Einstein-Yang-Mills theory with a massive dilaton and axion: String-inspired regular and black hole solutions

Christopher M. O'Neill
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853
(Received 8 November 1993)

We study the classical theory of a non-Abelian gauge field [gauge group SU(2)] coupled to a massive dilaton, massive axion, and Einstein gravity. The theory is inspired by the bosonic part of the low-energy heterotic string action for a general Yang-Mills field, which we consider to leading order after compactification to $3+1$ dimensions. We impose the condition that spacetime be static and spherically symmetric, and we introduce masses via a dilaton-axion potential associated with supersymmetry breaking by gaugino condensation in the hidden sector. In the course of describing the possible non-Abelian solutions of the simplified theory, we consider in detail two candidates: a massive dilaton coupled to a purely magnetic Yang-Mills field, and a massive axion field coupled to a non-Abelian dyonic configuration, in which the electric and magnetic fields decay too rapidly to correspond to any global gauge charge. We discuss the feasibility of solutions with and without a nontrivial dilaton for the latter case, and present numerical regular and black hole solutions for the former.

PACS number(s): 04.20.Jb, 04.70.Bw, 11.25.Mj

I. INTRODUCTION

Following the early investigation of a variety of field theories coupled to Einstein gravity [1–4], it was widely believed that only the charges carried by massless gauge fields could characterize the exterior of a black hole. The notion that mass, angular momentum, and "electric" and "magnetic" charges are the only distinguishing features outside the horizon became known as the no-hair conjecture.

In light of the no-hair results and several no-go theorems for classical glueball solutions with [5] and without [6] gravity, the recent discovery of both black hole [7–9] and smooth [10] solutions of SU(2) gauge theory coupled to Einstein gravity came as quite a surprise. The fields in such solutions decay sufficiently quickly that no global gauge charges are present, and hence no imprint at spatial infinity is required for the existence of nontrivial gauge field structure. It was later shown that static solutions *with* global electric or magnetic gauge charges can only occur in the embedded Abelian sector of this theory, and that non-Abelian dyons and dyonic black holes are prohibited [11]. Further analysis has also established a sphaleron interpretation of some smooth solutions which bridge topologically distinct Yang-Mills vacua [12,13], and the inherent instability of such saddle-point field configurations may help explain the generic instability of all Einstein-Yang-Mills (EYM) solutions against collapse into Schwarzschild black holes [14–17]. Despite their lack of stability, these non-Abelian solutions still present a challenge to the no-hair results, which are not based on the issue of stability, but rather follow from the careful analysis of several theories which are fundamentally different in character from EYM theory.

In fact, the advent of these solutions has helped inspire a rethinking of the no-hair conjecture, as well as a wealth of other solutions incorporating non-Abelian structure. In [18], a distinction is drawn between primary hair, such as the structure arising from the familiar continuous gauge charges, and secondary hair [19], which exists solely as a result of primary hair sources and hence does not constitute a fundamentally new characteristic.[1] This distinction is well illustrated by the two recent approaches to Einstein-Yang-Mills-Higgs (EYMH) theory: for the "black holes inside magnetic monopoles" of [20–22], the 't Hooft–Polyakov monopole charge supports a core of secondary (triplet) Higgs hair outside the horizon, while for the case of SU(2) coupled to a Higgs doublet (the massive vector theory of the standard model less hypercharge) [23], gauge and Higgs hair exist near the horizon without global gauge or topological charges. Thus, a distinct observer in the latter case would not be able to distinguish such an object from a Schwarzschild black hole of the same mass, which motivates an alternate definition of primary hair: when the properties of a black hole are no longer completely determined *within a given theory* by the mass, angular momentum and continuous gauge charges, the additional parameters required to describe the black hole expand the space of states and give rise to primary hair [18].

Although the latter case and the original EYM black holes are examples of such primary hair, neither seem to share the stability properties of the well-known primary

[1]We do not discuss quantum hair in this paper, although this distinction applies equally well to the quantum and classical cases.

hair solutions. The black hole solutions of the spontaneously broken gauge theory appear to be unstable because of their similarity to the sphaleronlike EYM solutions of [10,7–9], and their interpretation as gravitating generalizations of the familiar SU(2) sphalerons: the weak-gravity limit of one class of solutions is equivalent to the YMH configuration of [24,25]. On the other hand, when we ignore Hawking radiation, the secondary hair solutions of [20–22] *are* stable for the same reasons flat-space monopoles are stable. Thus, the physically important condition of stability appears to be more closely tied to the stability properties of corresponding flat-space solitons (when such solutions exist) than to the classification of structure as primary or secondary hair. Another illustration of this correlation between black hole and flat-space soliton stability, the linearly stable [26] black hole solutions to Einstein-Skyrme theory [27,28], merits special attention. Although stable flat-space and gravitating Skyrmions carry nontrivial topological charge (the winding number), black holes with chiral hair are topologically *trivial* [28]. Hence [27,28] provide examples of stable black holes with primary hair which, unlike the monopole black holes of [20–22], are asymptotically indistinguishable from Schwarzschild black holes. Thus such solutions violate even the "weak" version of the no-hair conjecture, which posits the uniqueness (within a given theory) of *stable* black hole solutions for particular values of mass, angular momentum, and global gauge charges. Although a systematic approach to the existence of black hole solutions with solitonic flat-space counterparts has been formulated [29], a systematic treatment of black hole stability and its relationship to soliton stability is currently lacking. The search continues for stable black hole solutions to physically relevant theories, even several years after the first challenge to the no-hair conjecture opened the door to rich, new structure in the exterior of black holes.

On a separate front, some recent progress in black hole physics has stemmed from the generic modifications to gravity mandated by string theory, a promising candidate for a consistent theory of quantum gravity which also provides predictions that challenge general relativity well below Planck scale curvatures. In particular, the presence at low energies of the dilaton and axion, two scalars with unusual couplings which appear in the same supersymmetric multiplet as the graviton, has precipitated a host of new black hole solutions with secondary hair and interesting properties. In charged dilaton black holes [30–35], the Maxwell field acts as a source for dilaton hair, which leads to modifications of causal structure that help shed light on several puzzles peculiar to the Reissner-Nordström spacetime, as well as some mysteries of the later stages of Hawking evaporation. Because the axion couples to $F\tilde{F}\sim \mathbf{E}\cdot\mathbf{B}$, black holes with both electric and magnetic charge can support axion hair [36,37]. Another axion coupling is of the Lorentz Chern-Simons form, so that background metrics reflecting nonzero angular momentum can give rise to axion hair [38], which in turn acts as a source for dilaton hair [39], without the need for U(1) charges. The more general case of dilaton and axion hair for Kerr-Newman black holes [40,41] combines all of these scenarios.[2] There have also been recent studies of the more physically interesting case of a massive dilaton coupled to an Abelian charged black hole [44,45]. It is widely believed (but not required) that the dilaton acquires a mass when supersymmetry (SUSY) is broken: a precisely massless dilaton violates the equivalence principle [33,44], and the dilaton cannot have a mass with SUSY intact. Since SUSY is broken at low energies in any event, it seems essential that all of the above scenarios be reexamined with a massive dilaton, though the details of the SUSY-breaking mechanism and the dilaton potential are not yet well understood.

