

Fermi coordinates for weak gravitational fields

Karl-Peter Marzlin*

Fakultät für Physik der Universität Konstanz, Postfach 5560 M 674, D-78434 Konstanz, Germany

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We derive the Fermi coordinate system of an observer in arbitrary motion in an arbitrary weak gravitational field valid to all orders in the geodesic distance from the world line of the observer. In flat space-time this leads to a generalization of Rindler space for arbitrary acceleration and rotation. The general approach is applied to the special case of an observer resting with respect to the weak gravitational field of a static mass distribution. This allows us to make the correspondence between general relativity and Newtonian gravity more precise.

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I. INTRODUCTION

General relativistic studies of physical situations far away from black holes, neutron stars, or the big bang are often based on the linearization of Einstein's field equations and the assumption that the gravitational field is weak enough to allow a perturbational approach. The metric of space-time is written in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \quad (1)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ are small deviations which are treated only to *first order wherever they occur*. Although this form of the metric restricts the metric to be nearly Minkowskian, one has the remaining freedom to choose a coordinate system. For instance, to calculate gravitational waves as small perturbations in the vacuum, one usually uses the freedom of coordinate (or gauge) transformations to impose the restriction $h_{\mu,\nu} - \frac{1}{2}h_{\alpha,\mu}{}^\alpha = 0$ (harmonic gauge) on the gravitational field, thus fixing to a certain amount the coordinate system (see, e.g., Ref. [1]). As discussed by Faraoni [2], this choice of coordinates is very convenient for the study of gravitational waves, but it is difficult to describe the motion of the observer or the detector in this case.

The study of the observer is best done in his Fermi coordinate system [3,1]. This is in some sense a generalization of the notion of a frame of reference for an inertial observer in flat space to curved space-time. In these coordinates the metric tensor itself contains at least in the approximations which have been examined only quantities which are invariant under coordinate transformations such as the proper time of the observer, geodesic distances from the world line, and components of tensors with respect to a tetrad.

For gravitational waves and an observer in geodesic motion, the transformation to Fermi coordinates was done by Fortini and Gualdi [4]. Because actual observers are often accelerated and because the applications of the weak-field limit of general relativity is not restricted to gravitational waves, it is of interest to generalize their re-

sults to the case of an observer in arbitrary motion in an arbitrary weak gravitational field. It is the purpose of this paper to give this generalization. To incorporate the acceleration and the rotation of the observer, we follow the approach of Ni and Zimmermann [5].

After the performance of the general construction in Sec. II, we apply the result to an observer resting with respect to an arbitrary mass distribution. This physical situation is mostly studied in the context of Newtonian gravity, and it is often used to demonstrate that the Newtonian picture is a certain limit of general relativity. While the last claim is certainly true, it was never studied in the context of Fermi coordinates, i.e., in some kind of reference system of the observer. We will give the corresponding analysis in Sec. III. In Sec. IV, the results are reviewed and some comments are made regarding the existing work on gravitational waves. Our metric conventions are that of Ref. [1], i.e., the signature of the metric is $+2$. Greek indices run from 0 to 3, latin indices from 1 to 3. Tetrad indices are underlined>. We use units with $c=1$.

II. DERIVATION OF THE GENERAL RESULT

The observer is assumed to move (in the coordinate system y^μ) on the world line $z^\mu(\tau)$ governed by the equation

$$\ddot{z}^\mu + \Gamma^\mu_{\nu\lambda} \dot{z}^\nu \dot{z}^\lambda = a^\mu, \quad (2)$$

where τ is the proper time of the observer and a^μ is the four-acceleration. A dot denotes the derivative with respect to τ . To construct the hypersurfaces of constant τ , we consider the family of geodesics $y^\mu(\tau, s)$ with geodesic length s starting at $y^\mu(\tau, 0) = z^\mu(\tau)$ which satisfy the equation

$$y''^\mu + \Gamma^\mu_{\nu\lambda} y'^\nu y'^\lambda = 0, \quad (3)$$

where a prime denotes the derivative with respect to s . Their tangent vector on the world line has to be perpendicular to the tangent vector of the world line $\dot{z}^\mu(\tau)$; that is,

