Low energy states of (1+1)-dimensional Φ^6 field theories via the coupled cluster method

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The boson field theory with Φ^4 and Φ^6 coupling in (1+1) dimensions is investigated. Rigorous duality relations covering the whole range of coupling constants are derived. Vacuum, one-, and two-particle states are computed via the coupled cluster method in the four-particle approximations. Outside a critical region around second-order phase transitions the method works very well even in regions with first-order phase transitions and bound states.

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I. INTRODUCTION

The coupled cluster method (CCM) is one of the nonperturbative tools for dealing with quantum field theories. It has been applied to the Φ^4 field in 1+1 dimensions [1–4], both in the "S_4" [or "four-particle subsystem" (SUB4) approximation], as well as more recently in the "S_6 approximation" [5]. In this scheme all terms with more than 4 (6) virtual particles in the exponent operator $S=\sum S_n$ for the vacuum state have been omitted and the remaining equations for $S_1,\ldots,S_4(S_1,\ldots,S_6)$ have been solved exactly. The corresponding approximations also have been made in the equations for the one and two meson states. Except for a certain "critical region" around the phase transition this method works quite well. There are some recent papers applying the CCM in modified form to lattice gauge theories [6–8] with very encouraging results. (For a rather complete list of the relevant literature see [8].) Some confidence in this method stems from its wide use and its successes in quantum chemistry [9].

In the present paper an application of this method in the SUB4 approximation to the same field theory, but with both Φ^4 as well as Φ^6 coupling, will be described. This model (from now on called the "Φ^6 model" for short) is a nontrivial extension of the Φ^4 model. It is known to support two-particle bound states [10,11], whereas the Φ^4 field very likely does not [12,13]. It certainly has phase transitions of partially unknown character [14] and some complex duality relations [15].

This paper is organized as follows. First, the model is described together with some exact results which were only partially known before. In Sec. III a short overview of the Gauss approximation is given, followed in Sec. IV by a description of the CCM as applied to the vacuum and one- and two-particle states. The results are presented in Sec. V and summarized in Sec. VI.

II. HAMILTONIAN, BOGOLUBOV TRANSFORMATION, AND DUALITY

The Hamiltonian density is

$$H=N_m\left[\frac{1}{2}(\nabla\Phi)^2+\frac{1}{2}\Pi^2+\frac{m^2}{2}\Phi^2+\lambda_4\Phi^4+\lambda_6\Phi^6\right]. \tag{2.1}$$

(Note that there is no factor $\frac{1}{4}$ in front of λ_4 as occurs in most papers on the Φ^4 theory. Thus the present λ_4 is $\frac{1}{4}$ times the λ_4 of the older papers.) Here N_m means normal ordering with respect to the mass parameter m. It has been introduced to remove a trivial infinite constant. It renormalizes the Φ^2 and Φ^4 terms. The Hamiltonian then is

$$\mathcal{H}=\int dx\, H\ . \tag{2.2}$$

All spatial integrations are from $-L/2$ to $L/2$, where L is the normalization volume. The recipes for performing the transition from non-normal ordering to normal ordering are standard [16]. The non-normal ordered form of the Hamiltonian density (2.1) is

$$H=\frac{1}{2}(\nabla\Phi)^2+\frac{1}{2}\Pi^2+\frac{m_B^2}{2}\Phi^2+\lambda_B\Phi^4+\lambda_6\Phi^6-D\ . \tag{2.3}$$

Here the "bare parameters" λ_B and m_B are given by

$$\lambda_4=\lambda_B+15\lambda_6\int\frac{dk}{4\pi\omega_k} \tag{2.4}$$

and

$$m^2=m_B^2+12\lambda_4\int\frac{dk}{4\pi\omega_k}-90\lambda_6\left[\int\frac{dk}{4\pi\omega_k}\right]^2, \tag{2.5}$$

with

$$\omega_k=\sqrt{k^2+m^2}\ . \tag{2.6}$$

D is an irrelevant infinite constant. To obtain a finite lower bound of the energy only the case $\lambda_6\geq 0$ and $\lambda_4>0$ for $\lambda_6=0$ will be considered. For the field amplitudes and the usual creation and annihilation operators in

$$\Phi=\int dk\frac{1}{\sqrt{4\pi\omega_k}}(a_k e^{ikx}+a_k^\dagger e^{-ikx}) \tag{2.7}$$

the standard boson commutation relations are assumed. To optimize the starting point for calculations and to learn something about duality, a Bogolubov transformation to new creation and annihilation operators is performed:

$$a_k=\frac{\omega_k+\epsilon_k}{2\sqrt{\omega_k\epsilon_k}}b_k+\frac{\omega_k-\epsilon_k}{2\sqrt{\omega_k\epsilon_k}}b_{-k}^\dagger+\sqrt{\pi\omega_0}t\delta(k)\ . \tag{2.8}$$

There is a new Fock vacuum $|\Phi_b\rangle$ with $b_k|\Phi_b\rangle=0$. t is the wave operator shift introduced to facilitate a nonvanishing field expectation value (i.e., symmetry breaking),

$$\Phi=\Phi_{\text{new}}+t \ , \tag{2.9}$$

and ϵ_k is a new (theoretical) single-particle energy. Both t and ϵ_k will be determined later. From now on we work only with Φ_{new} and thus omit the subscript new. With some standard manipulations [16], including normal ordering N_b with respect to the b_k and $b_k^\dagger$, one now obtains the Hamiltonian in the form

$$\begin{aligned}\mathcal{H}=&LV+\kappa(\mathbf{b}_0+\mathbf{b}_0^\dagger)+\int dk\ \hat{E}_k\mathbf{b}_k^\dagger\mathbf{b}_k+\int dk\ \Omega_k(\mathbf{b}_k^\dagger\mathbf{b}_{-k}^\dagger+\mathbf{b}_k\mathbf{b}_{-k})\\&+f_3N_b\int dx\ \Phi^3+f_4N_b\int dx\ \Phi^4+f_5N_b\int dx\ \Phi^5+\lambda_6N_b\int dx\ \Phi^6 \ .\end{aligned} \tag{2.10}$$