The convergence of these separate efforts in black hole physics was inevitable. A natural question to ask is whether non-Abelian gauge fields in the low-energy string context lead to black holes with primary hair, and if so, whether the sphaleron nature of previous non-Abelian solutions is modified enough by the "stringy" scalar fields to yield stable solutions. It was the desire to answer these questions, as well as to explore more general black hole solutions to what could be *the* physically relevant theory, which motivated the present work. While this paper was being completed, however, we became aware of recent work in Einstein-Yang-Mills-Dilaton (EYMD) theory [46–50] which in part grew out of solutions to the Yang-Mills-Dilaton (YMD) system [51,52]. Although these efforts involve strictly massless dilatons, some of our numerical results overlap with those of [49] in which the authors examine a special case of the more general dilaton coupling γ explored in [46,47]. We draw comparisons to these numerical results wherever appropriate, and discuss the implications of this recent body of work for the stability of our solutions.

This paper is organized as follows. In Sec. II we introduce the bosonic part of the low-energy heterotic string action, which we take to first order in the inverse string tension after compactification to $3+1$ dimensions. We specify the generic form of dilaton-axion potential which arises when SUSY is broken by gaugino condensation, and obtain a simplified string-inspired theory by requiring spherical symmetry and staticity, and by assuming that the characteristic curvature of solutions is small compared to the Planck curvature. The spherically symmetric metric and SU(2) connection *Ansatz* are then used to fully specify the theory, which is rewritten in terms of dimensionless parameters and variables before the general field equations are derived in Sec. III. In Sec. IV we classify all possible non-Abelian solutions to the theory and ignore the embedded Abelian solutions, which correspond to some of those discussed above but with dilaton and axion masses included. Our analysis indicates that only two scenarios can admit solutions: a massive dilaton coupled to a single magnetic Yang-Mills degree of freedom, and the full theory of a massive dilaton and massive axion coupled to non-Abelian electric and magnetic fields. Although the latter theory is numerically intractable we outline a possible solution scenario before extensively analyzing and presenting numerical regular and black hole solutions to the former theory in Sec. V. In

[2]For reviews of these and related developments, see [42,43].

the course of analyzing this theory, which we label EYMD theory, we also note the equivalence of scaling arguments for the existence of solutions [53] and a judicious combination of the field equations. In Sec. VI we speculate further on solutions to the most general non-Abelian scenario, briefly address the issue of stability, and offer our conclusions.

II. PRELIMINARIES

A. Low-energy string action

Our starting point is the bosonic part of the low-energy heterotic string action [54];

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2}-\frac{1}{6}e^{-2\gamma D}H_{\mu\nu\lambda}H^{\mu\nu\lambda}-\frac{1}{2}\partial_\mu D\partial^\mu D-2V(D,s)-\frac{\alpha'}{16\kappa^2}e^{-\gamma D}(2g^2F^{(i)}_{\mu\nu}F^{(i)\mu\nu}-\hat{R}^2)\right] \tag{2.1}$$

which is expressed in the Einstein frame for a metric with signature $(-+++)$. The action has been expanded to first order in the inverse string tension $\alpha'=2\kappa^2/g^2$, where g is the gauge coupling for the Yang-Mills (YM) curvature $F=dA+gA\wedge A$ and $\kappa^2=8\pi G$. $H_{\mu\nu\lambda}$ is the field strength tensor associated with the three-form

$$H=dB+\frac{\alpha'}{8\kappa}(\Omega_{3L}-g^2\Omega_{3Y})\ , \tag{2.2}$$

where B is the two-form potential in the gravitational supersymmetric multiplet. Ω_{3L} and Ω_{3Y} are the Lorentz Chern-Simons (LCS) and Yang-Mills Chern-Simons (YMCS) three-forms,

$$\Omega_{3L}=\mathrm{Tr}[\omega\wedge R-\tfrac{1}{3}\omega\wedge\omega\wedge\omega]\ , \tag{2.3}$$

$$\Omega_{3Y}=\mathrm{tr}[A\wedge F-\tfrac{1}{3}A\wedge A\wedge A]\ , \tag{2.4}$$

which arise in string theory in order to remove gauge and gravitational anomalies. Here tr and Tr denote trace over the suppressed gauge and Lorentz indices, respectively, and the normalization for Ω_{3Y} is chosen for gauge generators satisfying $\mathrm{tr}(T^iT^j)=-2\delta^{ij}$. R in Ω_{3L} is not the first curvature scalar $R_{\mu\nu}g^{\mu\nu}$ that appears in the action; it is the curvature two-form

$$R_{\mu\nu}=d\omega_{\mu\nu}+\omega^{\alpha}_{\mu}\wedge\omega_{\alpha\nu}\ , \tag{2.5}$$

where $\omega_{a\mu\nu}=(e_\mu)^b\nabla_a(e_\nu)_b$ is the spin connection for the tetrad $(e_\mu)_b$. The other gravitational scalar appearing in (2.1) is the Gauss-Bonnet (GB) curvature combination:

$$\hat{R}^2=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2\ , \tag{2.6}$$

which also helps to cancel anomalies and is second-order in derivatives of $g_{\mu\nu}$. The dilaton field D couples to other fields through exponentials with coupling strength γ and has a self-interaction V whose form will be specified below. The normalization of V has been chosen to accommodate a choice of coupling and a field rescaling: we take $\gamma=\sqrt{2}\kappa$ and define the dimensionless dilaton field $\phi\equiv\kappa D/\sqrt{2}$, so that the dilaton kinetic and potential terms assume the form

$$\frac{1}{2\kappa^2}[-2\partial_\mu\phi\partial^\mu\phi-4\kappa^2V(\phi,s)]\ . \tag{2.7}$$

It is important to note that γ is the only coupling parameter that we fix in our analysis; all others (including the κ^2 factor now appearing in front of V) will be absorbed in the definition of other dimensionless fields and parameters.

