant Haar measure on the space of unit vectors $|\psi\rangle$. Here, the counterpart to their work when AB is in a random mixed state is desired. The average will then be defined with respect to the unitarily and reparametrization-invariant measure $|\rho|^{2q+1}$ over the $q\times q$ density matrices (ρ) [9], taking $q=mn$. (It should be noted that as $|\rho|=0$ for a pure state, such states are assigned null measure.) Previous to the recent work [9], the opinions had been expressed that "There does not seem to be any natural measure on the set of all mixed states" [10] and "the more realistic experimental case of mixed input states does not admit such a nice Bayesian treatment (the problem lies in selecting a good prior on mixed density matrices)" [11].

The relative (unnormalized) probability of the density matrix of an m-dimensional subsystem A of an mn-dimensional system AB in a random pure state, having eigenvalues $p_1, \ldots, p_m$, was expressed in Ref. [8] as

$$P(p_1,\ldots,p_m)\prod_{i=1}^{m} dp_i = \delta\left(1-\sum_{i=1}^{m} p_i\right)\prod_{1\le i<j\le m}(p_i-p_j)^2\prod_{k=1}^{m}(p_k^{n-m}dp_k). \quad (1)$$

Note that if n is set equal to $3m+1$, the rightmost product is simply the suggested measure over the $m\times m$ density matrices, $|\rho_A|^{2m+1}$, the determinant of a matrix equaling the product of its eigenvalues. It should also be observed that the other product, of ordered squares of differences of eigenvalues, is present in the joint probability density function for the eigenvalues of matrices from a Gaussian unitary ensemble (the ensemble of Hermitian matrices with equally probable real and imaginary parts) ([12], Theorem 3.3.1). The probabilities of such matrices are invariant under unitary transformations.

In Ref. [8] the entropy of the eigenvalues, $-\sum_{i=1}^{m}p_i\ln p_i$, was weighted by (1) and integrated over the $(m-1)$-dimensional simplex of possible vectors of eigenvalues. With an appropriate normalization, it was shown that

$$S_{m,n}=\sum_{k=n+1}^{mn}\frac{1}{k}-\frac{m-1}{2n} \quad (m\le n), \quad (2)$$

as conjectured by Page [7]. For large m and n [7],

$$S_{m,n}\simeq \ln m-\frac{m}{2n} \quad (1\ll m\le n). \quad (3)$$

Expression (1) for the relative probability of the eigenvalues of ρ_A can be compared with the joint distribution of the eigenvalues of an $m\times m$ matrix ρ_A having a central complex Wishart distribution with n degrees of freedom and scale or covariance matrix Σ. This joint distribution takes the form [[13], (3.8)]

*Electronic address: slater@sbitp.itp.ucsb.edu

TABLE I. Average entropies for selected m and n. (1) $S_{mn,3mn+1}=\tilde{S}_{mn}$. (2) $S_{m,2mn+m+1}$. (3) $S_{n,2mn+n+1}$.