Here,

$$V=\int\frac{dk}{8\pi\epsilon_k}(\omega_k-\epsilon_k)^2+\frac{m^2}{2}t^2+\lambda_4t^4+\lambda_6t^6+\Delta(6\lambda_4t^2+15\lambda_6t^4)+3\Delta^2(\lambda_4+15\lambda_6t^2)+15\lambda_6\Delta^3 \ , \tag{2.11}$$

$$\kappa=\left[\frac{\pi}{\epsilon_0}\right]^{1/2}\frac{\partial V}{\partial t} \ , \tag{2.12}$$

$$\hat{E}_k=\frac{1}{2\epsilon_k}[\omega_k^2+\epsilon_k^2+12\lambda_4t^2+30\lambda_6t^4+12\Delta(\lambda_4+15\lambda_6t^2)+90\lambda_6\Delta^2] \ , \tag{2.13}$$

$$\Omega_k=\frac{\hat{E}_k-\epsilon_k}{2} \ , \tag{2.14}$$

$$f_3=t[4\lambda_4+20\lambda_6(t^2+3\Delta)] \ , \tag{2.15}$$

$$f_4=\lambda_4+15\lambda_6(t^2+\Delta) \ , \tag{2.16}$$

$$f_5=6\lambda_6t \ , \tag{2.17}$$

$$\Delta=\frac{1}{4\pi}\int dk\left[\frac{1}{\epsilon_k}-\frac{1}{\omega_k}\right] . \tag{2.18}$$

V is the expectation value of the Hamiltonian density with respect to the new Fock vacuum $|\Phi_b\rangle$.

The Bogolubov transformation can be used to derive rigorous duality relations which in turn are quite desirable for checking the numerical reliability of the calculations described below. The reason for the occurrence of dualities is the fact that the solutions of Eqs. (2.4) and (2.5) are not unique. In other words, two Hamiltonians $\mathcal{H}_i$ and $\mathcal{H}_j$ are equal, except for a constant, if the bare parameters are equal:

$$(\lambda_6)_i=(\lambda_6)_j=\lambda_6 \ , \quad (\lambda_B)_i=(\lambda_B)_j \ , \tag{2.19}$$

or

$$(\lambda_4)_i-15\lambda_6\left[\int\frac{dk}{4\pi\omega_k}\right]_{m_i}=(\lambda_4)_j-15\lambda_6\left[\int\frac{dk}{4\pi\omega_k}\right]_{m_j} \ ; \quad (m_B^2)_i=(m_B^2)_j \ , \tag{2.20}$$

or

$$m_i^2-12(\lambda_B)_i\left[\int\frac{dk}{4\pi\omega_k}\right]_{m_i}-90\lambda_6\left[\int\frac{dk}{4\pi\omega_k}\right]_{m_i}^2=m_j^2-12(\lambda_B)_j\left[\int\frac{dk}{4\pi\omega_k}\right]_{m_j}-90\lambda_6\left[\int\frac{dk}{4\pi\omega_k}\right]_{m_j}^2 . \tag{2.21}$$

Substitution of (2.20) into (2.21) leads to

$$m_j^2=m_i^2+12(\lambda_4)_i\Delta_{j-i}+90\lambda_6\Delta_{j-i}^2 \ , \tag{2.22}$$

with

$$\Delta_{j-i}=\int\frac{dk}{4\pi}\left[\frac{1}{\sqrt{k^2+m_j^2}}-\frac{1}{\sqrt{k^2+m_i^2}}\right]=\frac{1}{4\pi}\ln\frac{m_i^2}{m_j^2} \ . \tag{2.23}$$

Using this in (2.20) we obtain

$$(\lambda_4)_j=(\lambda_4)_i+15\lambda_6\Delta_{j-i} \ . \tag{2.24}$$

The Hamiltonians are rescaled by introducing the dimensionless coupling constants

$$(\hat{\lambda}_4)_i=\frac{(\lambda_4)_i}{m_i^2}, \quad (\hat{\lambda}_6)_i=\frac{\lambda_6}{m_i^2} \ . \tag{2.25}$$

From now on Hamiltonians always are meant to be rescaled in this way. It also is convenient to introduce one of the Hamiltonians, say $\mathcal{H}_1$ ("model 1"), as a reference Hamiltonian (if there is more than one). Then (2.22) becomes [15]

$$0=1-e^{z_j}-2\alpha z_j+\beta z_j^2 \ , \tag{2.26}$$

with

$$z_j=\ln\left[\frac{m_j^2}{m_1^2}\right], \quad \alpha=\frac{3}{2\pi}(\hat{\lambda}_4)_1, \quad \beta=\frac{45}{8\pi^2}(\hat{\lambda}_6)_1 \ . \tag{2.27}$$

This equation has either one or three solutions (crossing of an exponential with a parabola). It always has the solution $z=0$ with $m_j^2=m_1^2$ used as the reference Hamiltonian. From (2.5) and (2.22) it also follows that if to a "model 1" with a pair of coupling constants $(\hat{\lambda}_4)_1$, $(\hat{\lambda}_6)_1$ (with mass parameter m_1) there exist two additional solutions ν_1 and μ_1 of (2.22), then there exist two dual models: "model 2" with $z_2\neq 0 \Longleftrightarrow \mu_1^2\neq m_1^2$ and $(\hat{\lambda}_6)_2\neq(\hat{\lambda}_6)_1$ and "model 3" with $z_3\neq 0\Longleftrightarrow \nu_1^2\neq m_1^2$ and $(\hat{\lambda}_6)_3\neq(\hat{\lambda}_6)_1$. Here, from (2.25),

$$\frac{(\hat{\lambda}_6)_1}{(\hat{\lambda}_6)_2}=\frac{\mu_1^2}{m_1^2} \quad \text{and} \quad \frac{(\hat{\lambda}_6)_1}{(\hat{\lambda}_6)_3}=\frac{\nu_1^2}{m_1^2} \ , \tag{2.28}$$

with [from (2.24)]

$$\begin{aligned}(\hat{\lambda}_4)_2&=(\hat{\lambda}_6)_2\left[\frac{(\hat{\lambda}_4)_1}{(\hat{\lambda}_6)_1}+\frac{15}{4\pi}\ln\frac{(\hat{\lambda}_6)_2}{(\hat{\lambda}_6)_1}\right] ,\\ (\hat{\lambda}_4)_3&=(\hat{\lambda}_6)_3\left[\frac{(\hat{\lambda}_4)_1}{(\hat{\lambda}_6)_1}+\frac{15}{4\pi}\ln\frac{(\hat{\lambda}_6)_3}{(\hat{\lambda}_6)_1}\right] .\end{aligned} \tag{2.29}$$