The field s appearing in V is the dimensionless pseudoscalar Kalb-Ramond axion, the only truly dynamical mode of the three-form field which we introduce via

$$H_{\mu\nu\lambda}=\frac{1}{2\kappa}e^{4\phi}\epsilon_{\mu\nu\lambda\sigma}\partial^\sigma s\ . \tag{2.8}$$

With this relation and the dual of the Bianchi identity

$$dH=\frac{\alpha'}{8\kappa}[\,\mathrm{Tr}(R\wedge R)-g^2\,\mathrm{tr}(F\wedge F)] \tag{2.9}$$

which follows from (2.2) we can express the three-form field strength as a sum of axion kinetic and topological current contributions

$$-\tfrac{1}{6}e^{-4\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}=\frac{1}{2\kappa^2}\left[-\frac{1}{2}e^{4\phi}\partial_\mu s\partial^\mu s-2\kappa\nabla_\mu({}^*H)^\mu\right]\ , \tag{2.10}$$

where $\kappa\nabla_\mu({}^*H)^\mu=(\alpha'/8)[\nabla_\mu({}^*\Omega_{3L})^\mu-g^2\nabla_\mu({}^*\Omega_{3Y})^\mu]$ is comprised of the four-divergence of two topological currents,

$$\nabla_\mu({}^*\Omega_{3L})^\mu=-\tfrac{1}{2}\nabla_\rho\,\mathrm{Tr}[(\omega_{[\lambda}R_{\mu\nu]}-\tfrac{2}{3}\omega_{[\mu}\omega_{[\nu}\omega_{\lambda]]})\epsilon^{\rho\lambda\mu\nu}]\ , \tag{2.11}$$

$$\nabla_\mu({}^*\Omega_{3Y})^\mu=-\tfrac{1}{2}\nabla_\rho\,\mathrm{tr}[(A_{[\lambda}F_{\mu\nu]}-\tfrac{2}{3}gA_{[\mu}A_{[\nu}A_{\lambda]]})\epsilon^{\rho\lambda\mu\nu}]\ , \tag{2.12}$$

which can also be expressed $-\mathrm{Tr}(R_{\mu\nu}\tilde{R}^{\mu\nu})/2$ and $-\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})/2$, respectively. With this replacement, the action becomes

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{R-\tfrac{1}{2}e^{4\phi}\partial_\mu s\partial^\mu s-2\partial_\mu\phi\partial^\mu\phi-4\kappa^2V(\phi,s)+\frac{\alpha'}{4}[\tfrac{1}{4}s\epsilon^{\rho\lambda\mu\nu}(R_{\alpha\beta\rho\lambda}R_{\mu\nu}{}^{\alpha\beta}+2g^2F^{(i)}_{\rho\lambda}F^{(i)}_{\mu\nu}) -\tfrac{1}{2}e^{-2\phi}(2g^2F^{(i)}_{\mu\nu}F^{(i)\mu\nu}-\hat{R}^2)]\right\}\ . \tag{2.13}$$

The above equation is a general expression for the low-energy heterotic string action to first order in the inverse string tension for $\gamma=\sqrt{2}\kappa$. We now briefly examine some features of this form of the action while arriving at some useful simplifications.

Note that when $V=0$, all sources for the dilaton and axion fields are $O(\alpha')$, so the fields themselves are first order in α'. Furthermore, the sources are comprised of gauge field and higher-derivative curvature combinations on an equal footing. Even for fixed s, for which the topological current terms arising from the YMCS and LCS three-forms contribute nothing to the equations of motion, we must include the GB term $\hat{R}^2$ if we are to account for the gauge field strength. From this perspective, the Reissner-Nordström solution (corresponding here to the Abelian sector of some non-Abelian field strength and fixed dilaton field) should, for example, be viewed as an $O(\alpha')$ correction to the Schwarzschild solution, subject to GB curvature corrections at the same order [41]. The gravitational effect of such curvature terms for fixed ϕ has been examined in [55–57], although in $d=4$ the GB contribution enters purely as a boundary term and can be ignored. The inclusion of the dilaton, however, introduces an $\hat{R}^2$ source term in the dilaton field equation even in $d=4$, and such scenarios have also been studied [58–62]. Several authors have neglected the GB curvature contribution in their investigations of the dilaton while consistently keeping the gauge field strength source [31–33] by considering solutions whose mass scale is large compared to the Planck mass. In some circumstances, such as the extremal limit of charged dilaton black holes [31,63,64], one can satisfy the mass scale assumption but introduce a different inconsistency: in this regime, α' is necessarily large, so the dropping of higher order terms in the effective string action is no longer justified [32]. Mindful of these concerns, we neglect the $\hat{R}^2$ term in the action by assuming that the mass of solutions is large relative to the Planck scale, but that α' is small enough for (2.13) to remain reliable.

For a dynamical axion s with or without the dilaton we again encounter higher-order curvature and gauge field source terms. For spacetimes with rotation, the LCS combination gives nontrivial contributions to the dilaton and axion equations of motion, and analytical solutions for dilaton and axion hair outside Kerr [38,39] and Kerr-Newman [40,41] black holes have been obtained. For the four-dimensional (4D) Schwarzschild spacetime, spacetimes related by a conformal transformation, or any 4D spacetime with a maximally symmetric 2D subspace, the LCS three-form either vanishes or is exact [65]. Thus for the static, spherically symmetric spacetime we investigate below, the remaining $O(\alpha' R^2)$ term in the action can be ignored.

The inclusion of the potential V introduces an additional mass scale into the problem, so that the dilaton and axion need not be $O(\alpha')$. None of the other preceding observations are qualitatively altered by its inclusion, but in choosing a potential our discussion must move from generic features of heterotic string theory [in static, spherically symmetric (3+1)-dimensional spacetime] to a more specific model. Before doing so we summarize the simplifications outlined above by rewriting the action

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\Bigg\{R-\tfrac{1}{2}e^{4\phi}\partial_\mu s\partial^\mu s-2\partial_\mu\phi\partial^\mu\phi-4\kappa^2V(\phi,s)$$
$$+\frac{\alpha'}{4}\left[g^2\epsilon^{\rho\lambda\mu\nu}\partial_\rho s\left(A_\lambda^{(i)}F_{\mu\nu}^{(i)}-\tfrac{1}{3}g\epsilon_{ijk}A_\mu^{(i)}A_\nu^{(j)}A_\lambda^{(k)}\right)-e^{-2\phi}g^2F_{\mu\nu}^{(i)}F^{(i)\mu\nu}\right]\Bigg\}\,, \tag{2.14}$$

where we have used the fact that $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a four-divergence [Eq. (2.12)] in a topologically trivial spacetime to recast the axion-gauge field coupling in a more convenient form.

B. Dilaton-axion potential

We choose a potential of the form which arises when supersymmetry is broken by gaugino condensation in the hidden sector of the theory [66]:

$$V(\phi,s)=\mu^4\frac{\alpha_\infty}{\alpha}\left\{1+\frac{(\alpha+1)^2}{(\alpha_\infty+1)^2}e^{-(\alpha-\alpha_\infty)}-2\frac{\alpha+1}{\alpha_\infty+1}e^{-(\alpha-\alpha_\infty)/2}\cos\left[\frac{3}{2b_0}s\right]\right\}\,, \tag{2.15}$$

where $\alpha\equiv 3\exp(-2\phi)/b_0$, α_∞ corresponds to the dilaton field at a potential minimum, and b_0 is determined by the one-loop β function of Q, the subgroup of the hidden sector gauge group which precipitates supersymmetry breaking. We take Q to be the entire hidden sector gauge group that arises in these scenarios, E_8', for which $b_0=90/(16\pi^2)$. The parameter μ is a scale related to the vacuum expectation values of the gaugino pair $\chi\bar{\chi}$ and the three-form H_{mnp}, where m, n, and p are indices on the internal compact manifold K only; we treat it here as a free parameter. With the axion field set to its vacuum value, $s_\infty=b_0(4\pi n)/3$ for integer n, this potential has been used by some authors to investigate inflation and cosmology in the context of superstring theories (see, e.g., [62, 67–69]). A plot of V for $s=s_\infty$ and $\phi_\infty=0$ is shown in Fig. 1(a); it has a minimum at $\phi=\phi_\infty$ and achieves a local maximum at $\phi<\phi_\infty$ before $V\to 0$ for $\phi\to-\infty$.