m	n	ln mn	(1)	ln m	(2)	ln n	(3)	(2)+(3)−(1)
2	2	1.386 29	1.242 53	0.693 147	0.625 481	0.693 147	0.625 481	0.008 437 1
2	3	1.791 76	1.638 48	0.693 147	0.643 425	1.098 61	1.015 57	0.020 517 9
2	4	2.079 44	1.922 07	0.693 147	0.653 847	1.386 29	1.297 19	0.028 959 6
2	5	2.305 29	2.142 99	0.693 147	0.660 657	1.609 44	1.517 25	0.034 911 6
2	6	2.484 91	2.323 93	0.693 147	0.665 455	1.791 76	1.697 76	0.039 281 8
2	25	3.912 02	3.746 53	0.693 147	0.685 872	3.218 88	3.119 83	0.059 175
3	3	2.197 22	2.038 6	1.098 61	1.038 16	1.098 61	1.038 16	0.037 719 1
3	4	2.484 91	2.323 93	1.098 61	1.051 09	1.386 29	1.321 73	0.048 888 7
4	4	2.772 59	2.61	1.386 29	1.335 68	1.386 29	1.335 68	0.061 355 8
4	5	2.995 73	2.832 23	1.386 29	1.344 67	1.609 44	1.557 3	0.069 738 1
5	5	3.218 88	3.054 68	1.609 44	1.566 61	1.609 44	1.566 61	0.078 532 9
6	6	3.583 52	3.418 52	1.791 76	1.754 85	1.791 76	1.754 85	0.091 189 5
7	7	3.891 82	3.726 35	1.945 91	1.913 57	1.945 91	1.913 57	0.100 792
50	100	8.517 19	8.350 54	3.912 02	3.909 54	4.605 17	4.600 22	0.159 218
1000	1000	13.815 5	13.648 8	6.907 76	6.907 51	6.097 76	6.907 51	0.166 167
10^9	10^9	41.446 5	41.279 9	20.723 3	20.723 3	20.723 3	20.723 3	0.166 667
10^{10}	10^{20}	69.077 6	68.910 9	23.025 9	23.025 9	46.051 7	46.051 7	0.166 667

$$\frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}|\Sigma|^{-n}\exp(-\mathrm{tr}\Sigma^{-1}\rho_A)|\rho_A|^{n-m}\prod_{i>j}^{m}(p_i-p_j)^2, \tag{4}$$

where the complex multivariate gamma function [[14], (83)]

$$\tilde{\Gamma}_m(a)=\pi^{m(m-1)/2}\prod_{i=1}^{m}\Gamma(a-i+1). \tag{5}$$

(The Wishart distribution is a multivariate generalization of the univariate Γ distribution.) One can take Σ to be the diagonal density matrix corresponding to the fully mixed state, with diagonal entries $1/m$. Then, the exponent in (4), $-\mathrm{tr}\Sigma^{-1}\rho_A$, can be seen to serve as a constant, since the density matrices have unit trace. With the imposition of the δ function to ensure a unit trace, it is clear that (1) and (4) are proportional, differing by only some constant factor [cf. [8], (3), (14)]. With a choice of degrees of freedom, $n=3m+1$, the suggested measure $|\rho_A|^{2m+1}$ appears in (1) and (4). Then, we have the general result

$$\tilde{S}_m=S_{m,3m+1}, \tag{6}$$

where $\tilde{S}_m$ is the average entropy of an m-dimensional system, without regard to any possible imbedding in a larger system.

Let us consider the mn-dimensional system AB in a random mixed state to have an mn-dimensional complex central Wishart distribution with degrees of freedom $3mn+1$ and covariance matrix, the fully mixed $mn\times mn$ density matrix. The subsystem A will then have an m-dimensional marginal complex central Wishart distribution with degrees of freedom $2mn+m+1$ and the covariance matrix, the fully mixed $m\times m$ density matrix. Similarly, the n-dimensional subsystem B will have such a marginal distribution with degrees of freedom $2mn+n+1$ and the fully mixed $n\times n$ density matrix as its covariance matrix (cf. [15], Corollary 3.2.6; [16], sec. 6). With these choices of degrees of freedom, the three density matrices $(\rho_A,\rho_B,\rho_{AB})$ will all be raised to the power $2mn+1$ in their associated density functions, since $3mn+1-mn=2mn+m+1-m=2mn+n+1-n$. Now, by (6), the average entropy associated with AB (an mn-dimensional system in a random mixed state) will equal $S_{mn,3mn+1}$. By the results of Refs. [7] and [8], the average entropy associated with A will equal $S_{m,2mn+m+1}$ and with B, $S_{n,2mn+n+1}$. For certain selected values of m and n, the values of these three expected entropies have been tabulated (Table I). For $m,n\leq 7$, the exact formula (2) was employed, while for larger values, the asymptotic result (3) was utilized. From (3) and (6),

$$\tilde{S}_m\simeq\ln m-1/6 \quad (1\ll m\leq n). \tag{7}$$

Also, for large m and n, $S_{m,2mn+1+m}\simeq\ln m$ and $S_{n,2mn+1+n}\simeq\ln n$, which results explain the asymptotic limit $1/6\simeq0.166\,667$ in the last rows of the table.