$$\dot{z}^\mu y'_\mu \Big|_{s=0} = 0. \quad (4)$$

*Electronic address: peter@spock.physik.uni-konstanz.de

Decompose $y^\mu(\tau, s)$ into a Minkowski part y_M^μ and a part y_h^μ which is of the order of $h_{\mu\nu}$. The same can be done with the world line z^μ of the observer. The Minkowski part of the solution of Eq. (3) is easily found by noting that in flat space all geodesics are straight lines. Thus,

$$y_M^\mu(\tau, s) = z_M^\mu(\tau) + s v_M^\mu(\tau), \quad (5)$$

where v_M^μ is some vector which is perpendicular to \dot{z}_M^μ in the Minkowskian sense, i.e., $v_M^\mu \dot{z}_M^\nu \eta_{\mu\nu} = 0$. Inserting this into Eq. (3) leads to

$$y_h''^\mu + \eta^{\mu\rho} (h_{\rho\nu, \lambda} - \frac{1}{2} h_{\nu, \lambda, \rho}) v_M^\nu v_M^\lambda = 0. \quad (6)$$

Up to now, each factor of $h_{\mu\nu}$ was taken at the point $y^\mu = y_M^\mu + y_h^\mu$. By making a Taylor expansion around y_M^μ , one can see that all terms including derivatives of $h_{\mu\nu}$ are of higher order in $h_{\mu\nu}$ so that one can restrict the sum to the lowest order:

$$h_{\mu\nu}(y_M^\rho + y_h^\rho) = h_{\mu\nu}(y_M^\rho) + O((h_{\mu\nu})^2). \quad (7)$$

The general solution of Eq. (6) is then given by

$$y_h^\mu(\tau, s) = C_2^\mu(\tau) + s C_1^\mu(\tau) - \int_0^s v_M^\mu h_\nu^\mu(y_M^\mu(\tau, s')) ds' + \frac{1}{2} \eta^{\mu\rho} v_M^\nu v_M^\lambda \int_0^s ds' \int_0^{s'} ds'' h_{\nu, \lambda, \rho}(y_M^\mu(\tau, s'')). \quad (8)$$

The four-vectors C_i^μ have to be determined by the condition (4) together with $y^\mu(\tau, 0) = z^\mu(\tau)$ and $y'^\mu y'_\mu|_{s=0} = 1$. To make contact with Fermi coordinates and to give the result in a convenient form, we first introduce an orthonormal tetrad $e_{\underline{a}}^\mu(\tau)$ defined in the tangent space of $z^\mu(\tau)$ which satisfies $e_{\underline{0}}^\mu = \dot{z}^\mu(\tau)$ and has the equation of motion [1]

$$\frac{D e_{\underline{a}}}{d\tau} = -\hat{\Omega} \cdot e_{\underline{a}}, \quad (9)$$

with

$$\hat{\Omega}^{\mu\nu} = a^\mu \dot{z}^\nu - a^\nu \dot{z}^\mu + \dot{z}_\alpha \omega^\alpha \epsilon^{\beta\mu\nu}, \quad (10)$$

where ω^α is the four-rotation of the tetrad. If the condition (4) is satisfied, we can write

$$y'^\mu(\tau, 0) = v_M^\mu + C_1^\mu - v_M^\nu h_\nu^\mu(\tau) = \alpha^i e_i^\mu. \quad (11)$$

The parameters α^i determine the direction of the geodesic at the world line and $h_{\mu\nu}(\tau)$ is shorthand for $h_{\mu\nu}[y^\mu(\tau, 0)]$. Taken all together, the family of geodesics perpendicular to the tangent vector of the world line is parameterized by α^i and is given by

$$y^\mu(\tau, s) = z^\mu(\tau) + s \alpha^i [e_i^\mu + h_i^\mu(\tau)] - \alpha^i \int_0^s h_i^\mu ds' + \frac{1}{2} \eta^{\mu\rho} \alpha^i \alpha^j \int_0^s ds' \int_0^{s'} ds'' h_{ij, \rho}. \quad (12)$$