C. Metric

We parametrize the metric for a static, spherically symmetric spacetime as

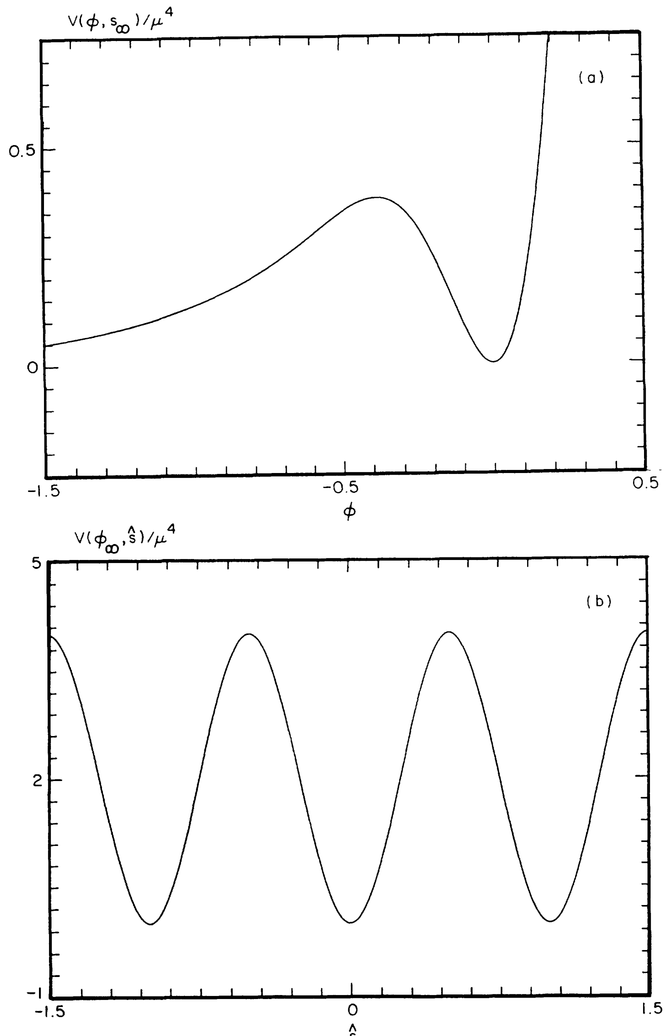

FIG. 1. The dilaton-axion potential (2.15), which is of the form used in the study of SUSY-breaking via gaugino condensation in string theory. In (a) the rescaled axion field $\hat{s}\equiv s(3/4\pi b_0)$ is fixed at one of the degenerate minima $\hat{s}=\hat{s}_\infty=n$ for integer n, and the dilaton at spatial infinity is chosen to be $\phi_\infty=0$. Solutions to Einstein-Yang–Mills-dilaton theory correspond to ϕ rolling monotonically to the minimum from the right, confined to a region where V is well approximated by leading-order $(\phi-\phi_\infty)$ behavior. In (b) the dilaton is fixed at $\phi=\phi_\infty$ and the potential assumes the form $V\propto 2[1-\cos(2\pi\hat{s})]$. A possible solution scenario for the full theory involves the axion traversing one of the maxima in the $\hat{s}$ direction and monotonically approaching an adjacent minimum as $\phi\to\phi_\infty$ from above. The fact that such non-Abelian dyonic solutions are conceivable critically depends on the presence of both the massive axion and massive dilaton fields.

$$ds^2=-T^{-2}(r)dt^2+R^2(r)dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\ , \qquad (2.16)$$

where $R(r)\equiv(1-2Gm(r)/r)^{-1/2}$, $m(r)$ is the total mass energy within the radius r, and we have set $c=1$. To describe black hole solutions we define $\delta\equiv-\ln(R/T)$ and rewrite (2.16) as

$$ds^2=-\left[1-\frac{2Gm(r)}{r}\right]e^{-2\delta}dt^2+\left[1-\frac{2Gm(r)}{r}\right]^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\ . \qquad (2.17)$$

Regularity at the origin requires $T(0)<\infty$ and $R'(0),T'(0)=0$, while regularity at the event horizon at $r=r_h$ is satisfied by $m(r_h)=r_h/2G$ and $\delta(r_h)<\infty$. We also impose the condition of asymptotic flatness, which implies $R(r),T(r)\to 1$ as $r\to\infty$ or, equivalently, $R(r)\to 1$ and $\delta(r)\to 0$. By exploiting the freedom to rescale the time coordinate, however, we can make the boundary conditions more suitable for integrating the Einstein equations. Rather than requiring $T_0\equiv T(0)$ and $\delta_0\equiv\delta(r_h)$ as initial conditions, consider rescaling t such that

$$R(0)=1,\quad T(0)=1\ ,\qquad m(r_h)=r_h/2G,\quad \delta(r_h)=0\ . \qquad (2.18)$$

The condition of asymptotic flatness then translates into $T(\infty)=1/T_0$ and $\delta(\infty)=-\delta_0$, and we can determine T_0 or δ_0 from the behavior of a solution as $r\to\infty$.

D. Gauge connection and YM curvature

In this paper we investigate the simplest choice of non-Abelian gauge group SU(2). The most general spherically symmetric SU(2) connection can be written in the form [70]

$$A=\frac{1}{g}\{a\hat{\tau}_r dt+b\hat{\tau}_r dr+[d\hat{\tau}_\theta-(1+c)\hat{\tau}_\varphi]d\theta+[(1+c)\hat{\tau}_\theta+d\hat{\tau}_\varphi]\sin\theta\, d\varphi\}\ , \qquad (2.19)$$

where g is the gauge coupling and $(\hat{\tau}_r,\hat{\tau}_\theta,\hat{\tau}_\varphi)$ is the anti-Hermitian SU(2) basis projected along the polar coordinate directions: $\hat{\tau}_r=\hat{\mathbf{r}}\cdot\tau$, etc., and the matrices satisfy $[\hat{\tau}_a,\hat{\tau}_b]=\epsilon_{abc}\hat{\tau}_c$ (we deviate from the gauge generator normalization used above for this section only). From (2.19) we note that c and d are in general dimensionless functions of r and t, while a and b have dimension $[L]^{-1}$. The connection has a residual gauge freedom under transformations of the form $U=\exp[\beta(r,t)\hat{\tau}_r]$, where $\beta(r,t)$ is an arbitrary real function, which we can use to set $b\equiv 0$ while preserving the form of the connection. Because the field equation for b becomes a potentially useful constraint equation after gauge fixing, we leave b nonzero for now and demand that the component functions of A depend on r only.

