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Collisions of boosted black holes: Perturbation theory prediction of gravitational radiation

Andrew M. Abrahams and Gregory B. Cook
Center for Radiophysics and Space Research, Cornell University, Ithaca, New York 14853
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We consider general relativistic Cauchy data representing two nonspinning, equal-mass black holes boosted toward each other. When the black holes are close enough to each other and their momentum is sufficiently high, an encompassing apparent horizon is present so the system can be viewed as a single, perturbed black hole. We employ gauge-invariant perturbation theory, and integrate the Zerilli equation to analyze these time-asymmetric data sets and compute gravitational waveforms and emitted energies. When coupled with a simple Newtonian analysis of the infall trajectory, we find striking agreement between the perturbation calculation of emitted energies and the results of fully general relativistic numerical simulations of time-symmetric initial data.

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I. INTRODUCTION

The collision of two black holes is expected to be an important source of gravitational radiation for gravity wave detectors currently under construction. A major theoretical effort is underway to compute the gravitational waveform from the orbiting inspiral and coalescence of black-hole binaries. Post-Newtonian theory should provide a sufficiently accurate waveform for much of the inspiral phase and enable the extraction of considerable information about the parameters of the system [1]. However, it is anticipated that a fully general relativistic treatment will be necessary to predict the waveform from the final stages of coalescence.

Recently, the seminal calculations of Smarr and Eppley [2] for the head-on collision of two equal-mass holes starting at rest from a finite separation have been redone [3] with the benefit of new numerical techniques and theoretical tools, as well as vastly increased computational resources. The basic conclusions reached by the modern calculations are remarkable for their similarity to those of the original study. The maximum amount of energy radiated is small, less than 0.1% of the mass of the system. Also, the waveforms are indistinguishable at the level of the numerical accuracy from quasinormal mode oscillations of a black hole.

Compelled by these results, Price and Pullin [4] analyzed the Misner initial data for two black holes at a moment of time symmetry as if it represented a single, perturbed black hole. Using gauge-invariant perturbation theory and the Zerilli equation, they were able to compute the initial distortion of the black hole, and the resulting asymptotic waveforms and energy fluxes as a function of separation. For small separation, when the approximation of the merged system as a single, perturbed black hole is expected to be most valid, the agreement between the radiated energy from perturbation theory and the results of fully relativistic simulations [4] is excellent, apparently within the error bars of the numerical calculation. The quadrupole waveforms are also remarkably similar when read off at the same radius. Only when the initial separation of the black holes is somewhat larger than the cutoff for encompassing apparent horizons is there substantial discrepancy between the perturbation theory and evolution results for radiation efficiency.

Previously (Ref. [5], hereafter paper 1), the current authors had used a similar perturbation theory analysis to study spurious radiation in time-symmetric and -asymmetric two-black-hole data sets in the opposite limit—that of large separation. These data sets represent the direct extension of Misner data allowing the two black holes to have nonvanishing *initial* linear and angular momenta. In this paper, we apply these gauge-invariant perturbation techniques, in the spirit of the Price and Pullin paper, to the close limit when the two black holes have an encompassing apparent horizon, and examine the gravitational radiation waveform and the amount of energy radiated for time-asymmetric initial data.

There are several motivations for this study of boosted black-hole initial data. In the final plunge phase of binary black-hole coalescence, the black holes are likely to have substantial infall velocities. Although the actual collision is not expected to be head-on, axisymmetric calculations of boosted head-on collisions are an interesting limiting case of full three-dimensional plunge simulations. In addition, these calculations extend the regime where perturbation calculations can be fairly compared with the fully relativistic simulations. It is clear that, for time-symmetric initial data with no event horizon, the perturbation theory calculation should greatly overestimate the distortion of the merged black hole, and thus the radiated energy. For these cases the merged black hole does not form until the individual black holes have evolved toward each other and have some infall momentum. It is this merged black hole that one would like to analyze with perturbation theory and compare against the evolution results.