Here and in the remainder of this paper we use the transformation of space-time indices to tetrad indices, e.g., $X_{\underline{a}} = X_\mu e_{\underline{a}}^\mu$. We use this notation also if the index is a derivative. It is now convenient to make a Taylor expansion of the integrals in Eq. (12). It is obvious that

$$\left[\frac{d}{ds} \right]^n h_i^\mu (y_M^\mu = z_M^\mu + s v_M^\mu) = h_{i, k_1 \dots k_n}^\mu (y_M^\mu) \alpha^{k_1} \dots \alpha^{k_n} \quad (13)$$

holds. We introduce the Fermi coordinate system x^μ in a weak gravitational field by setting $x^0 = \tau$ and $x^i = s \alpha^i$ as proposed by Manasee and Misner [3]. The transformation from the coordinate system y^μ to Fermi coordinates is then given by Eq. (12), or, after the Taylor expansion, by

$$y^\mu(x^\rho) = z^\mu(x^0) + x^i [e_i^\mu(x^0) + h_i^\mu(x^0)] - \sum_{l=0}^{\infty} \frac{1}{(l+1)!} h_{m, k_1 \dots k_l}^\mu (x^0) x^m x^{k_1} \dots x^{k_l} + \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{(l+2)!} h_{mn, \mu, k_1 \dots k_l} (x^0) x^m x^n x^{k_1} \dots x^{k_l}. \quad (14)$$

With the aid of this formula, it is straightforward to derive the metric in Fermi coordinates by using Eq. (9) and

$$g_{\alpha\beta}(x^\rho) = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta} g_{\mu\nu}(y^\lambda(x^\rho)). \quad (15)$$

Note that the factor of $g_{\mu\nu}$ on the right-hand side (RHS) must also be expanded in terms of x^i . The result is

$$g_{00} = -(1 + a_i x^i)^2 + (\omega \times \mathbf{x})^2 - \gamma_{00} - 2(\omega \times \mathbf{x})_i \gamma_{0i} - (\omega \times \mathbf{x})_i (\omega \times \mathbf{x})_j \gamma_{ij}, \quad (16)$$

$$g_{0i} = (\omega \times \mathbf{x})_i - \gamma_{0i} - (\omega \times \mathbf{x})_j \gamma_{ij},$$

$$g_{ij} = \delta_{ij} - \gamma_{ij}.$$

The expression $(\omega \times \mathbf{x})_i$ denotes $\epsilon_{ijk} \omega^j x^k$ and the coefficients $\gamma_{\mu\nu}$ are found to be

$$\gamma_{00} = \sum_{r=0}^{\infty} \frac{2}{(r+3)!} R_{0m0n, k_1 \dots k_r} x^m x^n x^{k_1} \dots x^{k_r} \times [(r+3) + 2(r+2)x^i a_i + (r+1)(x^i a_i)^2],$$

$$\gamma_{0i} = \sum_{r=0}^{\infty} \frac{2}{(r+3)!} R_{0min, k_1 \dots k_r} x^m x^n x^{k_1} \dots x^{k_r} \times [(r+2) + (r+1)x^i a_i],$$

$$\gamma_{ij} = \sum_{r=0}^{\infty} \frac{2(r+1)}{(r+3)!} R_{imjn, k_1 \dots k_r} x^m x^n x^{k_1} \dots x^{k_r}, \quad (17)$$

where the linearized Riemann tensor is given by

$$R_{\mu\nu\alpha\beta} = \frac{1}{2} [h_{\mu\beta, \nu\alpha} + h_{\nu\alpha, \mu\beta} - h_{\nu\beta, \mu\alpha} - h_{\mu\alpha, \nu\beta}]. \quad (18)$$

Equation (16) is the main result of this letter. It agrees with Eq. (49) of Fortini and Gualdi [4] for the case of a gravitational wave and with the more general result of Li and Ni [6] in absence of any rotation or acceleration. It is also in concordance with the expansion to third order in the geodesic distance s from the world line derived by Li and Ni for an accelerated and rotating observer [7].