Following [70] we express the c and d degrees of freedom in the connection in complex scalar form:

$$c(r)-id(r)=f(r)\exp[i\beta(r)] , \tag{2.20}$$

which will make the non-Abelian character of the system more transparent. The YM curvature $F=dA+gA\wedge A$ for the connection (2.19) is then

$$\begin{aligned}F=&\frac{1}{g}\left[af\frac{T}{r}\hat{d}t+(b-\beta')\frac{f}{rR}\hat{d}r\right]\wedge[(\cos\beta\hat{\tau}_\theta-\sin\beta\hat{\tau}_\varphi)\hat{d}\theta+(\sin\beta\hat{\tau}_\theta+\cos\beta\hat{\tau}_\varphi)\hat{d}\varphi]\\&+\frac{1}{g}\left[\frac{f'}{rR}\hat{d}r\right]\wedge[-(\sin\beta\hat{\tau}_\theta+\cos\beta\hat{\tau}_\varphi)\hat{d}\theta+(\cos\beta\hat{\tau}_\theta-\sin\beta\hat{\tau}_\varphi)\hat{d}\varphi]\\&+\frac{1}{g}\left[-a'\frac{T}{R}\hat{\tau}_r\hat{d}t\wedge\hat{d}r+\frac{1}{r^2}(f^2-1)\hat{\tau}_r\hat{d}\theta\wedge\hat{d}\varphi\right]\end{aligned} \tag{2.21}$$

which we have expressed in a convenient orthonormal tetrad basis:

$$\begin{aligned}&\hat{d}t\equiv\frac{1}{T}dt,\quad \hat{d}r\equiv R\,dr ,\\&\hat{d}\theta\equiv r\,d\theta ,\\&\hat{d}\varphi\equiv r\sin\theta\,d\varphi .\end{aligned} \tag{2.22}$$

Note that the dependence of F and A on the gauge coupling indicates that g only enters the simplified form of the action (2.14) through the inverse string tension α'.

III. GENERAL FIELD EQUATIONS

With the choice of *Ansatz* (2.19) for the gauge connection and (2.15) for the dilaton-axion potential we are in a position to express the action in terms of the axion, dilaton, metric, and gauge degrees of freedom. Before proceeding we relax the condition $c=1$ and examine the dimensionful quantities in the action in order to cast our equations of motion in dimensionless form.

Noting that $[g]=[T][M]^{-1/2}[L]^{-3/2}$, $[\mu^2]=[T]^{-1}[M]^{1/2}[L]^{-1/2}$, and that $\kappa=\sqrt{8\pi G}/c^2$ has dimensions $[\kappa]=[T][M]^{-1/2}[L]^{-1/2}$, we observe that the parameters appearing explicitly in the action have dimensions

$$[\alpha']=[\kappa^2/g^2]=[L]^2,\quad [\kappa^2\mu^4]=[L]^{-2} , \tag{3.1}$$

and that from (2.13) all terms in $2\kappa^2\mathcal{L}$ have dimensions $[L]^{-2}$. Thus, if we factor $(\alpha'/4)^{-1}$ out of $2\kappa^2\mathcal{L}$, define $\hat{g}\equiv 2/\sqrt{\alpha'}$, and define the dimensionless quantities

$$\begin{aligned}&\hat{\mu}^2\equiv\kappa^2\mu^2/\hat{g},\quad \hat{r}\equiv\hat{g}r ,\\&\hat{a}\equiv a/\hat{g},\quad \hat{b}\equiv b/\hat{g} ,\end{aligned} \tag{3.2}$$

then $\sqrt{-g}\,(2\kappa^2\mathcal{L})$ can be written purely in terms of dimensionless fields and parameters. To do this explicitly we also define a dimensionless mass-energy function $\hat{m}$ based on the metric (2.17):

$$\frac{1}{R^2(r)}=\left[1-\frac{2Gm}{c^2r}\right]=\left[1-\frac{2\hat{m}}{\hat{r}}\right]$$

$$\text{for } \hat{m}(\hat{r})\equiv\hat{g}Gm(gr)/c^2 . \tag{3.3}$$

Expressing the curvature scalar $R_{\mu\nu}g^{\mu\nu}$ in terms of the metric functions $R(\hat{m},\hat{r})$ and $T(\hat{r})$ we find the following expression for the gravitational and matter action of our static, spherically symmetric system:

$$S_G=\frac{c^4}{\hat{g}G}\int dt\,d\hat{r}\left[\frac{1}{2}\left[\frac{1}{R^2}-1\right]\hat{r}\frac{d}{d\hat{r}}\left[\frac{R}{T}\right]\right]=\frac{c^4}{\hat{g}G}\int dt\,d\hat{r}\left[-\hat{m}\frac{d}{d\hat{r}}(e^{-\delta})\right] , \tag{3.4}$$

$$\begin{aligned}S_M=\frac{c^4}{\hat{g}G}\int dt\,d\hat{r}\Bigg[&-\left[\frac{1}{8}e^{4\phi}\frac{(\hat{r}s')^2}{R^2}+\frac{1}{2}\frac{(\hat{r}\phi')^2}{R^2}+\hat{r}^2\hat{V}(\phi,s)\right]\frac{R}{T}-s'\hat{a}(f^2-1)\\&+e^{-2\phi}\left[T^2\left[\frac{1}{2}\frac{(\hat{r}\hat{a}')^2}{R^2}+f^2\hat{a}^2\right]-\frac{1}{R^2}\left[f'^2+f^2(\hat{b}-\beta')^2\right]-\frac{1}{2\hat{r}^2}(1-f^2)^2\right]\frac{R}{T}\Bigg] ,\end{aligned} \tag{3.5}$$

where the prime denotes derivative with respect to $\hat{r}$ and $\hat{V}\equiv\kappa^2V$. The solutions to the dimensionless field equations obtained from this action give us solutions for any $\hat{g}>0$ (or $0<\alpha'<\infty$) through scaling relations

$$a_{\hat{g}}(r)=\hat{g}\hat{a}(\hat{g}r),\quad m_{\hat{g}}(r)=\frac{c^2}{\hat{g}G}\hat{m}(\hat{g}r),\quad \mathcal{F}_{\hat{g}}(r)=\mathcal{F}(\hat{g}r)\ , \tag{3.6}$$

where $\mathcal{F}$ denotes any of the functions $\{s,\phi,f,\beta,R,T,\delta\}$ and we have ignored b since it will be eliminated by gauge fixing. Hence the radial structure of solutions for a given value of $\hat{g}$ is the same as that obtained from (3.5), but it occurs at a physical radius $r=\hat{r}/\hat{g}$ with physical scales given by (3.6) and $\mu^2=\hat{g}\hat{\mu}^2/\kappa$. For notational simplicity throughout the remainder of the paper, we drop the carets on dimensionless quantities with the understanding that everything is now dimensionless unless otherwise specified.