Without loss of generality one may assume $\mu_1^2>\nu_1^2$. Let μ_2^2, ν_2^2, τ_2^2 (for $z_2\neq 0$) and μ_3^2, ν_3^2, τ_3^2 (for $z_3\neq 0$) be the solutions dual to the model 1 with solutions μ_1^2, ν_1^2, τ_1^2 (for $z=0$). Since all three (unscaled) Hamiltonians are identical, one has

$$\mu_1^2=\mu_2^2=\mu_3^2, \quad \nu_1^2=\nu_2^2=\nu_3^2, \quad \tau_1^2=\tau_2^2=\tau_3^2 \ . \tag{2.30}$$

From the first and second equations (2.28) then follows

$$\mu_2^2=m_2^2>\nu_2^2 \quad \text{and} \quad \mu_3^2>\nu_3^2=m_3^2 \ . \tag{2.31}$$

In the coupling constant region where there are no solutions with $\mu^2=m_1^2$ of (2.22) there is no dual Hamiltonian. Now it is convenient to inspect the potential $V(m^2,t)$ of (2.11) at the point $t=0$. Call this $V_0(z)$. It is given by

$$\frac{V_0(z)}{m^2}=\frac{1}{8\pi}(e^z-1-z+\alpha z^2-\tfrac{1}{3}\beta z^3) \ . \tag{2.32}$$

The extrema of $V_0(z)$ are given by $dV_0/dz=0$, which is the same as (2.26). Now $V_0(z)\to\infty$ for $z\to\pm\infty$. Thus $V_0(z)$ has either only one minimum or two local minima and a local maximum in between, corresponding to the zeros of (2.26).

Returning to (2.31), let us assume that we have the maximum of V_0 at $\tau_1^2=m_1^2$ and the two minima at μ_1^2 and ν_1^2. From (2.30) it follows that model 1 is dual to model 2 and model 3 with minima of V_0 at m_2^2 and m_3^2. If μ_1^2 is a global minimum, model 2 has the global minimum at m_2^2 and model 3 at $\mu_3^2>m_3^2$. Similar statements are valid if ν_1^2 is the global minimum. Taking all things together one arrives at the following list of regions in the parameter space.

One minimum : region "0" ,

Two minima:

$$\begin{aligned}&\mu^2_{\text{loc min}}>\nu^2_{\max}>\tau^2_{\text{glob min}} : \ \text{region "1"} ,\\ &\tau^2_{\text{glob min}}>\mu^2_{\max}>\nu^2_{\text{loc min}} : \ \text{region "2"} ,\\ &\mu^2_{\text{glob min}}>\nu^2_{\max}>\tau^2_{\text{loc min}} : \ \text{region "3"} ,\\ &\tau^2_{\text{loc min}}>\mu^2_{\max}>\nu^2_{\text{glob min}} : \ \text{region "4"} ,\\ &\mu^2_{\text{glob min}}>\tau^2_{\max}>\nu^2_{\text{loc min}} : \ \text{region "5"} ,\\ &\mu^2_{\text{loc min}}>\tau^2_{\max}>\nu^2_{\text{glob min}} : \ \text{region "6"} .\end{aligned}$$

Here all $\tau^2=m_1^2$ and for instance "loc min" means "local minimum". Thus in regions "1", "4", "6" the Hamiltonians are mutually dual and the same is true in the regions "2", "3", and "5". Furthermore, selecting one region of each triple plus the region "0" gives a complete description of the Hamiltonian. This was observed before in Ref. [15]. For instance, the combined regions "0","1", and "2", or as well "0", "2", and "4" will contain all information. It is quite remarkable that the small "weak coupling" regions "2" and "4" (plus region "0") describe the Hamiltonian also for "strong coupling". The concepts of "strong" or "weak coupling" clearly have lost their meaning as strong or weak physical interaction. The different duality regions can be identified numerically by scanning $V(z)$ for the parameters α and β. Most boundaries can be determined analytically, however. Increasing α and β, starting from the origin, in addition to $z=0$ two additional solutions show up. Continuity requires the birth of a saddle point, i.e., the first and second derivative of $V(z)$ must vanish at this boundary. This yields the parametrization

$$\alpha=\frac{e^z(z/2-1)+1}{z} \quad \text{and} \quad \beta=\frac{e^z(z-1)+1}{z^2} \ . \tag{2.33}$$

This defines the border line of region "0" with "1" and "2" in Fig. 1. (Note that in this figure the region "2" is too small to be distinguishable from the line.) Increasing further the distance from the origin, the new minimum μ^2 becomes a global minimum, i.e., $V_0(z=0)$ becomes negative. This yields

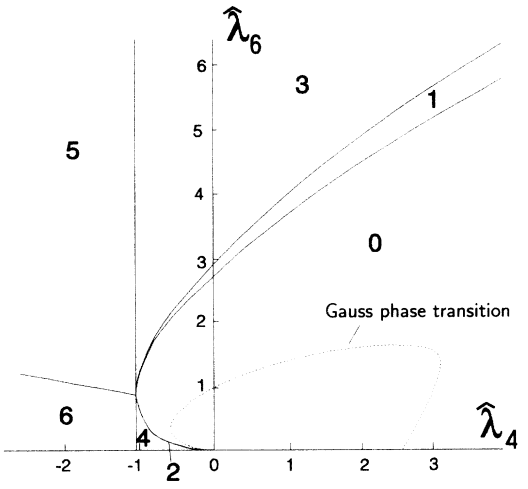

FIG. 1. Duality regions for the Φ^6 model. Region 2 is too small to be distinguishable from the line.

$$\alpha=\frac{2z+3+(z-1)e^z}{z^2},\quad \beta=\frac{3[z+2+(z-2)e^z]}{z^3}\ . \tag{2.34}$$

In Fig. 1 this defines the boundaries between "1" and "2" on the one hand with "3" and "4" on the other. The boundary between "5" and "6" with "3" and "2" appears at $\alpha=-\frac{1}{2}$ where the second derivative of $V_0(z=0)$ changes sign because $\mu^2=m^2$ becomes a maximum. Thus, for $\alpha<-1/2\Leftrightarrow\hat{\lambda}_4<-\pi/3$ there are minima at $\mu^2>m^2$ and $v^2<m^2$. The transition between the region "5" and "6" corresponding to the cases $\mu^2=$global minimum and $v^2=$global minimum could be determined numerically only. The region "0" is the only one with no duality partner.

