

Reduced canonical quantization of the induced two-dimensional gravity

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The quantization of the induced two-dimensional gravity on a compact spatial section is carried out in three different ways. In the three approaches the supermomentum constraint is solved at the classical level but they differ in the way the Hamiltonian constraint is imposed. We compare these approaches establishing an isomorphism between the resulting Hilbert spaces.

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I. INTRODUCTION

Generally covariant theories in a two-dimensional (2D) space-time collect the advantages of both being much simpler than the corresponding theories in 3+1 and 2+1 dimensions and of having a sufficiently rich structure which can shed light on the issues that appear in quantizing higher dimensional theories. Several years ago Jackiw and Teitelboim [1,2] proposed the equation

$$R + \frac{\Lambda}{2} = 0 \quad (1)$$

as the natural analogue of the vacuum Einstein equations with a cosmological term. This equation can be obtained from a local variational principle if a scalar field, playing the role of a Lagrangian multiplier, is incorporated in the theory. The above equation can also be derived from the induced 2D gravity [3]:

$$S = \frac{c}{96\pi} \int \sqrt{-g} (R \square^{-1} R + \Lambda) . \quad (2)$$

This action is nonlocal, but it is preferable to convert it into a local one by introducing an auxiliary scalar field Φ . The action can be written as

$$S = \frac{1}{2} \int \sqrt{-g} (g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + 2R \Phi + \Lambda) . \quad (3)$$

The aim of this paper is to carry out a canonical analysis of the induced 2D gravity theory (2) in three different ways (we shall restrict ourselves to the case of a compact spatial section). The first one is presented in Sec. III and it was spelled out in [4]. It is based on the covariant formulation of the canonical formalism [5]. The reduced phase space of the theory turns out to be a two-dimensional cotangent bundle and the corresponding (geometric) quantization permits us to determine the Hilbert space. In Sec. III we introduce the Arnowitt-Deser-Misner (ADM) formulation of the theory. By gauge fixing and imposing the supermomentum constraint we can reduce the theory to a finite-dimensional system. At

this point one can choose different ways to quantize the theory. One way is to look for the reduced Hamiltonian (this requires a complete gauge fixing) and then to impose the corresponding Schrödinger equation. This is our second approach and it is developed in Sec. IV. The third approach (Sec. V) is based on the (reduced) Wheeler-DeWitt equation. Throughout the paper we set up the equivalence of these approaches establishing an isomorphism between the corresponding Hilbert spaces.

II. COVARIANT PHASE-SPACE QUANTIZATION

The covariant definition of the reduced phase space [5] has been very useful in determining the phase space of a variety of field theories [6-8]. In this approach, the reduced phase space is defined as the set of all solutions of the classical theory, modulo gauge transformations (see below). The symplectic form is defined as follows.

Let us consider a field theory with fields Ψ^α and Lagrangian \mathcal{L} . If we vary the fields in the Lagrangian we get

$$\delta \mathcal{L} = \partial_\mu j^\mu + (E - L)_\alpha \delta \Psi^\alpha . \quad (4)$$

If we now regard δ as an exterior derivative operator in the space of classical solutions $\Psi^\alpha(x)$, we can, in a natural way, pull back (4) to the space of all solutions of the equations of motion, $(E - L) = 0$ and consider j^μ as a vector-valued one-form on this space.

Since δj^μ is a conserved current $\partial_\mu \delta j^\mu = 0$, it is natural to define the (pre)symplectic form ω as the corresponding conserved charge:

$$\omega = - \int_\Sigma \delta j^\mu d\sigma_\mu \quad (5)$$

(Σ is any spatial hypersurface of the space-time). With this definition, we prevent ω from depending on Σ or on the time coordinate. Because δj^μ is exact, so is ω and thus is closed.

The only property we cannot ensure for ω is nondegenerateness since ω as defined above can have a nontrivial

kernel. We define now the gauge transformations as the ones generated by the kernel of ω . If now, in the space of all solutions, we take modulus by the gauge transformations, we get a symplectic space which is called the reduced (or physical) phase space.