By varying (3.5) with respect to the fields we obtain the dimensionless, static field equations

$$\frac{d}{dr}\left[\frac{r^2\phi'}{RT}\right]-\frac{\partial V(\phi,s)}{\partial\phi}r^2\frac{R}{T}-\frac{1}{2}e^{4\phi}\frac{(rs')^2}{RT}+2e^{-2\phi}\left[\frac{1}{R^2}[f'^2+f^2(\beta'-b)^2]+\frac{(1-f^2)^2}{2r^2}-T^2\left[\frac{1}{2}\frac{(ra')^2}{R^2}+f^2a^2\right]\right]\frac{R}{T}=0, \tag{3.7}$$

$$\frac{d}{dr}\left[\frac{1}{4}\frac{e^{4\phi}}{RT}r^2s'\right]-\frac{\partial V(\phi,s)}{\partial s}r^2\frac{R}{T}-\frac{d}{dr}(a[1-f^2])=0\ , \tag{3.8}$$

$$\frac{d}{dr}\left[e^{-2\phi}\frac{T}{R}r^2a'\right]-2e^{-2\phi}f^2aRT-s'(1-f^2)=0\ , \tag{3.9}$$

$$\frac{d}{dr}\left[\frac{e^{-2\phi}}{RT}f'\right]+e^{-2\phi}\left[\frac{1-f^2}{r^2}f-\frac{[\beta'-b]^2}{R^2}f+T^2a^2f\right]\frac{R}{T}-s'af=0\ , \tag{3.10}$$

$$\frac{d}{dr}\left[\frac{e^{-2\phi}}{RT}f^2[\beta'-b]\right]=0\ , \tag{3.11}$$

and the constraint equation

$$\frac{e^{-2\phi}}{RT}f^2(\beta'-b)=0\ . \tag{3.12}$$

Even without using the remaining gauge freedom to set $b\equiv0$, we find that $f^2(\beta'-b)$ disappears from the field equations. The constraint equation with gauge fixing does give us additional information, however: it implies that our gauge choice eliminates an additional degree of freedom (c and d are related by a multiplicative constant) when we insist $f\not\equiv0$, a criterion for non-Abelian solutions. In the Abelian sector $f\equiv0$, it and the β field equation indicate that β' is an arbitrary function of radius, which reflects the fact that $c\equiv d\equiv0$ in (2.20) entirely eliminates the need for the complex scalar phase. In either case we can now eliminate $f^2(\beta'-b)$ from our analysis.

To obtain the Einstein equations we can either utilize the energy-momentum tensor

$$\frac{\kappa^2}{\hat{g}^2}T_{\mu\nu}=e^{-2\phi}[2F^{(i)}_{\mu\gamma}F^{(i)\gamma}_{\ \ \nu}-\tfrac{1}{2}g_{\mu\nu}F^{(i)}_{\mu\nu}F^{(i)\mu\nu}]+\tfrac{1}{2}e^{4\phi}\partial_\mu s\partial_\nu s+2\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}[\tfrac{1}{4}e^{4\phi}\partial_\rho s\partial^\rho s+\partial_\rho\phi\partial^\rho\phi+2V(\phi,s)]\ , \tag{3.13}$$

or use the explicit dependence of (3.4) and (3.5) on some pair of independent metric functions (m and δ will do as well as R and T) to derive the gravitational equations directly. By whatever route, we find the (tt) and (rr) Einstein equations can be expressed in the form

$$m'=e^{-2\phi}\left[\left[1-\frac{2m}{r}\right]f'^2+\frac{(1-f^2)^2}{2r^2}+T^2\left[\frac{1}{2}\left[1-\frac{2m}{r}\right](ra')^2+f^2a^2\right]\right]$$
$$+\left[1-\frac{2m}{r}\right]\left[\frac{1}{8}e^{4\phi}(rs')^2+\frac{1}{2}(r\phi')^2\right]+V(\phi,s)r^2\ , \tag{3.14}$$

$$r\left[1-\frac{2m}{r}\right]\frac{T'}{T}=e^{-2\phi}\left[-\left[1-\frac{2m}{r}\right]f'^2+\frac{(1-f^2)^2}{2r^2}+T^2\left[\frac{1}{2}\left[1-\frac{2m}{r}\right](ra')^2-f^2a^2\right]\right]$$
$$-\left[1-\frac{2m}{r}\right]\left[\frac{1}{8}e^{4\phi}(rs')^2+\frac{1}{2}(r\phi')^2\right]+V(\phi,s)r^2-\frac{m}{r}\ . \tag{3.15}$$

For black hole solutions we replace the T' equation with

$$\delta'=-\frac{2}{r}\left[e^{-2\phi}\left[f'^2+\left[1-\frac{2m}{r}\right]^{-2}e^{2\delta}f^2a^2\right]+\frac{1}{8}e^{4\phi}(rs')^2+\frac{1}{2}(r\phi')^2\right]\ , \tag{3.16}$$

which indicates that δ monotonically decreases with radius. From this fact and the boundary condition $\delta(\infty)=0$, we can conclude that $\delta\geq 0$ and thus $T(r)\geq R(r)\geq 1$. Another metric relation useful for predicting solution properties is

$$\frac{d}{dr}\left[\frac{r^2}{R}\left[\frac{1}{T}\right]'\right]=\left[2e^{-2\phi}\left[\left[1-\frac{2m}{r}\right]f'^2+\frac{(1-f^2)^2}{2r^2}\right]-2V(\phi,s)r^2\right]\frac{R}{T}+2e^{-2\phi}\left[\frac{1}{2}\left[1-\frac{2m}{r}\right](ra')^2+f^2a^2\right]RT \tag{3.17}$$

which may be obtained by combining the field equations. In preparing the equations for integration we will also require the relation

$$r^2\frac{T}{R}\frac{d}{dr}\left[\frac{1}{RT}\right]=\left[2m-e^{-2\phi}\left[r\left[1-\frac{2m}{r}\right](ra')^2T^2+\frac{(1-f^2)^2}{r}\right]-2V(\phi,s)r^3\right] . \tag{3.18}$$

Finally, in classifying the possible solutions of this system of equations, it is useful to rewrite (3.9) as

$$\frac{1}{2}\frac{d}{dr}\left[e^{-2\phi}\frac{T}{R}r^2(a^2)'\right]=e^{-2\phi}\frac{T}{R}(ra')^2+2e^{-2\phi}f^2a^2RT+s'(1-f^2)a . \tag{3.19}$$

The positivity properties of the right-hand side of this equation, coupled with boundary conditions, can be applied to establish no-dyon results in the non-Abelian sector analogous to those for EYM [11] and EYMH [23] theories.

IV. TAXONOMY OF NON-ABELIAN SOLUTIONS

In previous studies of SU(2) gauge theories coupled to Einstein gravity, many authors have noted that $f\equiv 0$ in the gauge connection ansatz gives a theory with Coulombic [i.e., U(1)] magnetic and electric charges. As discussed in the introduction, a great deal of work has recently been done on the corresponding Abelian sector of the low-energy heterotic string action. Although the addition of the SUSY-breaking potential (2.15) could provide interesting new features for some Abelian sector solutions (see, for example, [44,45]), we choose not to explore them here. Our focus is the non-Abelian sector of the theory (2.14), and in this section we consider the various possibilities for static, spherically symmetric solutions with $f\not\equiv 0$.