Stevenson [15] used an alternative method based on the Gaussian effective potential [14] in the regions "0", "1", and "2". The remainder of the parameter space can be transformed into this area via a renormalization group transformation.

III. GAUSS APPROXIMATION

The Gauss approximation corresponds to the Hartree approximation of many body theory, minimizing the energy expectation value for a suitable Fock state. In the present model this means variation of V in (2.11) with respect to t and ϵ_k as necessary conditions. There are many applications of this approach in the literature [17–19]. The most complete one for the Φ^6 theory is due to Stevenson and Roditi [15]. The CCM as applied in the present work uses this approximation as a starting point. Using the conditions

$$\frac{\partial V}{\partial t}=0,\quad \frac{\delta V}{\delta\epsilon_k}=0\ , \tag{3.1}$$

with V from (2.11), it then is straightforward to compute V for the t and ϵ_k determined by (3.1). It turns out that near the origin (in the coupling constant plane) the minimum corresponds to $t=0$. Since $t=\langle\Phi_b|\Phi|\Phi_b\rangle\equiv\langle\Phi\rangle$ this implies that there is the symmetric phase corresponding to field expectation value zero. Going away from the origin there in any direction shows up a local minimum and finally a global minimum with $t\neq0$; i.e., there is a transition to the symmetry broken phase with a field expectation value different from zero. In Fig. 1 the boundary between both regions is shown. This phase transition is of first order. This is known to be wrong for the Φ^4 model [13] and therefore to some extent must be wrong for the Φ^6 field theory, too.

One-particle states in the Gauss approximation are just generated by the creation operators as

$$|1^k\rangle=\mathbf{b}_k^\dagger|\Phi_b\rangle\ . \tag{3.2}$$

Two-particle states for total momentum zero are of the form

$$|2^{k=0}\rangle=\int dp\, f(p)\mathbf{b}_p^\dagger\mathbf{b}_{-p}^\dagger|\Phi_b\rangle\ . \tag{3.3}$$

$f(p)$ then can be determined by varying the expectation value of the energy with respect to $f(p)$. In this way there have been found bound states of two particles (for $\lambda_6\neq0$ and $t=0$ only) as well as the two particle continuum [11,15,19]. The results of the present paper later will be compared with the Gaussian ones.

IV. CCM APPROXIMATIONS

A. Vacuum

The application of the CCM to the vacuum has been described in much detail in Ref. [3]. Thus here only the essentials will be put down without any details. The vacuum state is written in the usual exponential form

$$|\Psi_{\rm vac}\rangle=\exp(\mathbf{S})|\Phi_b\rangle \tag{4.1}$$

with

$$\mathbf{S}=\sum_n^\infty\mathbf{S}_n\ , \tag{4.2}$$

where $\mathbf{S}_n$ creates n virtual particles:

$$\mathbf{S}_n=\frac{1}{n!}\int dq_1\cdots dq_n\mathbf{S}_n(q_1\cdots q_n)\mathbf{b}_{q_2}^\dagger\cdots\mathbf{b}_{q_n}^\dagger\ . \tag{4.3}$$

Then the (standard) CCM equations are

$$\begin{aligned}&\langle\Phi_b|\exp(-\mathbf{S})\mathcal{H}\exp(\mathbf{S})|\Phi_b\rangle=E_{\rm vac}\ ,\\ &\vdots\\ &\langle\Phi_{q_1\cdots q_n}|\exp(-\mathbf{S})\mathcal{H}\exp(\mathbf{S})|\Phi_b\rangle=0\ ,\\ &\vdots\ .\end{aligned} \tag{4.4}$$

Here

$$|\Phi_{q_1\cdots q_2}\rangle=\mathbf{b}_{q_1}^\dagger\mathbf{b}_{q_2}^\dagger\ldots\mathbf{b}_{q_n}^\dagger|\Phi_b\rangle \tag{4.5}$$

are Fock states of n particles. The S_4 approximation takes all $\mathbf{S}_n$ with $n\leq4$ into account, thus leading to just the first five equations (4.4). Furthermore, the "max-

imum overlap" condition is imposed like in Ref. [3]. Thus $\mathbf{S}_1$ and $\mathbf{S}_2$ are replaced by the field shift t and single-particle energies ϵ_k. It does not make sense to write down the explicit equations: there are 69 terms (before symmetrization), but their derivation is based on a straightforward reduction of the matrix elements via Wick's theorems. The analytic form of the terms can be found in Ref. [20].

B. One-particle states and physical mass

One-particle states in field theory technically correspond to closed shell plus one-particle states in many body theory [21–23]. The adaption of the method to quantum fields has been described elsewhere [24] and even applied before to the Φ^4 field theory [4]. Thus, again only few details will be given here. One particle with momentum k has the state

$$|\Psi^k\rangle=\hat{\mathbf{F}}_k \exp(\mathbf{S})|\Phi_b\rangle \ , \tag{4.6}$$

with

$$\hat{\mathbf{F}}_k=F_0^k+\mathbf{F}_k \ . \tag{4.7}$$

Here F_0^k is a number, and $\mathbf{F}_k$ is the operator

$$\mathbf{F}_k=F_1^k\mathbf{b}_k^\dagger + \sum_{n=2}^{\infty}\frac{1}{n!}\int dk_1\cdots dk_n\delta(k_1+\cdots+k_n-k) \times F_n^k(k_1,\ldots,k_{n-1})\mathbf{b}_{k_1}^\dagger\cdots\mathbf{b}_{k_n}^\dagger \ . \tag{4.8}$$

The operator $\mathbf{S}$ is taken from the vacuum problem. The ansatz (4.6) is quite general. It contains as ingredients the physical vacuum and dressed 1,2,. . . particle states. The one virtual particle state should be the leading contribution. This will be incorporated by the solution technique. One then proceeds by writing the eigenvalue equations as

$$\exp(-\mathbf{S})\mathcal{H}\hat{\mathbf{F}}_k \exp(\mathbf{S})|\Phi_b\rangle=E^k\hat{\mathbf{F}}_k|\Phi_b\rangle \ . \tag{4.9}$$