Let us apply the program above to the induced 2D gravity. The equations of motion obtained from (2) imply the vanishing of the stress tensor $T_{\mu\nu}$, which is given by

$$T_{\mu\nu} = -\nabla_\mu \Phi \nabla_\nu \Phi + 2\nabla_\mu \nabla_\nu \Phi + \frac{1}{2} g_{\mu\nu} \nabla^\alpha \Phi \nabla_\alpha \Phi - g_{\mu\nu} (2R + \frac{1}{2} \Lambda), \quad (6)$$

and the relation of Φ with the curvature

$$\square \Phi = R. \quad (7)$$

If we use now the gauge invariance of the metric under diffeomorphisms to bring it to a conformally flat form

$$ds^2 = -2e^\rho dx^+ dx^- \quad (8)$$

($x^+ = t+x, x^- = t-x$ are the light-cone coordinates), the equations of motion split into the relation of Φ with the curvature,

$$\square \Phi \equiv 2e^{-\rho} \partial_+ \partial_- \Phi = R \equiv -2e^{-\rho} \partial_+ \partial_- \rho, \quad (9)$$

the Liouville equation

$$0 = T_{+-} = 2\partial_+ \partial_- \rho - \frac{\Lambda}{2} e^\rho, \quad (10)$$

and the constraints

$$0 = T_{++} = -(\partial_+ \Phi)^2 + 2\partial_+^2 \Phi - 2\partial_+ \rho \partial_+ \Phi = 0, \quad (11)$$

$$0 = T_{--} = -(\partial_- \Phi)^2 + 2\partial_-^2 \Phi - 2\partial_- \rho \partial_- \Phi = 0. \quad (12)$$

The general solution for the metric field $g_{\mu\nu}$ and the dilaton field Φ can be found easily [4] and written as

$$ds^2 = -2 \frac{\partial_+ A \partial_- B}{\left[1 - \frac{\Lambda}{8} AB\right]^2} dx^+ dx^-, \quad (13)$$

$$\Phi = \ln \lambda \frac{\left[1 - \frac{\Lambda}{8} AB\right]^2}{[(d-a)A+b]^2 \left[\frac{\Lambda}{8} B\right]^2}, \quad (14)$$

and

$$\Phi = \ln \lambda \frac{\left[1 - \frac{\Lambda}{8} AB\right]^2}{\left[-b\frac{\Lambda}{8} B + (d-a)\right]^2}, \quad (15)$$

where a, b, d are such that $M = \begin{pmatrix} ab & \\ & da \end{pmatrix}$ belong to the affine subgroup of $\text{PSL}(2, R)$, i.e., $a = d^{-1}$, and $A = A(x^+)$ and $B = B(x^-)$ verify the monodromy transformation properties (we choose the length of the circle equal to unity)

$$A(y+1) = \frac{aA(y)+b}{d} \equiv M(A(y)), \quad (16)$$

$$-\frac{\Lambda}{8} B(y-1) = \frac{-d\frac{\Lambda}{8} B(y)}{b\frac{\Lambda}{8} B(y)+a} \equiv M^{-1T} \left[-\frac{\Lambda}{8} B(y) \right]. \quad (17)$$

If we choose the spatial hypersurface as the one defined by $t = t_0$, the symplectic form can be written as

$$\omega = \int_x^{x+1} [-\delta\Phi \delta(\partial_+ + \partial_-)(\Phi + \rho) + \delta(\partial_+ + \partial_-)\Phi \delta\rho]. \quad (18)$$

The projection onto the space of classical solutions takes a special form:

$$\omega = \frac{1}{2} \int_x^{x+1} (\partial_+ - \partial_-) W, \quad (19)$$

where W is given by

$$W = \frac{1}{2} \left[\delta \ln \frac{\partial_+ A}{\partial_- B} \left[\frac{\frac{\Lambda}{8} B}{(d-a)A+b} \right]^2 \right. \\ \times \delta \ln \lambda \frac{\left[1 - \frac{\Lambda}{8} AB\right]^2}{[(d-a)A+b]^2 \left[\frac{\Lambda}{8} B\right]^2} \\ \left. + \delta \ln \left[1 - \frac{\Lambda}{8} AB\right]^2 \delta \ln \frac{A}{B} \left[\frac{\frac{\Lambda}{8} B}{(d-a)A+b} \right]^2 \right. \\ \left. + \delta \ln \frac{[(d-a)A+b]^2}{\left[\frac{\Lambda}{8}\right]^2} \delta \ln \frac{(d-a)^2}{\left[\frac{\Lambda}{8} B\right]^2} \right] \quad (20)$$

for the solution (14) for Φ and by

$$W = \frac{1}{2} \left[\delta \ln \frac{\partial_+ A}{\partial_- B} \left[b\frac{\Lambda}{8} B - (d-a) \right]^2 \right. \\ \times \delta \ln \frac{\left[1 - \frac{\Lambda}{8} AB\right]^2}{\left[b\frac{\Lambda}{8} B - (d-a)\right]^2} \\ \left. + \delta \ln \left[1 - \frac{\Lambda}{8} AB\right]^2 \delta \ln \frac{A}{B} \left[b\frac{\Lambda}{8} B - (d-a) \right]^2 \right. \\ \left. + \delta \ln \left[b\frac{\Lambda}{8} \right]^2 \delta \ln \left[b\frac{\Lambda}{8} B - (d-a) \right]^2 \right] \quad (21)$$

for the solution (15) for Φ .