II. METHODOLOGY

In this paper, we will be concerned with two equal-mass nonrotating black holes, each with axisymmetric inward-pointing momentum P (the slice has zero net-momentum). Initial-data sets representing one or more black holes with individually specifiable linear and angular momenta are constructed using the conformal-imaging approach developed by

York and co-workers [6]. The case of two black holes with axisymmetric momenta was implemented numerically by Cook [7]. The data sets used for this study were constructed using a code based on the Čadež-coordinate approach described in that work. The reader should refer there for details regarding the construction of the data sets and for further descriptions of the parametrization of the data sets described below. The separation of the holes is parametrized by β, which is related to the bispherical-coordinate separation parameter μ_0 by the relation $\mu_0=\operatorname{arccosh}(\beta/2)$ in the case of equal-mass holes. The code is used to compute the inversion-symmetric (with minus isometry condition) extrinsic curvature K_{ij} and to solve the Hamiltonian constraint for the conformal factor ψ. Once the full initial data are computed, several physical quantities characterizing the system are computed. Of interest in this paper are the Arnowitt-Deser-Misner (ADM) mass of the initial-data slice M, the proper separation of the holes ℓ, and the masses of the individual holes $m_1=m_2$ defined in terms of the area of the marginally outer-trapped surface associated with each hole. We also define the total or bare mass of the system by $m=m_1+m_2$. Note that the difference between m and M is due to the binding energy of the system. Given an initial-data set, we use the boundary-value-problem method described in paper 1 to locate all marginally outer-trapped surfaces surrounding the two holes (if they exist) and identify the apparent horizon(s). We should note that there is nothing unique about our choice of initial data for representing two colliding black holes. One could imagine, for example, considering data with Euclidean topology with the black holes represented by boosted matter collapsed inside its horizon. Our initial data were chosen for the convenience of their highly refined numerical treatment [7] and earlier physical exploration (paper 1).

Like Price and Pullin [4], for purposes of our analysis we treat the spacetime as a perturbation (but not necessarily a time-symmetric one) of Schwarzschild. First, we establish a Schwarzschild-like coordinate system around the two black holes in terms of the (background space) isotropic coordinates used in the numerical solution $r=r_i(1+M/2r_i)^2$. Note that the background space of the numerical solution can be directly parametrized by the isotropic radial coordinate r_i even though the numerical solution is found in Čadež coordinates. The total ADM mass of the slice M is used as the Schwarzschild background mass. Tortoise coordinates are also defined in the usual way: $r_*=r+2M\ln(r/2M-1)$. Computation of wave perturbations involves the calculation of multipole amplitudes by surface integrals. These are performed over constant Schwarzschild radial-coordinate two-spheres. The integrands involve the conformal factor ψ, Schwarzschild-coordinate extrinsic-curvature components K_{ij}, and their Schwarzschild-coordinate radial derivatives. Calculation of these quantities at their required locations is achieved with bicubic spline interpolations and a series of coordinate transformations.

The gauge-invariant function $Q_{\ell m}$ is formed out of multipole projections of ψ and $\psi_{,r}$ computed by numerical integrations over a coordinate two-sphere (cf. Refs. [8–10]). For this paper we compute only the case of $Q\equiv Q_{20}$. We also require the Schwarzschild time derivative of the gauge-invariant function $\partial_t Q$. This time derivative is computed as

$$\partial_t Q=\alpha\mathscr{L}_{\mathbf{n}}Q, \tag{1}$$

where $\mathscr{L}_{\mathbf{n}}$ is the Lie derivative along the slice-normal congruence $\mathbf{n}$ and the factor $\alpha=\sqrt{(1-2M/r)}$ comes about from the transformation from the slice-normal time coordinate to the Schwarzschild time coordinate. The Lie derivative of Q is calculated using the extrinsic curvature (and its radial derivative) via the definition

$$\mathscr{L}_{\mathbf{n}}g_{ij}=-2K_{ij}. \tag{2}$$

The gauge-invariant perturbation function and its time derivative, known as a function of radius surrounding the merged black hole, serves as initial data for integration of the Zerilli equation. The numerically generated initial perturbation is interpolated onto a fine grid (typically 8000–16 000 zones) that is even in r_* and extends from $r_*=-500M$ to $r_*=2000M$. The Zerilli equation (cf. Refs. [8,10,4]) is then integrated forward in time until the whole perturbation has been propagated to $|r_*|\to\infty$. Approximate asymptotic waveforms and energy fluxes are computed at large radii.