The most general class of possible non-Abelian solutions to the field equations corresponds to $\{f,a,s,\phi\}$ behaving as nontrivial functions of radius. By studying the asymptotic behavior of the field equations we find that the condition $f\not\equiv 0$ requires $f^2(\infty)=1$ and $a(\infty)=0$, so the asymptotic values of the gauge functions for fundamentally non-Abelian solutions are identical to those of pure EYM theory [10,7–9]. This important result follows directly from (3.9) and (3.10) when we make some physically reasonable assumptions. Namely, we assume that $f(r)$ and $a(r)$ are bounded and admit expansions in powers of $(1/r)$ as $r\rightarrow\infty$, and that the massive fields ϕ and s exponentially approach finite asymptotic values fixed by V [recall that we are free to choose ϕ_∞ but the axion must assume the value $s_\infty=b_0(4\pi n)/3$ for some integer n]. The latter assumption allows us to ignore ϕ' and s' at large r, so the f and a equations resemble those of EYM theory asymptotically. With this simplification, the proof of the desired result can be divided into three parts. If we take $a(\infty)\neq 0$ in the asymptotic form of the f equation, then we find that the only nontrivial solutions are oscillatory about $f=0$ and infinite in energy. If we take $a(\infty)=0$, on the other hand, then $f(\infty)=0$ implies $f(r)\equiv 0$ to all orders in $(1/r)$, and nontrivial solutions are possible only for $f^2(\infty)=1$ and $\lim_{r\rightarrow\infty}r^2a'=0$. Finally, if we take $f^2(\infty)=1$ in the asymptotic form of the a equation and solve, the result $a(r)\approx\text{const}\times 1/r^2$ satisfies the above constraint on r^2a', and we have established a consistent set of asymptotic conditions for fundamentally non-Abelian solutions. Although a mathematically rigorous proof in the spirit of [71] for EYM theory might be possible, the analysis is complicated by the s' term in the alternate form of the a equation (3.19); as we discuss below, the fact that this term can be negative at finite r precludes the no-dyon result [$a\equiv 0$ for $f^2(\infty)=1$] which simplifies the EYM proof. Note that since the decay of a is too rapid to give a net electric charge and $f^2(\infty)=1$ corresponds to zero magnetic charge, the most general non-Abelian solutions are not dyons in the usual sense but do possess both electric and magnetic fields.

To actually obtain a general non-Abelian solution, we must also study the behavior of the field equations as $r\rightarrow 0$ for regular particlelike solutions and $r\rightarrow r_h$ for black hole solutions. Such an analysis demonstrates that the initial data $\{f''(0),a'(0),s(0),\phi(0)\}$ or $\{f(r_h),a'(r_h),s(r_h),\phi_h)\}$ are required to integrate regular or black hole solutions, respectively; in either case finite energy density restrictions force the initial value of a to vanish. Exploring a system with four integration parameters is impractical numerically, but we can demonstrate analytically that solutions are not forbidden to exist, and we can anticipate some of the properties of potential solutions. The analysis of the general case is best done, however, after the simpler possibilities are surveyed. To describe the less general non-Abelian solutions, we reduce the number of integration parameters by ansatzing each of the fields $\{f,a,s,\phi\}$ to an appropriate constant value in turn. We will find that only the $a\equiv$const and $s\equiv$const cases admit nontrivial solutions, though the $\phi\equiv$const *Ansatz* provides much insight into the characteristics of the most general non-Abelian solutions.

A. $f \equiv$const

The asymptotic behavior of the field equations restrict the acceptable values to $f^2 \equiv 1$, and the alternate version of the a equation (3.19) can then be used to establish $a \equiv 0$. The proof relies on the fact that the right-hand side of (3.19) is non-negative for $f^2 \equiv 1$, so that a^2 is strictly increasing beyond the initial radius for nontrivial solutions. The boundary conditions $a(0)=a(r_h)=a(\infty)=0$, however, require that a^2 decrease toward $a^2=0$ asymptotically. It then follows that the only solution consistent with the a field equation and the boundary conditions is the trivial solution $a \equiv 0$, and we are left with a theory of two gravitating scalar fields. Through the metric relation (3.17) and the same line of reasoning used to establish $a \equiv 0$, we can now show that only trivial solutions follow from $f \equiv$const. Since the right-hand side of that relation is never positive, nontrivial solutions obey the condition $T'>0$ beyond the initial radius. Because we also noted in the discussion of (3.16) that $T(r) \geq T(\infty)=1$, the only way to reconcile monotonically increasing T with the boundary conditions is to require $V(\phi,s) \equiv 0$: only the trivial solution $\phi \equiv \phi_\infty$, $s \equiv s_\infty$ is compatible with both the boundary conditions and the field equations.

B. $a \equiv$const or $s \equiv$const: EYMD theory

By setting either a or s to a constant value we arrive at a theory with one gauge degree of freedom coupled to a massive dilaton and Einstein gravity, which we denote Einstein-Yang-Mills-Dilaton (EYMD) theory. In the $s \equiv$const case, the potential requires $s \equiv s_\infty$, and the a equation (3.19) with $s'=0$ subsequently yields $a \equiv 0$ when we employ the reasoning introduced in the $f \equiv$const discussion. Alternately, if we take $a \equiv$const the boundary conditions require $a \equiv 0$, and the field equations then imply $s \equiv s_\infty$ for $f^2 \not\equiv 1$. Note that the right-hand side of the metric relation (3.17) again involves a $-V$ contribution, but now there are positive f-dependent terms which make possible the decrease of $T(r)$ toward $T(\infty)=1$ required for nontrivial solutions. To obtain regular or black hole solutions to EYMD theory, we must fix ϕ_∞ and choose the integration parameters $\{f''(0),\phi(0)\}$ or $\{f(r_h),\phi(r_h)\}$, respectively, such that the fields match the appropriate boundary conditions upon integration. We provide a detailed discussion of this procedure, which is often referred to as a two-parameter "shooting" procedure, and obtain numerical solutions in the next section.

It is interesting to note that in the absence of the dilaton-axion potential, solutions to EYMD theory are the most general non-Abelian solutions to the full field equations: dyonic non-Abelian solutions and nontrivial axion solutions are prohibited when $V \equiv 0$. The s field equation (3.8) in this case implies

$$s'=4\frac{RT}{r^2}e^{-4\phi}[a(1-f^2)+c_{s'}] , \tag{4.1}$$

where $c_{s'}$ is a constant which acts asymptotically like a Coulombic charge, giving a contribution to the total mass-energy proportional to $c_{s'}^2/r^2$ at large radius. Though nonzero $c_{s'}$ appears acceptable at large r, it violates regularity of the metric at the origin for regular solutions and at the event horizon for black hole solutions, since $m'(0)$ and $\delta'(r_h)$ diverge as $(rs')^2$ diverges. When (4.1) with $c_{s'}=0$ is substituted into the alternate form of the a equation (3.19), the right-hand side is non-negative and $a \equiv 0$ by the arguments outlined above. It then follows from (4.1) that $s \equiv$const, the value of which is no longer fixed by the minimum of V, and the general theory reduces to EYMD theory for a massless dilaton. A closer look at the field equations, however, reveals that $a \equiv 0$ and $s \equiv$const also follows from the weaker condition $(\partial V/\partial s) \equiv 0$: the no-dyon result is a consequence of having an axion-independent potential, rather than no potential at all. The possibility of non-Abelian solutions other than the EYMD class relies crucially on the presence of a massive (or at least self-interacting) axion field in the theory.