The Araki-Lieb inequality [17] requires that the entropy of ρ_{AB} be no greater than that of ρ_A plus that of ρ_B (that is, subadditivity must be satisfied). The results of the table, obtained by averaging over such entropies, respect this inequality, as the last column has all positive entries.

For a system AB in a random pure state, Page [3] studied, in detail, the case in which the product mn of the dimensions of the black hole and radiation subsystems equaled $2^4 3^6 5^2=291\,600\sim e^{4\pi}$ (about the number of states very naively expected for a black hole near the Planck mass). (Then, because of the assumed purity, the entropy of subsystem A had to equal that of B.) He plotted and analyzed the average information in the m-dimensional (radiation) subsystem A:

$$I_{m,n}=\ln m-S_{m,n} \quad (m\leq n), \tag{8}$$

$$\simeq m/2n \quad (1\ll m\leq n), \tag{9}$$

$$=\ln m-\ln n+I_{n,m} \quad (n\leq m), \tag{10}$$

$$\sim\ln m-\ln n+n/2m \quad (1\ll n\leq m), \tag{11}$$

and concluded that "if all the information going into gravitational collapse escapes gradually from the apparent black hole, it would likely come at initially such a slow rate or be so spread out (requiring so many measurements) that it could never be found or excluded by a perturbative analysis."

In the mixed state case under consideration here,

$$I_{m,n} = \ln m - S_{m,2mn+m+1}, \tag{12}$$

$$\simeq m/(4mn+2m+2) \quad (1 \ll m). \tag{13}$$

(12) holds for all m, whether smaller or larger than n, since m is always less than $2mn+m+1$, and (2) is applicable. For $n=1$, (13) approaches 1/6 as m increases. [In comparison, for $n=1$, (11) approaches $\ln m$ ([3], Fig. 1).] Since (13) is uniformly less than (9), Page's conclusion regarding the initial slow rate of information escape is reinforced in the case of an initial mixed state (cf. [18]).

The suggested measure [9], $|\rho|^{2d+1}$, over the $d\times d$ density matrices (ρ) is induced by the (Fisher) information metric [19]. There are, of course, nondenumerable other measures over the d-dimensional quantum states which are similarly unitarily invariant. Two examples of special interest are the measures induced by the Bures metric [20] and by the von Neumann entropy [21]. These two appear to lend themselves less readily to exact integrations. In addition, such alternatives lack the feature of reparametrization invariance, a desideratum first formulated by Jeffreys [22]. (His objective had been to broaden the applicability of Bayes' theorem [23] by developing a general principle for generating suitable prior distributions.) In the context of the study here, the term "reparametrization" refers to the parameters of a d-dimensional complex multinormal distribution having a zero mean vector and ρ as its covariance matrix [24]. Let us present the underlying argument in the following manner.

The $d\times d$ density/covariance matrix ρ can be transformed to a $2d\times 2d$ covariance matrix

$$\Omega = \frac{1}{2}\begin{pmatrix} Q_1 & -Q_2 \\ Q_2 & Q_1 \end{pmatrix}$$

of a $2d$-dimensional *real* multinormal distribution [25]. (The real symmetric part of ρ is Q_1 and its skew-symmetric part is Q_2, so $\rho = Q_1 + Q_2$.) Then, the Jacobian of the transformation $\Omega \to \Omega^{-1}$ is $|\Omega|^{2d+1}$ ([15], Theorem 2.1.8). The Jeffreys' (diffuse, vague, noninformative, invariant) prior distribution [23(a)], which is the volume element on the Riemannian manifold of possible probability distributions with respect to the information metric [19], is then taken to be the square root of this Jacobian, that is $|\Omega|^{(2d+1)/2}$. Since $|\Omega| = (2^{-d}|\rho|)^2$, the suggested (unnormalized) form $|\rho|^{2d+1}$ for the prior measure over the $d\times d$ density matrices is obtained [9(b)].