In any case, since ω does not depend on the coordinate x in (19), it cannot depend on either of the functions A or B . So ω will depend only on the classes of monodromy transformations to which the functions A and B belong and on the parameter λ in (14) and (15). Moreover, if we transform the functions A and B as

$$A \rightarrow h(A), \quad (22)$$

$$-\frac{\Lambda}{8}B \rightarrow h^{-1T} \left[-\frac{\Lambda}{8}B \right], \quad (23)$$

where h is a constant affine matrix acting as a Möbius transformation, we get the same solution of the equations of motion. Under the transformation (22) and (23) the monodromy parameters transform as

$$M \rightarrow hMh^{-1}. \quad (24)$$

Thus two solutions which differ on a transformation of the type (24) are the same point of the reduced phase space. The only invariant quantity under the transformation (24), and hence the only allowed monodromy dependence in ω , is the parameter a .

A direct computation from (20) and (21) leads to

$$\omega = 2 \frac{\delta\lambda}{\lambda} \frac{\delta a}{a} \quad (25)$$

for the solutions in (20) and

$$\omega = -2 \frac{\delta\lambda}{\lambda} \frac{\delta a}{a} \quad (26)$$

for the solutions in (21). From (25), (26), and previous considerations, we are tempted to assume that the reduced phase space is of the form

$$T^*(G/\text{ad}G) \cup T^*(G/\text{ad}G), \quad (27)$$

where G is the affine subgroup of $\text{PSL}(2, \mathcal{R})$. This would lead us to a Hilbert space of the form

$$\mathcal{H} = \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)}, \quad (28)$$

where

$$\mathcal{H}^{(+)} = \mathcal{H}^{(-)} = L^2(\mathcal{R}^+) \oplus \mathcal{C}^2. \quad (29)$$

The result (28), in which the Hilbert space has a continuum and a discrete sector, is in accordance with some results obtained by Becchi-Rovet-Stora-Tyutin methods [9]. However, there is also some evidence that the discrete sector cannot be endowed with a well-defined inner product. This result is achieved here by showing that, in fact, the discrete sector actually does not appear. This is a consequence of the additional symmetry $A \rightarrow -A$, $B \rightarrow -B$, $a \rightarrow a$, and $b \rightarrow -b$ that identifies the otherwise distinct parabolic ($a=1$) solutions.

Let us write the classical solutions (13)–(15) in a more explicit form. To this end we should completely fix the space-time coordinates by imposing an additional “gauge-type” condition. From (9) we observe that

$$\partial_+ \partial_- (\Phi + \rho) = 0. \quad (30)$$

Therefore we can (and we will) choose a spatially homogeneous conformal gauge by imposing

$$\Phi + \rho = \epsilon + 2pt, \quad (31)$$

where ϵ and p are constant parameters.

If $a \neq 1$, i.e., if the monodromy class is hyperbolic, the solutions take the form

$$ds^2 = -2 \frac{8}{|\Lambda|} \frac{p^2 e^{2pt}}{(1 - \text{sgn}\Lambda e^{2pt})^2} dx^+ dx^-, \quad (32)$$

$$\Phi = \ln \lambda \frac{(1 - \text{sgn}\Lambda e^{2pt})^2}{4(\sinh p/2)^2 e^{4pt}}, \quad (33)$$

$$\Phi = \ln \lambda \frac{(1 - \text{sgn}\Lambda e^{2pt})^2}{4(\sinh p/2)^2}, \quad (34)$$

where $e^p = a = d^{-1}$. On the other hand, if $a=1=d$, the unique parabolic solution takes the form

$$ds^2 = -\frac{2}{\Lambda} \frac{1}{t^2} dx^+ dx^-, \quad (35)$$

$$\Phi = \ln 4\lambda t^2. \quad (36)$$

We can easily see that (33) and (34) transform into each other when we make the replacement $a \leftrightarrow a^{-1} = d$, under which also (25) and (26) transform into each other. Moreover, when $\Lambda > 0$, (35) has the right signature and can be obtained from (32)–(34) in the limit $p \rightarrow 0$ ($a \rightarrow 1$). So we can conclude that the phase space for $\Lambda > 0$ is just

$$T^*(\mathcal{R}) \cup T^*(\mathcal{R}), \quad (37)$$

with the symplectic form

$$\omega = 2\delta(\ln \lambda)\delta p. \quad (38)$$

The two sectors in (37) correspond to whether the scalar field is expanding or contracting.