It is worth noting that, in the absence of curvature, i.e., in flat space, all $\gamma_{\mu\nu}$ vanish and Eq. (16) becomes exact even for strong accelerations or rotations which may depend on the proper time. In this case Eq. (16) can be considered as the generalization of Rindler space-time which describes an observer with constant acceleration in two dimensions.

III. THE RESTING OBSERVER IN THE FIELD OF A STATIC MASS DISTRIBUTION

In addition to gravitational waves, it is of interest to study the Fermi coordinates of an observer who is at rest with respect to a static mass distribution. This situation is equivalent to those usually treated in Newtonian gravity. Since Fermi coordinates are in this context close to the concept of an inertial system in flat space, their analysis may give the notion of the Newtonian limit of general relativity a more accurate form.

The gravitational field is caused by a static mass density $\rho(\mathbf{y})=T_{00}(\mathbf{y})$. All other components of the energy-momentum tensor $T_{\mu\nu}$ are assumed to vanish. In the harmonic gauge the solution of the linearized Einstein equations is then given by [1]

$$h_{00}(\mathbf{y})=h_{i(i)}(\mathbf{y})=2G \int \frac{\rho(\mathbf{y}')}{|\mathbf{y}-\mathbf{y}'|} d^3y', \quad (19)$$

where G is Newton's constant and the index i in parentheses denotes that no summation is understood. Obviously, the components $h_{\mu\nu}$ are related to Newton's potential ϕ by $\phi(\mathbf{y})=-h_{00}/2$. The curvature tensor has in this coordinate system the components

$$\begin{aligned} R_{0i0m} &= \phi_{,im} , \\ R_{0lim} &= 0 , \\ R_{imjn} &= \delta_{mn}\phi_{,ij} + \delta_{ij}\phi_{,mn} - \delta_{in}\phi_{,jm} - \delta_{mj}\phi_{,in} . \end{aligned} \quad (20)$$

We take the observer to be fixed to the spatial origin of the coordinate system, $z^i=0$. By the normalization condition $\dot{z}^\mu \dot{z}_\mu = -1$ and Eq. (2), his acceleration is found to be

$$a^0=0, \quad a^i=\partial_i\phi|_0, \quad (21)$$

which is the negative of Newton's acceleration. This is reasonable since the Newtonian acceleration describes the apparent acceleration of freely falling objects (the apple) as seen by the observer and is therefore the negative of the actual acceleration of the observer. An appropriate tetrad is

$$e_{\underline{\alpha}}^\mu = \delta_{\alpha}^\mu + O(h_{\mu\nu}). \quad (22)$$

We can neglect the first-order part since we will be concerned with tensors which are already of first order in $h_{\mu\nu}$. It follows that the components of the curvature tensor and the acceleration with respect to the tetrad are the same as those taken with respect to the coordinates y^μ .

We now turn to the calculation of the quantities $\gamma_{\underline{\mu}\underline{\nu}}$. A glance at Eqs. (17) and (20) shows that γ_{0i} vanishes. The derivation of γ_{00} is also not difficult. Inserting Eq.

(20) in Eq. (17), one sees immediately that it is given by the Taylor expansion of Newton's potential without the first two terms. Assuming that we are in the range of convergence of the Taylor series, we thus find

$$\gamma_{00}=2[\phi(\mathbf{x})-\phi(0)-x^i\phi_{,i}|_0]. \quad (23)$$

Note that all acceleration-dependent contributions to $\gamma_{\underline{\mu}\underline{\nu}}$ are neglected since they are of higher order in $h_{\mu\nu}$. To get a closed expression for the remaining components γ_{ij} , we first introduce the functions

$$v_m[f](\mathbf{y}):=\frac{1}{r^{m+1}} \int_0^r r'^m f(\mathbf{y}r'/r) dr', \quad (24)$$

where f is some function of \mathbf{y} and $r=|\mathbf{y}|$ is the absolute value of the three-vector y^i in the usual Cartesian sense. One can show by induction and by using de l'Hôpital's rule that the Taylor coefficients of these functions are given by

$$v_{m,k_1,\dots,k_n}[f]|_0 = \frac{1}{n+m+1} f_{,k_1\dots k_n}|_0. \quad (25)$$

After the insertion of R_{imjn} from Eq. (20) into Eq. (17), one can see that the resulting series is just the Taylor expansion of

$$\begin{aligned} \gamma_{ij} &= 2\mathbf{x}^2\{v_1[\phi_{,ij}]-v_2[\phi_{,ij}]\} + 2\delta_{ij}\{\phi(\mathbf{x})-2v_0[\phi]+\phi(0)\} \\ &\quad - 2[x^i\{2v_1[\phi_{,j}]-v_0[\phi_{,j}]\} \\ &\quad + x^j\{2v_1[\phi_{,i}]-v_0[\phi_{,i}]\}]. \end{aligned} \quad (26)$$

