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Radiation collapse and gravitational waves in three dimensions

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Two nonstatic solutions for three-dimensional gravity coupled to matter fields are given. One describes the collapse of radiation that results in a black hole. This is the three-dimensional analogue of the Vaidya metric, and is used to construct a model for mass inflation. The other describes plane gravitational waves for coupling to a massless scalar field.

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Lower dimensional gravity has often been used as an arena for investigating various problems that arise in four dimensions, but are not solvable there. Among those that have been substantially investigated include quantum gravity in three dimensions [1,2] and black hole evaporation in two dimensions [3].

Obtaining classical solutions in lower dimensions is often a first step in these models. In three-dimensional gravity the solutions for point masses were the first to be studied [4]. More recently a black hole solution has been given by Banados *et al.* [5], which also provides an arena for investigating black hole evaporation.

There are two classical problems in general relativity in four dimensions that have recently attracted some attention, and a three-dimensional version of them may be useful to address.

The first problem has to do with the inner (Cauchy) horizon of the Kerr and Reissner-Nordström black holes. This horizon is believed to be unstable to time-dependent perturbations because it is a surface where infalling radiation is infinitely blueshifted. The question is what effect the back reaction of the blueshifted radiation has on the internal geometry. More precisely one would like to know what type of singularity develops at or before the Cauchy horizon as a result of this back reaction. This question is important for the cosmic censorship hypothesis, for if the Cauchy horizon can be crossed, the timelike curvature singularity in such spacetimes becomes naked. This question may be asked of plane wave spacetimes, which also have Cauchy horizons.

Recent approaches to this problem, within spherical symmetry, take into account nonlinear perturbations at the Cauchy horizon as well as their back reaction on the geometry. The results suggest that the singularity has a null portion, where the internal mass function of the black hole diverges [6,7]. However, it is not yet known what type of singularity replaces the Cauchy horizon under general perturbations.

The second problem is the investigation of the collapse of matter fields to form black holes. This has been studied numerically and the results are intriguing [8,9]. It has been found that when the initial matter field is an ingoing pulse, the collapsing matter forms a black hole with mass given by $M=K(c-c_*)^\gamma$, where K is a constant, c is anyone of the parameters in the initial data for the matter field, c_* is the critical value of this parameter (that gives a zero mass black hole), and $\gamma\sim 0.36$. In particular, no black hole is formed when $c<c_*$. An important feature of this result is that it appears to be independent of spherical symmetry and the type of matter fields, with the same numerical exponent γ appearing in all cases studied to date. This seems to reflect a universal property of the Einstein equations in strong field regions. So far there is no analytical understanding of this result. It would be interesting to see if a similar result is true in lower dimensions, and whether it can be better understood there, perhaps analytically.

This paper is concerned with the first problem, and two metrics in three dimensions are given that have Cauchy horizons. One has the Vaidya form and describes collapsing spherically symmetric radiation. It allows the construction of the Ori model for mass inflation [7] in three dimensions. The other describes plane gravitational waves for coupling to a massless scalar field. This metric may also be used as a starting point for studying perturbations of Cauchy horizons [10].

The Vaidya form of a three-dimensional metric may be written using an advanced time coordinate v, and polar coordinates r,θ in the plane. It is

$$ds^2=-f(r,v)dv^2+2drdv+r^2d\theta^2. \quad (1)$$

The total energy-momentum tensor we use contains contributions

$$I_{\alpha\beta}=\frac{\rho(v)}{4\pi r}\,\partial_\alpha v\,\partial_\beta v \quad (2)$$

for infalling radiation with luminosity $\rho(v)$, and

$$E_\alpha^\beta=\frac{q^2}{4\pi r^2}\ \mathrm{diag}(-1,-1,1) \quad (3)$$

[in the coordinates (v,r,θ)] for the external electric field due to a charge q.

The Einstein equations with a cosmological constant Λ,

$$G_{\alpha\beta}+\Lambda g_{\alpha\beta}=2\pi(I_{\alpha\beta}+E_{\alpha\beta}), \quad (4)$$

have, with the ansatz (1), the solution

$$f(r,v)=-[\Lambda r^2+g(v)+q^2\ln r], \quad (5)$$

where $g(v)$ is given by

$$\frac{dg(v)}{dv} = \rho(v). \tag{6}$$

This gives the three-dimensional analogue of the charged Vaidya metric [11].