Subtracting the vacuum equation multiplied with $\exp(-\mathbf{S})\hat{\mathbf{F}}_k$ one obtains

$$(E^k-E_{\rm vac})\hat{\mathbf{F}}_k|\Phi_b\rangle =\exp(-\mathbf{S})[\mathcal{H}\mathbf{F}_k-\mathbf{F}_k\mathcal{H}]\exp(\mathbf{S})|\Phi_b\rangle \ . \tag{4.10}$$

Now one again projects with the complete set of Fock states to obtain, for $n=0$,

$$(E^k-E_{\rm vac})F_0^k=\langle\Phi_b|\mathcal{H}\exp(\mathbf{S})\mathbf{F}_k|\Phi_b\rangle \ , \tag{4.11}$$

and, for $n>0$,

$$(E^k-E_{\rm vac})F_n^k(k_1,\ldots,k_{n-1})\delta(k_1+\cdots+k_n-k) =\langle\Phi_{k_1\cdots k_n}|\exp(-\mathbf{S})[\mathcal{H}\mathbf{F}_k-\mathbf{F}_k\mathcal{H}]\exp(\mathbf{S})|\Phi_b\rangle \ . \tag{4.12}$$

(4.11) occurs because the particle number is not conserved. It determines the (here irrelevant) constant F_0^k. (4.12) is a coupled set of linear equations for the energy difference and the amplitudes F_n^k. Because these equations are extremely large a matrix diagonalization is not feasible. Instead, an iterative procedure has been performed. As explained above, to hit upon physical one-particle states the natural starting value is $F_1^k=1$, i.e., the state of one bare particle. For $k=0$ the energy difference E^k-E_{vac} then is the physical particle mass. Above this energy begins the continuum describing single physical bosons moving with momenta k. Of course, approximations have to be made again. In this work all F_n^k with $n\le 4$ have been included. Again, the explicit equations (with 217 terms) will not be written down and can be found in Ref. [20]. Their derivation is straightforward, although quite cumbersome.

C. Two particle states

Considering only the subspace with total momentum zero, for two bosons due to the relative motion, there is a continuum of states beginning at twice the physical mass. The existence of bound states will lead to states with discrete energies smaller than twice this mass. It is remarkable that the CCM equations for two particles are practically the same as for one particle. The only difference is that one now has to use another starting point. Let $G_n(q)$ be the amplitudes for the two-particle states corresponding to the F_n^k. With $b_k^\dagger b_{-k}^\dagger|\Phi_b\rangle$ as lowest free state one has

$$G_2(q)=\delta(q)+\text{corrections} \ . \tag{4.13}$$

Thus, after discretization one starts with

$$G_2(q)=1,\text{if } q=0 \text{ and} G_2(q)\to 0 \text{ for coupling}\to 0 \text{ if} q\neq 0 \ . \tag{4.14}$$

V. RESULTS

A. Vacuum

The set of coupled equations (4.4) is solved by iteration, starting from the Gauss approximation, the procedure being practically the same as in Ref. [3]. Figure 2 gives

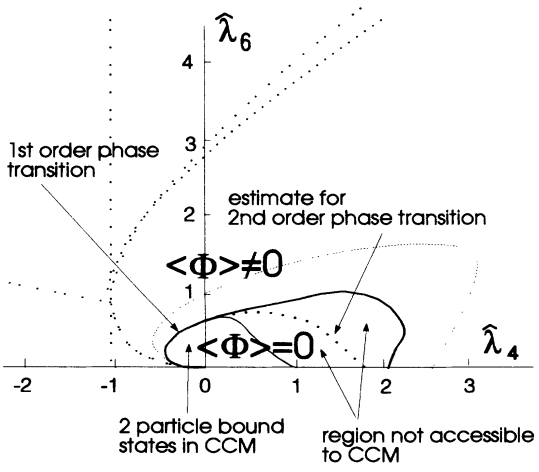

FIG. 2. Phases in the CCM approximation.

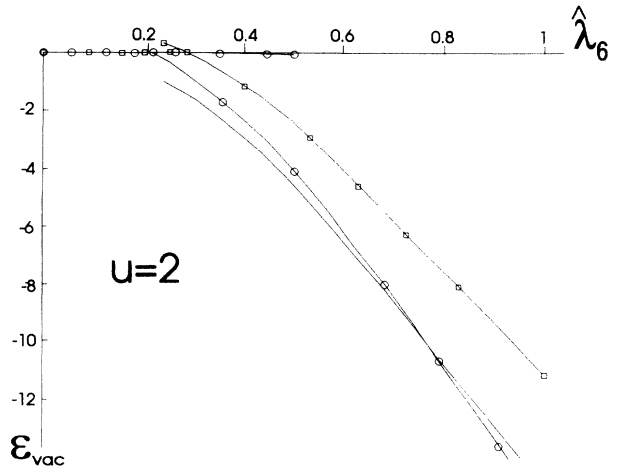

FIG. 3. Vacuum energy density for $u=2$. Circles, CCM; boxes, Gauss approximation; solid line, second-order perturbation theory.

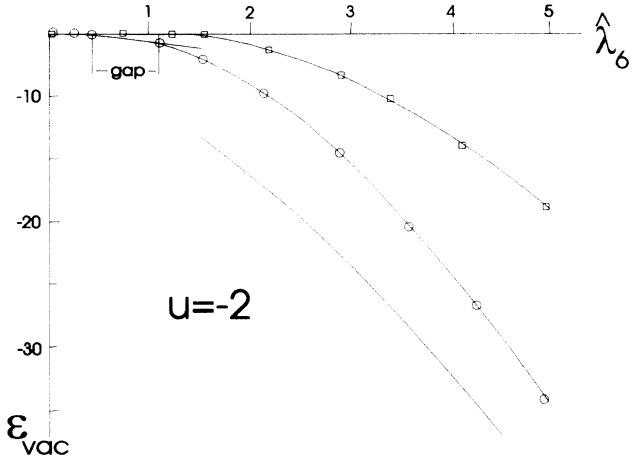

FIG. 5. Same as Fig. 3 for $u=-2$.