Our code for calculating the initial black-hole perturbation from numerically generated initial data was checked by comparing it against the time-symmetric results of Price and Pullin [4]. It should be noted that they analytically expanded the metric perturbation about Schwarzschild in powers of the parameter $\epsilon=1/|\ln\mu_0|$ and retain only the leading term in ϵ. In the limit of small separation, the initial perturbation we obtain numerically agrees closely with their analytic results (for $\beta\lesssim3.25$ the agreement is better than 5%). Not surprisingly, for larger separations the differences become larger. For the horizon cutoff point of $\beta=4.17$ or $\mu_0=1.36$ (the largest separation allowing an encompassing apparent horizon for time-symmetric initial data), the analytic prediction for the energy radiated is about 60% higher than the result from the full solution. It is interesting to note that neglecting the higher-order terms in ϵ always seems to lead to a greater amount of radiated energy.

III. WAVEFORMS AND ENERGY FLUX

The perturbed black-hole approximation assumed in this paper is not valid if the two black holes have not merged (have no common event horizon). For separations small enough that an encompassing apparent horizon exists, we find that the addition of inward-pointing linear momenta makes the encompassing apparent horizon more spherical and the maximum radiation efficiency (defined as the ratio of M minus the mass of the apparent horizon to M) decreases. Moreover, we find that the metric perturbation Q always gets smaller. We, therefore, contend that our treatment of time-asymmetric initial-data sets with inward-pointing linear momenta is always at least as valid as the study of the time-symmetric solution.

In paper 1 we located the horizon-formation line for boosted black-hole initial data. For a given separation parameter β, we searched for the smallest value of inward linear momentum for which an encompassing horizon surrounded the holes. The momentum as a function of proper hole separation for this horizon line is displayed in Fig. 1. The horizon cutoff point mentioned previously, at $\beta=4.17$ and $P=0$, lies

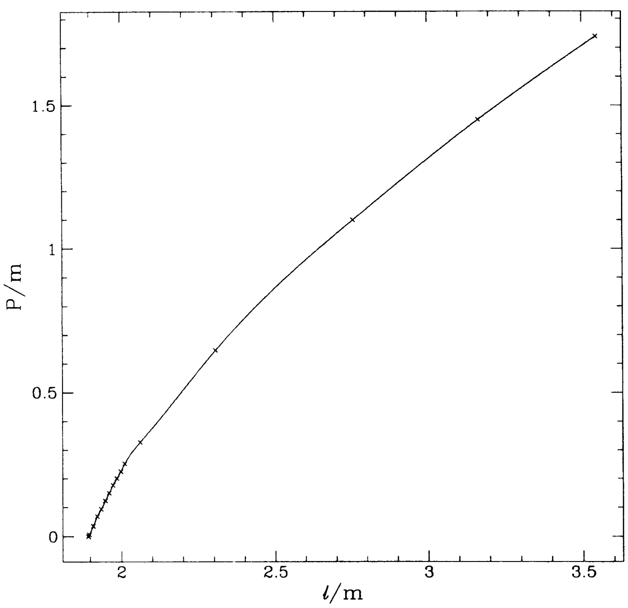

FIG. 1. The apparent horizon-formation line. The inward linear momentum on each hole P/m is plotted as a function of proper separation ℓ/m.