If asymptotically ($r \to \infty$ and $v \to -\infty$) the radiation inflow vanishes so that $g(v) = 0$, the metric assumes the "vacuum" form

$$ds^2 = (\Lambda r^2 + q^2 \ln r) dv^2 + 2 dv dr + r^2 d\theta^2. \tag{7}$$

This shows that $\Lambda \equiv -l^{-2}$ must be negative for the metric to be asymptotically Lorentzian.

The case $\rho(v) = 0$, $g(v) = M$, a constant, gives the spherically symmetric static black hole found by Banados *et al.* [5]. The event horizon (for $q=0$) is at

$$r_{\rm EH} = l\sqrt{M}. \tag{8}$$

When $\rho(v) \neq 0$, the mass of the black hole formed from the collapse depends on the parameters in the ingoing pulse. Asymptotically ($v \to \infty$), the apparent horizon becomes null and its radius gives the black hole mass. The radial coordinate of the apparent horizon $r_{\rm AH}$ is a measure of the black hole mass function $m(v)$, and is given (again for $q=0$) by

$$m(v) := r_{\rm AH}(v) = l\sqrt{g(v)}. \tag{9}$$

As an example, a "soliton" form for the radiation inflow,

$$\rho(v) = A\,{\rm sech}^2 \frac{v}{b}, \tag{10}$$

gives

$$g(v) = A b \tanh \frac{v}{b}, \tag{11}$$

where A, b are constants. The radius of the apparent horizon in the $v \to \infty$ limit gives the black hole mass formed in the collapse. The inflow (10) gives

$$m_{\rm BH} := \lim_{v \to \infty} r_{\rm AH} = l\sqrt{Ab}. \tag{12}$$

The charged case is interesting when $g(v) = 0$, since then

$$f(r) = (r/l)^2 - q^2 \ln r. \tag{13}$$

There is now the possibility of horizons even for zero mass depending on the value of the product ql. This is unlike the case for four-dimensional Reissner-Nordström black holes with a negative cosmological constant $\Lambda = -l^2$ where, for zero mass,

$$f = 1 + \left(\frac{q}{r}\right)^2 + \frac{1}{3}\left(\frac{r}{l}\right)^2, \tag{14}$$

and the singularity at $r=0$ is naked. In the present case, however, the equation for the horizon is

$$\left(\frac{r}{l}\right)^2 = q^2 \ln r, \tag{15}$$

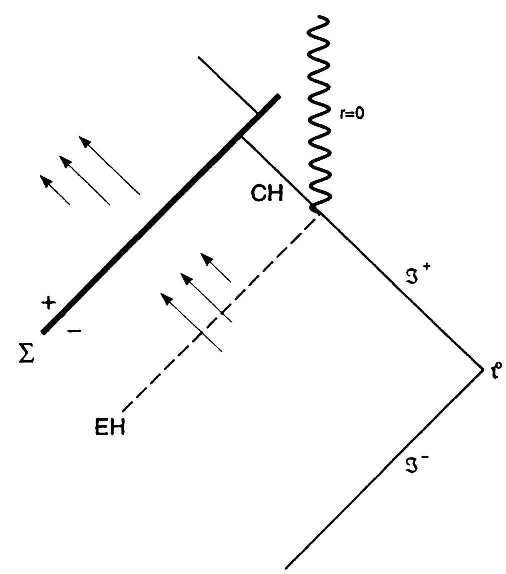

FIG. 1. Ori model for mass inflation: Two Vaidya metrics patched together along the outgoing null shell Σ. The arrows show infalling radiation

which first has a single solution for $ql = \sqrt{2e}$. This corresponds to a charged massless black hole without an inner horizon. For $ql < \sqrt{2e}$ the $r=0$ singularity is naked, whereas for $ql > \sqrt{2e}$ there is both an inner and outer horizon.

As mentioned above, there have been analytical studies of the nature of the Cauchy horizon designed to take into account the back reaction of blueshifted radiation on the internal black hole geometry. It has been found that the internal mass function diverges at the Cauchy horizon—a phenomenon which has been dubbed "mass inflation" [6].