The cotangent bundle structure of the phase space makes it easy to determine the Hilbert space of the quantum theory: it will be given by the square integrable functions on the configuration space. Hence in this case we shall have

$$\mathcal{H} = L^2(\mathcal{R}, dp) \oplus L^2(\mathcal{R}, dp). \quad (39)$$

For $\Lambda < 0$ (35) is not positive definite nor can (36) be obtained from (33) and (34) as a limiting case. So that the phase space is given by

$$T^*(\mathcal{R}^+) \cup T^*(\mathcal{R}^-), \quad (40)$$

with the symplectic form

$$\omega = 2\delta \ln \lambda \delta p. \quad (41)$$

The Hilbert space should now be of the form

$$\mathcal{H} = L^2 \left[\mathcal{R}^+, \frac{dp}{p} \right] \oplus L^2 \left[\mathcal{R}^-, \frac{dp}{p} \right]. \quad (42)$$

Although it is difficult to figure out how the Hilbert spaces (42) can actually be realized, we shall see in the following sections that this prediction for the Hilbert space is consistent with other quantization approaches.

III. ADM FORMULATION

In Sec. II we saw explicitly that the classical solutions of the theory are spatially homogeneous. As has been shown in [10] for a wide class of 2D dilaton gravity models, this is so because the theory (3) has a Killing vector whose flow determines a natural coordinate system on the

cylinder where the metric and the scalar field takes a homogeneous form. The existence of the Killing vector requires that the metric equations of motion be satisfied. At this point it is important to remark that one can indeed reduce the theory to a finite number of degrees of freedom by imposing the supermomentum constraint only.

To this end let us now present the basic ingredients of the ADM formulation of the induced 2D gravity (see also [11]). First, we introduce the standard parametrization of the two-dimensional metric

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_1 N^1 & N_1 \\ N_1 & a^2 \end{pmatrix}, \quad (43)$$

where N and N^1 are the lapse and shift functions, respectively. To derive the canonical form of the action we can use the two-dimensional identity

$$\sqrt{-g} R = -2\partial_t(aK) + 2\partial_x[a(KN^1 - a^{-2}N^1)], \quad (44)$$

where K is the extrinsic curvature scalar:

$$K = \frac{1}{a^2 N} (N_{1|1} - a\dot{a}). \quad (45)$$

Removing total time derivatives we arrive at

$$S = \int d^2x (\pi_a \dot{a} + \pi_\Phi \dot{\Phi} - N\mathcal{C} - N^1 \mathcal{C}_1), \quad (46)$$

where the canonical momenta are

$$\pi_a = \frac{4}{N} (\Phi' N^1 - \dot{\Phi}), \quad (47)$$

$$\pi_\Phi = \frac{2a}{N} (\Phi' N^1 - \dot{\Phi}) + \frac{4}{N} [(aN^1)' - \dot{a}] \quad (48)$$

and the supermomentum and Hamiltonian constraints are given by

$$\mathcal{C}_1 = \Phi' \pi_\Phi - \pi'_a a, \quad (49)$$

$$\mathcal{C} = \frac{1}{16} a \pi_a^2 - \frac{1}{4} \pi_a \pi_\Phi - a \Lambda - \frac{1}{a} \Phi'^2 + 4(a\Phi)'. \quad (50)$$

Making use of the spatial diffeomorphism invariance of the theory we can fix the space coordinate and assume that

$$a = a(t). \quad (51)$$

In addition to this, and due to the time reparametrization invariance, we can also make a choice of time. All the above considerations suggest the following class of spatially homogeneous definitions of the internal time variable:

$$\mathcal{T}(\Phi, a) = \chi(t), \quad (52)$$

where χ is a generic function. This implies that

$$\Phi = \Phi(t). \quad (53)$$

Now, if we impose the supermomentum constraint we easily obtain

$$\pi_a = \pi_a(t) \quad (54)$$

and also $N = N(t)$.

The momentum π_Φ is still a function of both t and x . However, we can integrate the action in (46) with respect to the compact coordinate and the resulting expression is

$$S = \int dt (\pi_a \dot{a} + \dot{\Phi} \int dx \pi_\Phi - N\mathcal{C}), \quad (55)$$

where now

$$\mathcal{C} = \frac{1}{16} a \pi_a^2 - \frac{1}{4} \pi_a \int dx \pi_\Phi - a \Lambda. \quad (56)$$

From now on π_Φ stands for the momentum conjugated to $\Phi(t)$, i.e., $\pi_\Phi(t) = \int dx \pi_\Phi(t, x)$. Although (55) corresponds to a minisuperspace approach to the theory, it must be regarded instead as a reduced form of the theory in an appropriate gauge choice and not as a mere approximation to the theory.