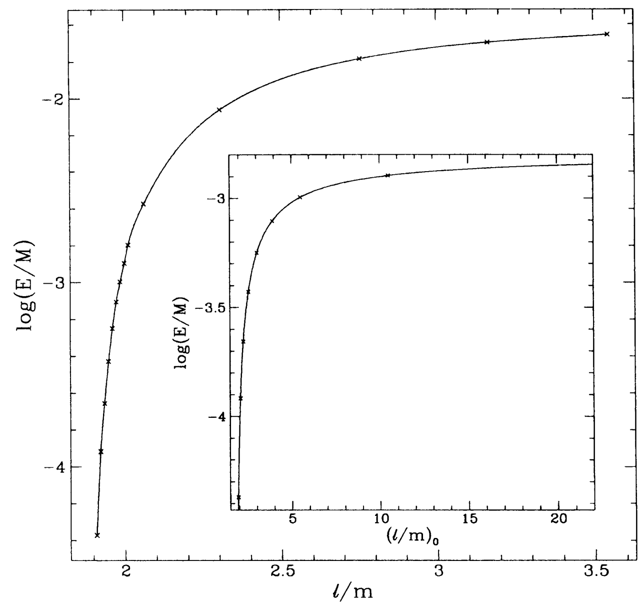

FIG. 2. Radiated energy along the apparent horizon-formation line. The radiation efficiency (the total radiated energy as a fraction of the ADM mass) computed with perturbation theory is plotted for initial data along the horizon line, parametrized by the proper separation of the holes ℓ/m. In the inset we show the radiation efficiency plotted as a function of the separation of the corresponding time-symmetric initial-data set, $(\ell/m)_0$, computed using Eq. (3). The logarithm is to base 10.

on this line. For larger values of β, it is necessary to give the holes inward momentum in order for an encompassing apparent horizon to exist. We note that this horizon line is only an estimate of where the actual encompassing *event* horizons will form. Along this horizon-formation line, we have computed the radiated energy for the initial-data sets using the gauge-invariant perturbation formalism and Zerilli-equation integration method described above. In Fig. 2, the radiated energy is plotted as a function of proper separation ℓ/m. For the points shown, the inward momentum on each hole ranges from $P/m=0.0355$ to $P/m=1.738$. One interesting feature is that the radiation efficiency appears to saturate at about 2%, substantially below the maximum radiation efficiency based on area theorem arguments. This suggests that it may be impossible to obtain high radiation efficiency for black-hole collisions, even if they merge with very large momenta. As a gauge of what constitutes a large momentum, we estimate below the momenta of two holes at the moment of horizon formation assuming a parabolic infall from rest at infinite separation.

At each point on the horizon-formation line, we have a value for the separation ℓ/m and momenta P/m of the holes. Treating the black holes as point particles and using Newtonian dynamics, we can estimate the separation $(\ell/m)_0$ at which the holes were at rest:

$$\left(\frac{\ell}{m}\right)_0=\frac{\ell/m}{1-8(P/m)^2(\ell/m)}\ . \tag{3}$$

Clearly, if we assume infall from rest, the maximum momenta the two holes can obtain at the point of horizon formation is estimated by the point on the horizon-formation line where the denominator of Eq. (3) vanishes, i.e., $(\ell/m)_0\to\infty$. We find this point to be $P/m=0.249$ and $\ell/m=2.01$. At this point, the implied radiation efficiency is less than 0.16%. In the inset of Fig. 2, we show the radiated energy from points on the horizon-formation line plotted as a function of the Newtonian estimate for their proper separation when at rest. We find striking agreement between our calculation of total energy radiated and the simulations of Anninos *et al.* [3]. For cases where their initial data had an encompassing horizon, it is clearly correct to compare with the perturbation analysis of Misner data. For cases with greater separation, the time-asymmetric analysis gives excellent results. For example, for $\mu_0=2.2$ or $(\ell/m)_0=3.97$, the radiation efficiency from perturbation theory of the corresponding horizon-line initial-data set is $E/M=7.9\times10^{-4}$, as compared with $E/M=1.7\times10^{-3}$ from the time-symmetric analysis of $\mu_0=2.2$ Misner data and $E/M\simeq5.5\times10^{-4}$ from the fully relativistic simulations. A post-Newtonian calculation of the infall trajectory might improve this comparison.

The agreement between the time-asymmetric perturbation theory and fully relativistic numerical simulations lends support to the notion that most of the radiation from head-on collisions is emitted when the black holes are close together (in the form of quasinormal modes) and that the long-wavelength infall radiation contributes little to the total energy. A simple calculation based on the quadrupole formula for Newtonian trajectories shows that about three quarters of the infall energy from infinity is emitted after the holes are within $3M$ of each other.