an overview of the results. The parameter u is defined by $u=-\hat{\lambda}_4/\hat{\lambda}_6$. Thus, $u=-\infty$ corresponds to the pure Φ^4 model, and $u=0$ to the pure Φ^6 case. There is a small region around the origin corresponding to the $\langle\Phi\rangle=0$ sector. To the right there is a region where no CCM solution could be found. The rest could be computed and corresponds to the $\langle\Phi\rangle\neq 0$ phase. For negative $\hat{\lambda}_4$ the phase transition is of first order. This can be calculated explicitly; see the energy density in Fig. 3 for $u=2$ as an example. This is in a region where the two phases coexist. The energy density defined as the lower of the two has a discontinuous first derivative. Decreasing u further, it so happens that there is a very small region where the CCM method fails, see Fig. 4. Here the energy density plot for $u=1$ has been enlarged to make this effect

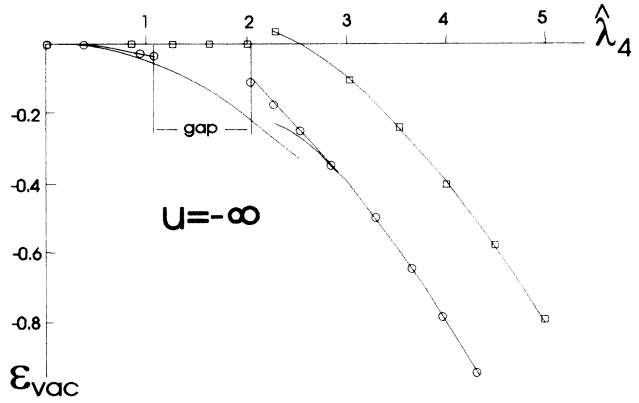

FIG. 6. Same as Fig. 3 for $u=-\infty$.

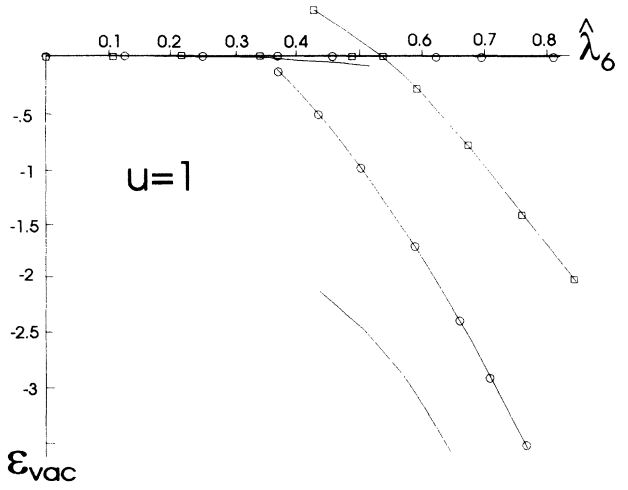

FIG. 4. Same as Fig. 3 for $u=1$.

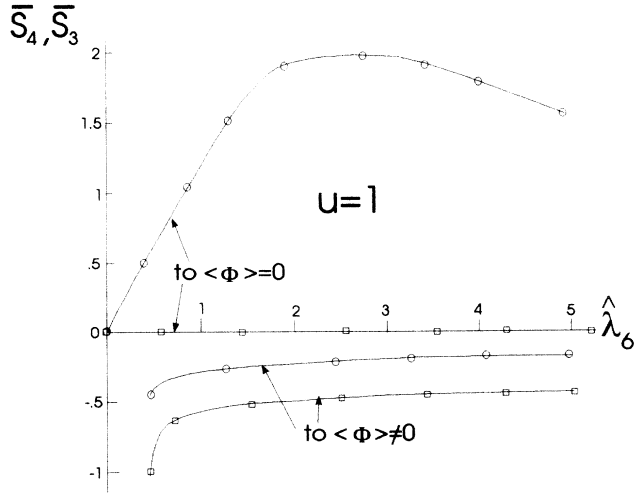

FIG. 7. Averages over S_3 and S_4 for $u=1$. Circles, $\bar{S}_4$; boxes, $\bar{S}_3$.

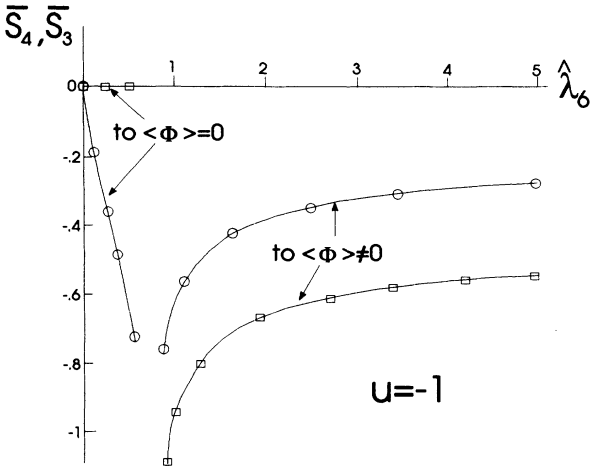

FIG. 8. Same as Fig. 7, for $u = -1$.

visible. But clearly it still can be assumed that the transition is of first order. For negative u there is a noncomputable gap increasing with $-u$ to the value corresponding to the Φ^4 model ($u = -\infty$), see Figs. 5 and 6. It very likely is related to the second-order phase transition as required for the Φ^4 case [13]. In these figures also the energy densities from the Gauss approximation have been plotted for comparison. Finally, the outcome of second-order perturbation theory is shown, demonstrating the superiority of the CCM in the neighborhood of phase transitions. Technical details concerning perturbation theory can be found in Ref. [20].

In Figs. 7 and 8 examples are shown for the size of the averaged correlation amplitudes S_3 and S_4. In the region with negative λ_4 (with first-order phase transition) they are finite everywhere, whereas for positive λ_4 they become infinite near the presumably second-order phase transition, reflecting the infinite correlation length.

Finally, the field expectation values (computed in the same way as in Ref. [3]) are shown in Fig. 9. They have

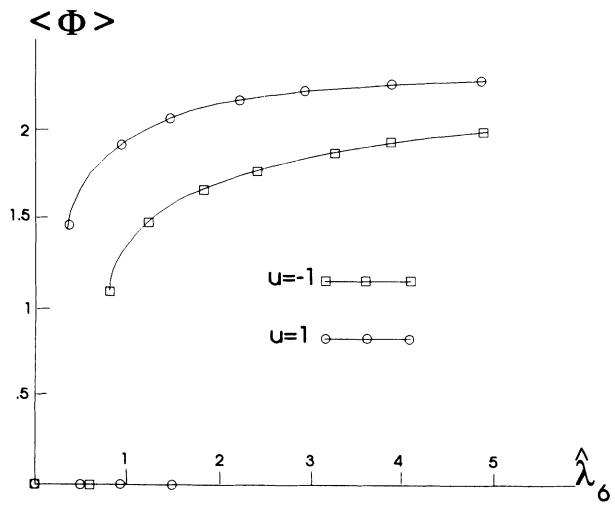

FIG. 9. Field expectation values.