For the sake of completeness we write down the equations of motion and the symplectic form obtained from (55):

$$\dot{\pi}_\Phi = 0, \quad \dot{\pi}_a = -N \left[-\frac{1}{16} \pi_a^2 + \frac{\Lambda}{2} \right], \quad (57)$$

$$\dot{\Phi} = N \pi_a, \quad \dot{a} = N \left[\frac{1}{8} a \pi_a - \frac{1}{4} \pi_\Phi \right], \quad (58)$$

$$\omega = \delta \pi_\Phi \delta \Phi + \delta \pi_a \delta a. \quad (59)$$

IV. REDUCED PHASE-SPACE QUANTIZATION IN THE CONFORMAL CHOICE OF TIME

In this section we shall develop a genuine Hamiltonian quantization of the reduced theory (55). In this approach the choice of time is done before quantization and the constraint $\mathcal{C}=0$ is solved classically (see, for instance, the review [12]). In this context, the choice of time is nothing other than a gauge-fixing condition. This gauge fixing is required to be complete in the sense that no further gauge freedom must be left, but also we must not lose information, i.e., actual solutions to the equation of motion.

Let us choose the conformal gauge

$$N = a, \quad (59)$$

which implies, according to the equations of motion (57) [see also (32)], the following implicit definition of the time variable:

$$a^2 = 4 \frac{\pi_\Phi^2}{|\Lambda|} \frac{e^{\pi_\Phi t}}{(1 - \text{sgn} \Lambda e^{\pi_\Phi t})^2}. \quad (60)$$

Solving now the constraint $\mathcal{C}=0$ for $a \pi_a$ we find the solutions

$$a \pi_a = 4 \pi_\Phi \frac{1}{1 + \text{sgn} \Lambda e^{\pi_\Phi t}} \quad (61)$$

and

$$a \pi_a = 4 \pi_\Phi \frac{1}{1 + \text{sgn} \Lambda e^{-\pi_\Phi t}}, \quad (62)$$

which remind us of the classical twofold solution for the field Φ .

Once the choice of time has been made, the effective Hamiltonian associated with it, i.e., the function that gives the proper *classical* time evolution for the remaining fields, is (minus) the conjugate momentum of time. Substituting (61) and (62) into (58) we find

$$\omega = \delta\pi_\Phi \delta\Phi - 2\pi_\Phi \frac{1}{1 + \text{sgn}\Lambda e^{\pm\pi_\Phi t}} \delta\pi_\Phi \delta t. \quad (63)$$

Since this two-form must project down to the (reduced) symplectic form of the model, the Hamiltonian flow of the vector field in the kernel of (63) should provide the remaining trajectories of motion (see, for instance, [13]). Therefore, the effective Hamiltonian should fit the expression

$$\omega = \delta\pi_\Phi \delta\Phi - \delta H \delta t. \quad (64)$$

So, we obtain

$$\begin{aligned} H &= \int^{\pi_\Phi} d\pi_\Phi 2\pi_\Phi \frac{1}{1 + \text{sgn}\Lambda e^{\pm\pi_\Phi t}} \\ &= \pi_\Phi^2 - (\pm)2 \frac{\pi_\Phi}{t} \ln(1 + \text{sgn}\Lambda e^{\pm\pi_\Phi t}) \\ &\quad - 2 \frac{1}{t^2} \text{Polylog}(2, -\text{sgn}\Lambda e^{\pm\pi_\Phi t}). \end{aligned} \quad (65)$$

The Hamiltonians in (65) can be converted into each other by means of the change $\pi_\Phi \leftrightarrow -\pi_\Phi$ or $t \leftrightarrow -t$. Thus, in this system, reversing the arrow of time is equivalent to changing the sign of the momentum π_Φ .

The quantum system will be described by the wave functions $\Psi(\pi_\Phi, t)$ that obey a time-dependent Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \Psi(\pi_\Phi, t) = H(\pi_\Phi, t) \Psi(\pi_\Phi, t). \quad (66)$$

Since the Hamiltonian functions at different times commute, the Schrödinger equation (66) can be solved immediately to give

$$\Psi(\pi_\Phi, t) = \Psi(\pi_\Phi) \exp \left[-\frac{i}{\hbar} \int^t dz H(\pi_\Phi, z) \right]. \quad (67)$$

The scalar product of two wave functions $\Psi(\pi_\Phi, t)$ and $\varphi(\pi_\Phi, t)$ will be taken as the natural one:

$$\langle \Psi | \varphi \rangle = \int d\pi_\Phi \Psi^*(\pi_\Phi, t) \varphi(\pi_\Phi, t) = \int d\pi_\Phi \Psi^*(\pi_\Phi) \varphi(\pi_\Phi). \quad (68)$$

The Hilbert space for $\Lambda > 0$ is hence given by

$$\begin{aligned} \mathcal{H} &= \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)} \\ &= L^2(R, d\pi_\Phi) \oplus L^2(R, d\pi_\Phi). \end{aligned} \quad (69)$$

The two sectors correspond to the double sign of the effective Hamiltonian (65) and represent whether the two-dimensional universe is expanding or contracting. Thus we recover the result of Sec. II. Note that the monodromy parameter $a = e^{2p}$ in Sec. II must be identified with the constant of motion e^{π_Φ} .