In Fig. 3 we show a typical waveform from a boosted head-on collision observed at a radius of $r=200M$. The case shown is for a separation of $\beta=4.275$, $P/m=0.1225$, and

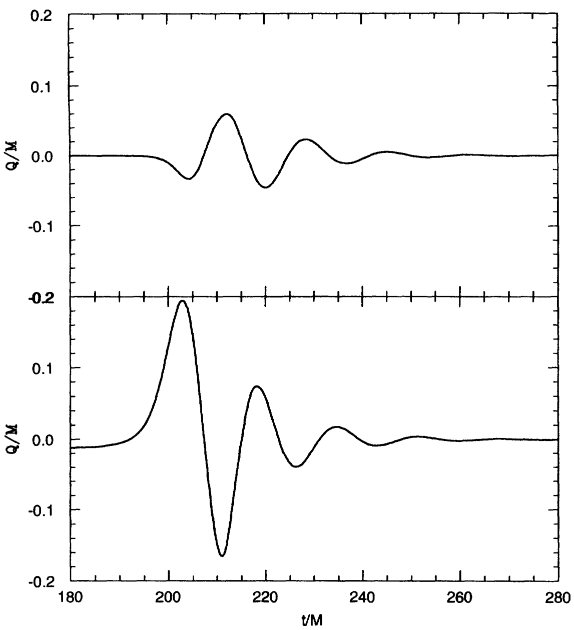

FIG. 3. Waveforms from time-symmetric and -asymmetric two-black-hole initial data. The top curve shows the quadrupole waveform from analysis of a time-symmetric data set with $\beta = 4.275$. The lower curve shows the quadrupole waveform from analysis of a time-asymmetric initial-data set with the same value of β and $P/m = 0.1225$. Both waveforms are extracted at a radius $r = 200M$.

$\mathscr{E}/m = 1.948$. For comparison we show the waveform from time-symmetric data with the same β. Both waveforms are dominated by normal mode oscillations within about $10M$ after the black-hole surface is causally apparent at the extraction radius. The addition of ingoing momentum considerably increases the amplitude of the oscillation and reverses the sign of the waveform. The presence of momentum (extrinsic curvature) on the initial slice causes a significant transient feature in the waveform; this qualitative effect is not seen by the numerical relativity simulations of time-symmetric initial data with $\mu_0 = 2.2$ because it takes finite time for the momentum component of the perturbation to propagate out to the extraction surface. It is possible that this feature will be present in the evolution of time-symmetric initial data starting at larger separations.

IV. DISCUSSION

Anninos *et al.* [3] have shown that for collisions resulting from large initial separations there is excellent agreement between the emitted energy and the well-known results for a test particle falling into Schwarzschild corrected for equal-mass objects, finite infall distance, and horizon heating. Price and Pullin [4] demonstrated that a perturbation analysis of time-symmetric initial data could reproduce the results of the fully relativistic simulation in the case that the black holes have small initial separation. Here we have shown that the perturbation analysis can be extended to larger separations, including the regime in which point particle analysis is valid, by considering appropriate time-asymmetric initial-data sets. Adopting this perspective, one can accurately predict the total emitted energies over the entire range of separation, from the close limit to parabolic trajectories starting at infinity.

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[1] C. Cutler *et al.*, Phys. Rev. Lett. **70**, 2984 (1993).
[2] L. L. Smarr, in *Eighth Texas Symposium on Relativistic Astrophysics*, edited by M. D. Papagiannis (The New York Academy of Sciences, New York, 1977), pp. 569–604.
[3] P. Anninos *et al.*, Phys. Rev. Lett. **71**, 2851 (1993).
[4] R. H. Price and J. Pullin, Phys. Rev. Lett. **72**, 3297 (1994).
[5] G. B. Cook and A. M. Abrahams, Phys. Rev. D **46**, 702 (1992).
[6] J. W. York, Jr., in *Sources of Gravitational Radiation*, edited by L. L. Smarr (Cambridge University Press, Cambridge, England, 1979), pp. 83–126; J. M. Bowen, Gen. Relativ. Gravit. **11**, 227 (1979); J. M. Bowen and J. W. York, Jr., Phys. Rev. D **21**, 2047 (1980); J. M. Bowen, Gen. Relativ. Gravit. **14**, 1183 (1982); A. D. Kulkarni, L. C. Shepley, and J. W. York, Jr., Phys. Lett. **96A**, 228 (1983).
[7] G. B. Cook, Phys. Rev. D **44**, 2983 (1991).
[8] V. Moncrief, Ann. Phys. (N.Y.) **88**, 323 (1974).
[9] C. Cunningham, R. Price, and V. Moncrief, Astrophys. J. **230**, 870 (1979).
[10] A. Abrahams *et al.*, Phys. Rev. D **45**, 3544 (1992).