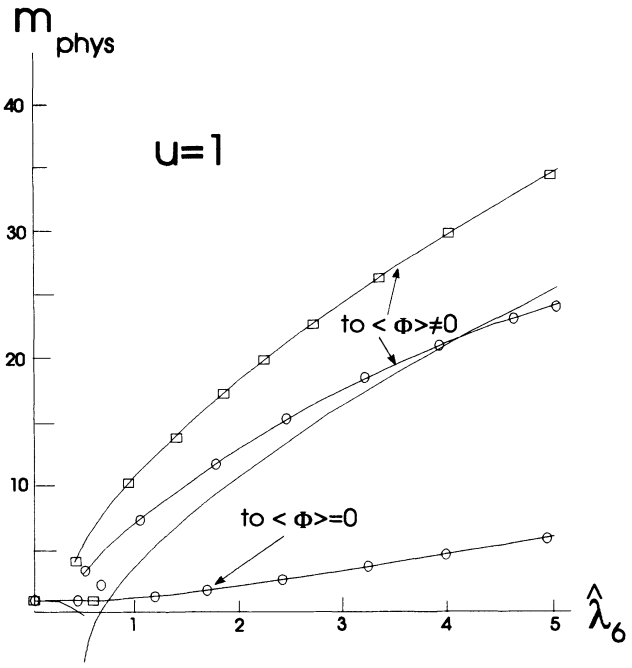

FIG. 10. Physical mass for $u = 1$. For notation see Fig. 3.

the expected behavior. The deviation from the Gauss approximation (not plotted here) always is small. By extrapolating the curves for $\langle \Phi \rangle$ to the $\langle \Phi \rangle = 0$ line one can obtain a rough estimate for the position of the phase transition. It has been indicated in Fig. 2.

For negative λ_4 there always is some region where the two phases coexist. Only the one with lower energy safely has a physical meaning, however.

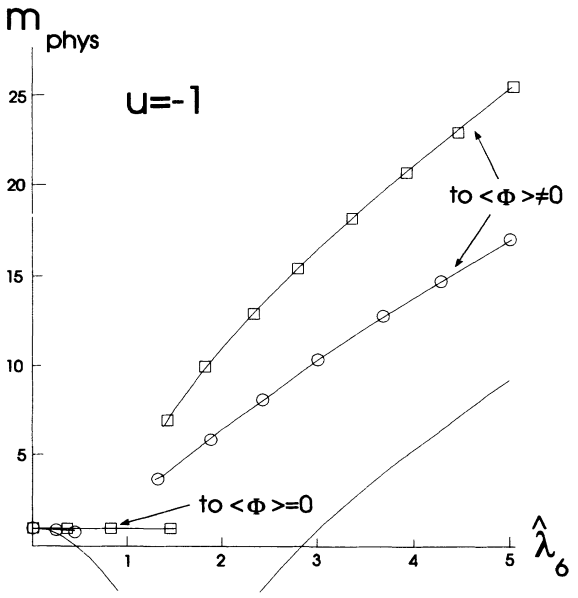

FIG. 11. Physical mass for $u = 1$. For notation see Fig. 3.

B. One-particle states and physical mass

Figures 10–12 present the physical mass m_{phys} for three typical cases: $u=1$, $u=-1$, and $u=-\infty$. The last case corresponds to the pure Φ^4 theory, which had been investigated before in a lower (F_3) approximation only [4]. It shows decrease of the mass approaching the critical region, as is required by the second-order phase transition, since m_{phys}^{-1} is the correlation length. Of course, again the critical region is inaccessible by the CCM, in contrast with the Gauss approximation with the wrong type of phase transition. The second-order perturbation theory is much worse, however. Turning to the case with negative λ_4 by way of the example $u=1$, there is again the coexistence of the two phases and everything is finite. The mass of the asymmetric phase jumps to a substantially higher value in the symmetry-breaking one. In the region with $\lambda_4>0$ the behavior becomes similar to the one of the Φ^4 field, see the example in Fig. 11.

C. Two-particle states

It is clear that the quality of all methods truncating at a certain number of excited virtual particles goes down with increasing energy. Also, since these states lie in the one-particle continuum, there necessarily are admixtures of those states, leading to oscillations in the numerical solutions. Thus one cannot expect too much from the CCM in the case of physical two-particle states. Especially, the unaccessible critical region will become larger.

Most interesting are the two-particle bound states. They occur only for negative λ_4 in the $\langle\Phi\rangle=0$ sector, see Fig. 2. Their energies in units of the physical mass are shown in Fig. 13 as function of λ_4 for various u (i.e., various λ_6). With increasing $|\lambda_4|$ the binding energy increases for all u. Finally, the symmetry breaking phase shows up on the figure marked by boldface arrows. Two-particle binding could be obtained somewhat beyond these points, as shown in the figure. But symmetry breaking implies that the energy of the nonsymmetric phase not supporting bound states becomes lower than these virtual bound states. This means that these states at best may be resonant states, if they have any physical meaning at all. The Gaussian curve in Fig. 13 is valid for arbitrary u due to the u independence in the Gauss approximation [15,19]. It is seen that even in the CCM approximation the variation with u is substantially weaker than with λ_4.

The fact that there are no bound states for the Φ^4 model in the $\langle\Phi\rangle\neq0$ phase is in agreement with Refs. [12] and [25]. The nonexistence of bound states in the condensed phase as found for this model in the present paper is new.

The two-particle continuum should be trivial in the sense that the states with zero center-of-mass momentum must have an excitation energy exactly twice as large as the physical mass. As an example the numbers for the Φ^4 field are presented in Table I. Clearly the results are excellent where they could be computed. However, for the reasons given above the two-particle CCM equations could be solved reliably only rather far away from the critical region of the vacuum state. With the present convention for the coupling constants the ranges are $0\leq\lambda_4\leq0.6$ and $7\leq\lambda_4\leq\infty$ instead of $0\leq\lambda_4\leq1.0$ and $2.15\leq\lambda_4\leq\infty$ for the vacuum.