However, for $\Lambda < 0$ we must prevent the wave functions from taking any non-null value in $\pi_\Phi = 0$ since at this point the gauge fixing condition (60) is not well defined. So, we must impose on the wave functions the restriction of vanishing at $\pi_\Phi = 0$,

$$\Psi(\pi_\Phi = 0, t) = 0, \quad (70)$$

a restriction that is preserved by the time evolution. Therefore, for $\Lambda < 0$, the Hilbert space will be given by

$$\begin{aligned} \mathcal{H} &= \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)} \\ &= L^2(R^+, d\pi_\Phi) \oplus L^2(R^+, d\pi_\Phi), \end{aligned} \quad (71)$$

which can be identified with (42).

V. QUANTIZATION VIA THE WHEELER-DEWITT EQUATION

In this section we shall quantize the reduced theory (55) without any identification of time prior to quantization. This essentially means to impose the operator version of the classical Hamiltonian constraint, i.e., the Wheeler-DeWitt equation. To propose the Wheeler-DeWitt operator $\hat{\mathcal{C}}$ for (56) we face at once the problem of the operator ordering ambiguities and the inequality $a > 0$ of the scale variable. The second difficulty can be solved by using the affine algebra $[\hat{a}, \hat{p}_a] = i\hbar a [\hat{p}_a = -i\hbar a (\partial/\partial a)]$, instead of the Heisenberg-Weyl algebra, as the basic one to define the quantization [14]. The reason is that the operator $\hat{\pi}_a = -i\hbar (\partial/\partial a)$ fails to be self-adjoint on $L^2(R^+, da)$, whereas the affine operator $\hat{p}_a = i\hbar a (\partial/\partial a)$ is self-adjoint in $L^2(R^+, da/a)$.

Imposing that the Wheeler-DeWitt operator be self-adjoint with respect to the measure $(da/a)d\Phi$ we can write the following expression for $\hat{\mathcal{C}}$:

$$\begin{aligned} \hat{\mathcal{C}} &= \left[\frac{1}{16} \hat{a}^{\alpha+i\beta} \hat{p}_a \hat{a}^{-1-2\alpha} \hat{p}_a \hat{a}^{\alpha-i\beta} \right. \\ &\quad \left. - \frac{1}{8} (\hat{a}^{\gamma+i\sigma} \hat{p}_a \hat{a}^{-\gamma-i\sigma-1} \right. \\ &\quad \left. + \hat{a}^{\gamma+i\sigma-1} \hat{p}_a \hat{a}^{-\gamma-i\sigma}) \hat{\pi}_\Phi + \Lambda \hat{a} \right], \end{aligned} \quad (72)$$

where α, β, γ , and σ are arbitrary factor-ordering parameters. We can separate variables in the Wheeler-DeWitt equation by expanding the wave function Ψ in $\hat{\pi}_\Phi$ eigenstates:

$$\Psi = \int dq e^{iq\Phi/\hbar} \Psi_q(a). \quad (73)$$

Inserting (73) into the equation $\hat{\mathcal{C}}\Psi = 0$, where $\hat{\mathcal{C}}$ is given by (72), we obtain that the functions $\Psi_q(a)$ obey the equation

$$\left[\frac{d^2}{da^2} + \frac{1}{a} (1-2\zeta) \frac{d}{da} + \frac{1}{a^2} (\zeta^2 - \nu^2) + 4 \frac{\Lambda}{\hbar^2} \right] \Psi_q(a) = 0, \quad (74)$$

where

$$\zeta = \frac{1}{2} \left[1 + 4 \frac{iq}{\hbar} + 2i\gamma \right], \quad (75)$$

$$\nu^2 = \frac{1}{4} \left[1 - 16 \frac{q^2}{\hbar^2} - 8\gamma^2 + 4\alpha(\alpha+1) + 16 \frac{q}{\hbar}(\sigma - \gamma) \right]. \quad (76)$$

The solutions of the above equation are

$$\Psi_q(a) = a^\zeta Z_\nu \left[\frac{2|\Lambda|^{1/2} a}{\hbar} \right], \quad (77)$$

where Z_ν are ordinary (modified) Bessel functions for $\Lambda > 0$ ($\Lambda < 0$) with order ν .

In constructing the Wheeler-DeWitt operator we required Hermiticity with respect to the standard inner product

$$\langle \Psi_1 | \Psi_2 \rangle = \int \frac{da}{a} d\Phi \Psi_1^* \Psi_2. \quad (78)$$

In canonical quantum gravity it is therefore natural to propose (78) as the scalar product for the solutions of the Wheeler-DeWitt equation. This proposal for the scalar product is problematic in the sense that we are integrating over one of the configuration variables that could have been defined as the ‘‘internal’’ time variable [12]. However, we shall insist on using it, but keep in mind that (78) could be divergent and therefore require some sort of regularization.