The inputs that yield this result for charged spherically symmetric black holes are (i) infalling radiation, and (ii) "backscattered" outgoing radiation, all within the event horizon. The effect of the outgoing radiation is that it displaces the Cauchy horizon on the outer side of the outgoing pulse relative to that on its inner side. This is crucial to obtaining mass inflation.

These ingredients are most easily realized in a model due to Ori [7] constructed by patching together two Vaidya metrics along an outgoing null surface, with each metric describing infalling radiation. The outgoing radiation is thus modeled as the null shell Σ where the metrics are glued (see Fig. 1).

This model may be constructed in three dimensions using the metric (1). The main steps are identical to the four-dimensional case [7]. The two relevant equations are (i) the equation of the outgoing null shell obtained from (1),

$$f_+ dv_+ = f_- dv_- = 2dr, \tag{16}$$

where the subscripts $\pm$ denote the inner (outer) Vaidya metrics, and (ii) the continuity of the inflow along the null shell. This last condition is obtained by requiring the continuity of the components of the energy-momentum tensor along the null rays,

$$n^{\alpha}_{\pm}=\left(\frac{2}{f_{\pm}},1,0\right): \tag{17}$$

$$T_{\alpha\beta}n^{\alpha}n^{\beta}|_{+}=T_{\alpha\beta}n^{\alpha}n^{\beta}|_{-}, \tag{18}$$

which gives

$$\frac{1}{f_{+}^{2}}\frac{dg_{+}}{dv_{+}}=\frac{1}{f_{-}^{2}}\frac{dg_{-}}{dv_{-}}. \tag{19}$$

Combining (16) and (19) gives the key relation that gives rise to mass inflation:

$$\frac{dg_{+}}{f_{+}}=\frac{dg_{-}}{f_{-}}. \tag{20}$$

At the Cauchy horizon on the outer side of the outgoing null shell ($v_{-}\rightarrow\infty$, $f_{-}\rightarrow 0$), the right-hand side diverges since g_{-}, which is determined by the inflow, is assumed to be bounded. Also f_{+} is bounded because of the presence of the outgoing shell, which shifts the vanishing of f_{+} to a smaller value of the radial coordinate (see Fig. 1). Thus g_{+}, which gives the internal mass, must diverge. As done in the four-dimensional case, g_{+} may be computed explicitly by assuming a specific power law fall off for ρ_{-}, and using the above equations. This shows that mass inflation also occurs in three dimensions. Similar results have been obtained for two-dimensional dilatonic black holes [12].

Another metric in three dimensions that contains Cauchy horizons is one that describes gravitational waves when there is coupling to matter fields. (Without matter fields, three-dimensional gravity has only a finite number of degrees of freedom so there can be no gravitational waves.)

Here we consider coupling to a massless scalar field and give a gravitational wave solution. The Einstein equation for scalar field coupling may be written as

$$R_{\mu\nu}=2\pi\phi_{\mu}\phi_{\nu}. \tag{21}$$

With the ansatz

$$ds^2=-dt^2+dx^2+a^2(x,t)dy^2, \tag{22}$$

the field equations become

$$\ddot{a}-a''=0, \tag{23}$$

$$\frac{a''}{a}=-2\pi(\phi')^2,\quad \frac{\ddot{a}}{a}=-2\pi\dot{\phi}^2,\quad \frac{\dot{a}'}{a}=-2\pi\dot{\phi}\phi', \tag{24}$$

where the dot and prime denote t and x derivatives. The solutions to these equations are ingoing or outgoing waves $a(v)$, $a(u)$ where $v=t+x$ and $u=t-x$. The scalar field ϕ and a are connected by the latter three equations, all of which reduce to one equation. An explicit example is the scalar field linear in v which gives sinusoidal waves. The Cauchy horizon is the null surface where $a(v)=0$.

When considering general nonlinear perturbations of the Cauchy horizon, arguments similar to those given in Ref. [10] for four-dimensional plane wave spacetimes apply in this case as well. But, unlike the black hole case, there is as yet no simple model for studying perturbations of the Cauchy horizon for these spacetimes.

In summary, the first metric given here generalizes the three-dimensional black hole [5] to the nonstatic case, and gives a model for mass inflation. The second metric gives gravitational wave solutions when there is coupling to a massless scalar field. As discussed above, these metrics may be of use in studying the effect of general perturbations on the Cauchy horizon in a simpler setting.

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