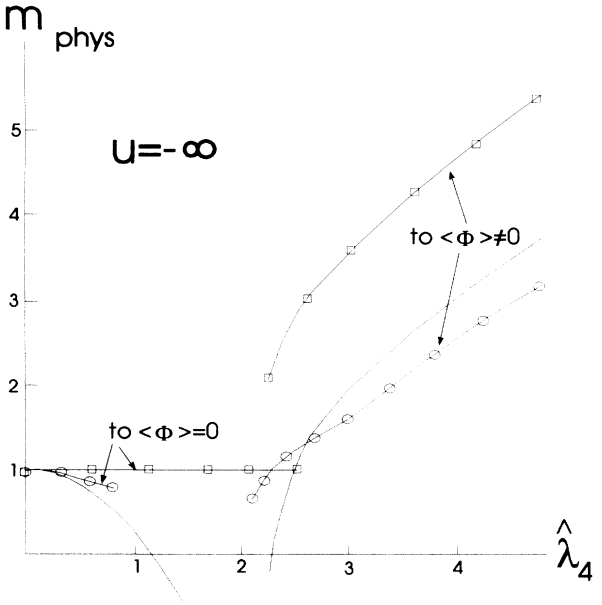

FIG. 12. Physical mass for $u=-\infty$ (Φ^4 field). For notation see Fig. 3.

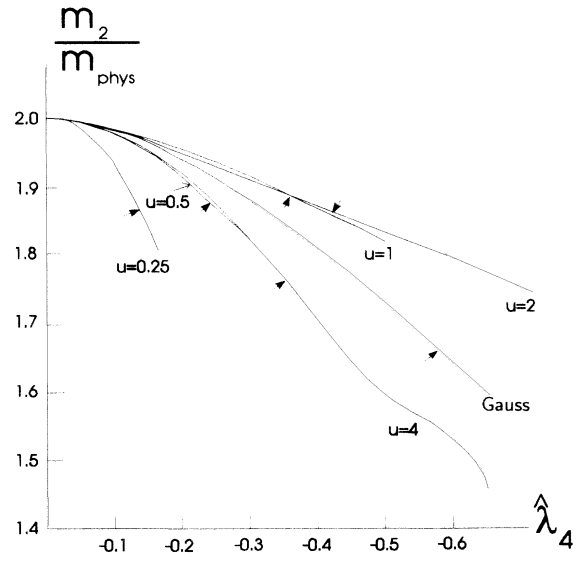

FIG. 13. Energy of two-particle bound states in units of physical mass. The points where the phase transition (to nonvanishing $\langle\Phi\rangle$) occurs are marked by boldface arrows.

TABLE I. Ratios of physical two-particle to one-particle masses for various $\hat{\lambda}_4$ (Φ^4 model).

$\hat{\lambda}_4$	m_{phys}	m_2/m_{phys}
0.1	0.9934	1.998
0.2	0.9768	1.994
0.4	0.9248	1.980
0.6	0.8590	1.963
. . .	. . .	. . .
7.0	4.838	2.048
8.0	5.477	2.032
9.0	6.079	2.021
10.0	6.650	2.013
50.0	20.754	1.975

TABLE II. Ratios t_2/t_1, $(\bar{S}_3)_2/(\bar{S}_3)_1$, etc. for dual partners of quantities $t, \bar{S}_3$ etc. given on first column. Second column, exact values; third column, CCM values.

Quantity	$\text{Ratio}_{\text{exact}}$	$\text{Ratio}_{\text{CCM}}$
t	1.0	1.008
$\bar{S}_3$	1.0	0.991
$\bar{S}_4$	1.0	0.982
m_{phys}	14.75	15.08
$\bar{F}_2$	0.2604	0.2557
$\bar{F}_3$	0.2604	0.2539
$\bar{F}_4$	0.2604	0.2526
m_2/m_{phys}	1.0	0.9997
G_1	3.840	3.835
$\bar{G}_2$	1.0	1.005
$\bar{G}_3$	1.0	0.991
$\bar{G}_4$	1.0	0.986

D. Using duality

For several cases the duality relations have been used to check the quality of the CCM approximations. As a typical example, in Table II the results are shown for the pair $((\hat{\lambda}_6)_2=10\ (\hat{\lambda}_4)_2=-10)$ and $((\hat{\lambda}_6)_1=0.045979$, $(\hat{\lambda}_4)_1=-0.341369)$ as the dual partner. They are located in regions 2 and 5 of Fig. 1. The exact duality ratios of t_2/t_1, etc., are represented in the second column of Table II for the quantities given in the first column. In the third column the CCM values are listed. It is seen that, even for the two-particle states, the agreement is excellent. For more technical details see Ref. [20].

VI. SUMMARY

This work has shown that the Φ^4 plus Φ^6 field theory in 1+1 dimensions has a rather rich structure. Some nontrivial features, such as the generation of two particle bound states, were known before. Here the CCM is quite helpful in establishing them even in post Gaussian approximations. The occurrence of first-order phase transitions in the bound state region apparently was not known before. The Gauss approximation as such cannot be used as an argument in favor of it, since it wrongly predicts first-order transitions elsewhere. Considering the complicated structure of this quantum field theory, the CCM comes out quite well. The fact that both bound states and first-order phase transitions can be obtained without any difficulty is even somewhat surprising. Second-order phase transitions are a different matter because of the infinite correlation length. Here the CCM certainly fails. One can fill the critical region around the phase transition by a suitable ansatz for the wave function and enforce the correct phase transition [26,27]. This approach has some merits. But this is not a derivation from first principles and the wave function so obtained probably is less realistic than the one obtained from the CCM. Furthermore, the CCM has passed all tests one could do at the level of the S_4 truncation. In a recent paper the S_6 approximation has been performed for the Φ^4 theory [5], giving some additional confidence in the truncation scheme of the CCM. Unfortunately, the combination of the CCM with the known renormalization procedures of realistic (non super-renormalizable) theories remains an unsolved problem [28]. In lattice formulations this problem does not exist. Thus in some recent papers the CCM has been successfully applied to lattice gauge field theories [6–8]. However, because of their complexity there is a long way to go until higher orders can be managed in lattice gauge fields.

ACKNOWLEDGMENTS

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