Let us analyze now the situation for the case of a negative cosmological constant. We can expand the general solution to the Wheeler-DeWitt equation in terms of the modified Bessel and Hankel functions \mathcal{J}_ν and \mathcal{H}_ν . However, because of the exponential behavior of the functions \mathcal{J} for large x [$x \equiv 2(|\Lambda|^{1/2}/\hbar)a$], they do not lead to normalizable wave functions and therefore should then be excluded from the physical Hilbert space. The physical wave functions should be of the form

$$\Psi = \int dq a^{iq\Phi/\hbar} a^\zeta C(q) \mathcal{H}_\nu(x). \quad (79)$$

To determine the Hilbert space we should find out the range of variation of the order ν . Because of the small x behavior or the modified Hankel functions, the wave functions will be normalizable when

$$\nu^2 < \frac{1}{4}. \quad (80)$$

To obtain the maximum range of variation for ν^2 as q varies over the real line we should choose the factor ordering parameters in such a way that (76) turns out to be of the form

$$\nu^2 = \frac{1}{4} \left[1 - 16 \frac{(q - q_0)^2}{\hbar^2} \right]. \quad (81)$$

The constant shift q_0 of q in (81) can be chosen according to the classical theory. On the covariant phase space the constant of motion π_Φ is proportional to the monodromy parameter $\ln a$. Because of the absence of classical solutions for $a=1$, the constant q_0 should vanish to exclude the quantum solution $\hat{\pi}_\Phi = 0$. Therefore we are finally led to the expression

$$\nu^2 = \frac{1}{4} \left[1 - 16 \frac{q^2}{\hbar^2} \right], \quad (82)$$

which corresponds to $\alpha = \beta = \gamma = \sigma = 0$ in (76).

Now we want to determine the Hilbert space when the cosmological constant is positive. According to (77) the general solution to the Wheeler-DeWitt equation can be expanded as ($\text{Re} \nu \geq 0, \text{Im} \nu \geq 0$)

$$\Psi = \int dq a^{1/2 + 2iq/\hbar} e^{iq\Phi/\hbar} [A(q) \mathcal{J}_\nu(x) + B(q) \mathcal{N}_\nu(x)], \quad (83)$$

where $A(q)$ and $B(q)$ are arbitrary complex functions and ν is given by (82). The norm of the wave function (83) with respect to (78) is given by [$k = 2(|\Lambda|^{1/2}/\hbar)$]

$$\langle \Psi | \Psi \rangle = \frac{\hbar}{k} \int_{-\infty}^{+\infty} dq \int_0^{+\infty} dx [|A(q)|^2 |\mathcal{J}_\nu(x)|^2 + |B(q)|^2 |\mathcal{N}_\nu(x)|^2 + A^*(q) B(q) \mathcal{J}_\nu^*(x) \mathcal{N}_\nu(x) + A(q) B^*(q) \mathcal{J}_\nu(x) \mathcal{N}_\nu^*(x)]. \quad (84)$$

Because of the asymptotic behavior of the Bessel functions for large x the above integral are divergent. We can define a regularized scalar product by substituting the integration measure dx in (84) by dx/x^ϵ ($\epsilon \geq 0$). In the limit $\epsilon \rightarrow 0$ the new inner product turns out to be

$$\langle \Psi | \Psi \rangle \sim \frac{\hbar}{\epsilon \rightarrow 0} \frac{\Gamma(\epsilon)}{2\pi k} \int_{-\infty}^{+\infty} dq \{ [\cos(\pi\nu)(|A|^2 + |B|^2) + \sin(\pi\nu)(B^* A - A^* B)] \Theta(-\nu^2) + [|A|^2 + |B|^2] \Theta(\nu^2) \}, \quad (85)$$

where Θ is the step function. We can eliminate the overall divergent factor $\Gamma(\epsilon)/2^\epsilon$ to define the physical scalar product. The elementary normalizable solutions with respect to the regularized scalar product can be classified immediately. They are \mathcal{J}_ν for $\nu \in [0, \frac{1}{2}]$ or $\text{Re} \nu = 0$, and \mathcal{N}_ν for $\nu \in [0, \frac{1}{2}]$ or $\text{Re} \nu = 0$. Note that the unique normalizable solution for $\nu = \frac{1}{2}$ is \mathcal{J}_ν .