It might be observed that the convenient and simplifying choice in (4) of Σ as the fully mixed density matrix corresponds to what could be called a quasi-infinite-temperature scenario. (The resultant Boltzmann-like exponential factor is then constant across states—although, of course, the underlying prior measure or quantum structure function, $|\rho|^{2d+1}$, itself is not.) It would appear, in line with basic properties of the Wishart distribution [15,25(a)], that Σ should be selected to be the mean of what is conceived to be the (cosmic) distribution of initial states. To a similar end, one could also seek the (posterior) distribution closest in the sense of relative entropy (information distance) to the normalized form of the prior, $|\rho|^{2d+1}$, which satisfies constraints on certain expected values.

I would like to express appreciation to the Institute for Theoretical Physics for computational support in this research.

[1] F. Flam, Science **259**, 1824 (1993).
[2] W. Hiscock, Nature (London) **363**, 397 (1993).
[3] D. N. Page, Phys. Rev. Lett. **71**, 3743 (1993).
[4] A. J. Coleman, Rev. Mod. Phys. **35**, 668 (1963).
[5] L. Ballentine, *Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1990).
[6] S. W. Hawking, Phys. Rev. D **14**, 2460 (1974).
[7] D. N. Page, Phys. Rev. Lett. **71**, 1291 (1993).
[8] S. K. Foong and S. Kanno, Phys. Rev. Lett. **72**, 1148 (1994).
[9] (a) P. B. Slater, Phys. Lett. A **171**, 285 (1992); (b) J. Math. Phys. **34**, 1794 (1993); (c) Nature (London) **367**, 328 (1994).
[10] W. K. Wootters, Found. Phys. **20**, 1365 (1990).
[11] K. R. W. Jones, Ann. Phys. (N.Y.) **207**, 140 (1991).
[12] M. L. Mehta, *Random Matrices* (Academic, New York, 1991).
[13] P. R. Krishnaiah, J. Multiv. Anal. **6**, 1 (1976).
[14] A. T. James, Ann. Math. Statist. **35**, 475 (1964).
[15] R. J. Muirhead, *Aspects of Multivariate Statistical Theory* (Wiley, New York, 1982).
[16] S. A. Andersson, H. K. Brøns, and S. T. Jensen, Ann. Statist. **11**, 392 (1983).
[17] H. Araki and E. Lieb, Commun. Math. Phys. **18**, 160 (1970); A. Wehrl, Rev. Mod. Phys. **50**, 221 (1978); S. M. Barnett and S. J. D. Phoenix, Phys. Rev. A **40**, 2404 (1989).
[18] J. D. Bekenstein, Phys. Rev. Lett. **70**, 3680 (1993).
[19] R. E. Kass, Statist. Sci. **4**, 188 (1989); N. N. Chentsov, *Statistical Decision Rules and Optimal Inference* (American Mathematical Society, Providence, RI, 1982).
[20] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994); M. Hübner, Phys. Lett. A **163**, 239 (1992).
[21] D. Petz, J. Math. Phys. **35**, 780 (1994); D. Petz and G. Toth, Lett. Math. Phys. **27**, 205 (1993).
[22] H. Jeffreys, *The Theory of Probability* (Clarendon, Oxford, 1939).
[23] (a) S. J. Press, *Bayesian Statistics: Principles, Models, and Applications* (Wiley, New York, 1989); (b) C. Howson and P. Urbach, *Scientific Reasoning: The Bayesian Approach* (Open Court, La Salle, 1989).
[24] A. Bach, J. Math. Phys. **21**, 789 (1980); J. Phys. A **14**, 125 (1981).
[25] (a) N. R. Goodman, Ann. Math. Statist. **34**, 152 (1963); (b) R. A. Wooding, Biometrika **43**, 212 (1956).