It follows that the metric in the Fermi coordinates of our observer is given by

$$\begin{aligned} g_{00} &= -1 + 2[\phi(0)-\phi(\mathbf{x})], \\ g_{0i} &= 0, \\ g_{ij} &= \delta_{ij} - \gamma_{ij}. \end{aligned} \quad (27)$$

This result has a clear physical interpretation. As is well known, the largest effect for objects with slow velocities comes from the g_{00} component of the metric. This effect is identical to that of the Newtonian potential normalized to be zero on the world line of the observer, as indicated by Eq. (27). This normalization condition has its origin in the use of the proper time of the observer as the time coordinate. Another choice of time would lead to a different normalization of the potential.

Less obvious is the interpretation of the g_{ij} components which can be tested by the measurement of spatial distances. Since the expression (26) for γ_{ij} involves the gradient and the second derivatives of the potential, we can infer that distances measured in the direction of the field gradient or of the main axes of the matrix ϕ_{ij} behave differently as in other directions. This may be more obvious for the (unnatural) case when $\phi(\mathbf{y})=\phi(r)$ is a function of the distance r to the observer only. After several partial integrations in Eq. (26), one gets

$$\gamma_{ij} = 2 \left[\frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3} \right] \int_0^r r' \frac{d\phi}{dr'} dr', \quad (28)$$

where we have slightly changed the notation so that r is now identical to $|\mathbf{x}|$. We see that the radial direction is indeed preferred.

IV. CONCLUSION

In this paper we have shown that the linearized construction of Fermi coordinates can be performed for arbitrary space-time geometries, arbitrary motion of the observer, and to all orders in the geodesic distance s from the world line. In particular, we have treated the case of a resting observer in the field of a static mass distribution. This enables us to make the correspondence between general relativity and Newtonian gravity more precise.

One advantage of the knowledge of the metric to all orders in the spatial geodesic distance s from the world line may be that one can considerably enlarge the range of validity of the Fermi coordinate system. In the usual expansion to second order in s , one has, in addition to others, to satisfy the condition [3]

$$|R_{\mu\nu\rho\sigma,k}| |x^k| / |R_{\mu\nu\rho\sigma}| \ll 1 .$$

For a gravitational wave with wavelength λ , for instance, the Riemann tensor is roughly proportional to $\exp(ikx)/\lambda^2$ so that this condition gives $|x^k|/\lambda \ll 1$. For laser detectors this may be restrictive since the wave-

length is often supposed to be in the order of 300 km. If one can calculate the whole sum in Eq. (17), the limit is much larger. For growing distance from the world line and certain directions, the factors γ_{ij} can grow roughly [8] like $A(x^l/\lambda)^2$, where A is the amplitude of the gravitational wave which is usually assumed to be smaller than about 10^{-18} . It should be stressed that this equation of Ref. [2], which was first derived by Baroni *et al.* [9], includes the first order in $h_{\mu\nu}$ but all orders in the geodesic distance s from the world line. The sum of Eq. (17) was given in a closed form in the same sense as in our Eq. (27). The only restriction is now that the corrections to the Minkowski metric remain weak. This means $|\gamma_{ij}| \ll 1$ or, equivalently $|x^l| \ll \lambda/\sqrt{A}$. We see that the knowledge of the whole sum has enlarged the range of validity by a factor of $1/\sqrt{A}$, which is about 10^9 in our example. In this case there is no problem to describe contemporary laser detectors of gravitational waves in Fermi coordinates. This last conclusion is not in concordance with Ref. [2].

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