Next we would like to relate the quantization obtained via the Wheeler-DeWitt equation with the approach

developed in the preceding sections. The main point is to see how the Hilbert space $L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+)$ [or $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, depending on the sign of Λ], obtained from the covariant and reduced phase-space quantizations, can be realized in terms of the normalizable solutions of the Wheeler-DeWitt equations. Let us first consider the case of negative cosmological constant. Any normalizable solutions Ψ of the form (79) can be decomposed as $\Psi = \Psi^{(+)} + \Psi^{(-)}$, where [we have redefined the

function $C(q)$

$$\Psi^{(+)} = \left[\frac{k}{\pi\hbar} \Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu) \right]^{1/2} \times \int_0^{\infty} dq a^{1/2+2i(q/\hbar)} e^{iq(\Phi/\hbar)} C^{(+)}(q) \mathcal{H}_{|\nu|}(x), \quad (86)$$

$$\Psi^{(-)} = \left[\frac{k}{\pi\hbar} \Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu) \right]^{1/2} \times \int_{-\infty}^0 dq a^{1/2+2i(q/\hbar)} e^{iq(\Phi/\hbar)} C^{(-)}(q) \mathcal{H}_{|\nu|}(x). \quad (87)$$

$$\Psi^{(+)} = \left[\frac{k}{2\hbar} \right]^{1/2} \int_{-\infty}^{+\infty} dq a^{1/2+2i(q/\hbar)} e^{iq(\Phi/\hbar)} A^{(+)}(q)$$

$$\times (\Theta(q) \mathcal{F}_{\nu}(x) + \Theta(-q) \{ \cos[\pi \operatorname{Im}(\nu)]^{1/2} \mathcal{N}_{\nu}(x) + \sin[\pi \operatorname{Im}(\nu)] \{ \cos[\pi \operatorname{Im}(\nu)] \}^{-1/2} \mathcal{F}_{\nu}(x) \}), \quad (89)$$

$$\Psi^{(-)} = \left[\frac{k}{2\hbar} \right]^{1/2} \int_{-\infty}^{+\infty} dq a^{1/2+2i(q/\hbar)} e^{iq(\Phi/\hbar)} A^{(-)}(q)$$

$$\times (\Theta(-q) \mathcal{F}_{\nu}(x) + \Theta(q) \{ \cos[\pi \operatorname{Im}(\nu)]^{1/2} \mathcal{N}_{\nu}(x) + \sin[\pi \operatorname{Im}(\nu)] \{ \cos[\pi \operatorname{Im}(\nu)] \}^{-1/2} \mathcal{F}_{\nu}(x) \}). \quad (90)$$

The resulting expression for the scalar product turns out to be

$$\langle \Psi | \Psi \rangle = \int_{-\infty}^{+\infty} dq [|A^{(+)}(q)|^2 + |A^{(-)}(q)|^2] \quad (91)$$

and shows the equivalence between the Hilbert space derived from the Wheeler-DeWitt equation and the one obtained from the covariant and reduced phase-space approach.

VI. FINAL COMMENTS

In this paper we have constructed the quantum theory of the induced 2D gravity in three different ways: (i) covariant phase-space quantization; (ii) reduced ADM phase-space quantization; and (iii) reduced Wheeler-DeWitt equation. We have explicitly shown the coincidence of the Hilbert space of these approaches. The first approach is based on the space of classical solutions and it permits us to determine the “size” of the Hilbert space. The other approaches lead to two different realizations of the Hilbert space. The comparison between the different approaches allows us to understand the role played by the classical solutions in the quantum theory.

The scalar product takes the form

$$\langle \Psi | \Psi \rangle = \int_0^{\infty} dq |C^{(+)}(q)|^2 + \int_{-\infty}^0 dq |C^{(-)}(q)|^2, \quad (88)$$

and this shows the coincidence with the Hilbert space derived in Secs. II and IV.

When the cosmological constant is positive, the Hilbert space of normalizable solutions of the Wheeler-DeWitt equation can also be decomposed into two orthogonal subspaces. Any normalizable solution Ψ of the form (83) can be split as $\Psi = \Psi^{(+)} + \Psi^{(-)}$, where

The absence of normalized wave functions for $\nu^2 = \frac{1}{4}$, when $\Lambda < 0$, can be understood as the absence of classical parabolic solutions. Furthermore, the existence of a unique wave function for $\nu^2 = \frac{1}{4}$, when $\Lambda > 0$, find its classical counterpart in the existence of a unique classical parabolic solution (up to an additive constant for the scalar field).

Finally we want to remark that one could obtain an inequivalent quantization if both the Hamiltonian and the supermomentum constraints were imposed at the quantum level. Solving the supermomentum constraint classically prevents the emergence of the “Schwinger term” in the algebra of surface deformations generated by \mathcal{C}_1 and \mathcal{C} (the central extension involves both Hamiltonian and supermomentum constraints [2]).

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