C. $\phi \equiv$const

At first glance, setting ϕ equal to its asymptotic value ϕ_∞ appears to reduce the number of independent matter fields from four to three. The exponential coupling of the dilaton to the gauge field and axion kinetic terms in (3.7) instead make the dilaton field equation a nontrivial constraint which must be satisfied by the remaining fields $\{f,a,s\}$. Using this constraint and some field redefinitions, it is possible to cast the resulting theory in a form which requires only two independent integration parameters. Although the constraint equation appears to make the theory numerically tractable, it is also responsible for the nonexistence of solutions: by differentiating with respect to r, we find that the constraint is incompatible with the $\phi \equiv$const field equations. Because this theory shares important features with the most general non-Abelian case, we develop the two-parameter formulation and show explicitly that solutions cannot exist.

The prospect of black hole solutions with nontrivial gauge degree of freedom a motivates us to define $\mathcal{A} \equiv aRT$: based on the expression (3.16) for δ', a must vanish at least as fast as $1/RT$ near the horizon for the a^2 term not to diverge and violate regularity requirements. It is also convenient to introduce $\mathcal{E} \equiv e^{\delta}a'$ in lieu of $\mathcal{A}'$, to which $\mathcal{E}$ is related by

$$\mathcal{E}=\left[\frac{\mathcal{A}}{R^2}\right]'-\delta'\frac{\mathcal{A}}{R^2}=\frac{1}{R^2}(\mathcal{A}'-\delta'\mathcal{A})-2\left[\frac{m}{r}\right]'\mathcal{A} . \tag{4.2}$$

In terms of these new variables, the self-interaction and kinetic contributions of a to the action interchange roles; the $1/R^2$ factor which appears with $(a')^2$ is absorbed by the definition of $\mathcal{E}$, and reappears in the f^2a^2 term:

$$T^2\left[\frac{1}{2}\left[1-\frac{2m}{r}\right](ra')^2+f^2a^2\right]=\frac{1}{2}r^2\mathcal{E}^2+\frac{f^2\mathcal{A}^2}{R^2} . \tag{4.3}$$

This combination also appears in the Einstein equations (3.14) and (3.15) and the metric relation (3.17), which are unchanged apart from the substitution of (4.3) and $\phi \equiv \phi_\infty$. The interchange of kinetic and potential roles is sensible when we consider the expression for δ',

$$\delta' = -\frac{2}{r}\left[e^{-2\phi_\infty}[f'^2 + f^2\mathcal{A}^2] + \frac{1}{8}e^{4\phi_\infty}(rs')^2\right], \quad (4.4)$$

which now involves only kinetic terms and is no longer explicitly dependent on the metric functions. Because the relation between $\mathcal{E}$ and $\mathcal{A}'$ depends explicitly on δ', it is convenient to use (4.4) in place of the Einstein equation (3.15) when considering regular as well as black hole solutions.

Through the metric derivative equation (3.18), which becomes

$$r^2 e^{\delta}\frac{d}{dr}\left[\frac{e^{-\delta}}{R^2}\right] = 2r\left[\frac{m}{r} - e^{-2\phi_\infty}\left[\frac{1}{2}r^2\mathcal{E}^2 + \frac{(1-f^2)^2}{2r^2}\right] - V(\phi_\infty, s)r^2\right] \quad (4.5)$$

for this case we can also express the axion and gauge field equations in a form independent of either T or δ:

$$\frac{r^2}{R^2}s'' + \left[\frac{2r}{R^2} + r^2 e^{\delta}\frac{d}{dr}\left[\frac{e^{-\delta}}{R^2}\right]\right]s' - 4e^{-4\phi_\infty}r^2\left[\frac{\partial V(\phi,s)}{\partial s}\right]_{\phi=\phi_\infty} - 4e^{-4\phi_\infty}\left[\mathcal{E}(1-f^2) - 2ff'\frac{\mathcal{A}}{R^2}\right] = 0, \quad (4.6)$$

$$r^2\mathcal{E}' + 2r\mathcal{E} - 2f^2\mathcal{A} - e^{2\phi_\infty}s'(1-f^2) = 0, \quad (4.7)$$

$$\frac{r^2}{R^2}f'' + \left[r^2 e^{\delta}\frac{d}{dr}\left[\frac{e^{-\delta}}{R^2}\right]\right]f' + \left[(1-f^2) + r^2\frac{\mathcal{A}}{R^2}(\mathcal{A} - e^{2\phi_\infty}s')\right]f = 0. \quad (4.8)$$

The potential for $\phi \equiv \phi_\infty$ may be written in the form

$$V(\phi_\infty, s) = 2\mu^4\left[1 - \cos\left[\frac{1}{2}\left[\frac{3}{b_0}s\right]\right]\right] = 4\mu^4\sin^2\left[\frac{1}{4}\left[\frac{3}{b_0}s\right]\right], \quad (4.9)$$

which is proportional to the derivative with respect to ϕ at ϕ_∞:

$$\left[\frac{\partial V(\phi,s)}{\partial\phi}\right]_{\phi=\phi_\infty} = A_{\phi_\infty}V(\phi_\infty, s) \equiv \left[\frac{\alpha_\infty(\alpha_\infty+1)+2}{(\alpha_\infty+1)}\right]V(\phi_\infty, s), \quad (4.10)$$

where $\alpha_\infty \equiv 3\exp(-2\phi_\infty)/b_0$ and we have introduced A_{ϕ_∞} to denote the constant of proportionality. This property of the potential allows us to write the dilaton constraint equation in the virial-like form

$$e^{-2\phi_\infty}\left[\frac{f'^2}{R^2} + \frac{(1-f^2)^2}{2r^2} - \frac{1}{2}r^2\mathcal{E}^2 - \frac{f^2\mathcal{A}^2}{R^2}\right] - \left[\frac{1}{4}e^{4\phi_\infty}\frac{(rs')^2}{R^2} + \frac{1}{2}A_{\phi_\infty}V(\phi_\infty, s)r^2\right] = 0. \quad (4.11)$$

We can use this constraint to help integrate the system comprised of the m' and δ' equations, Eqs. (4.6)–(4.8), and the $\mathcal{E}-\mathcal{A}'$ relation (4.2): it reduces the number of integration parameters to two and provides a check as we integrate the system.

There are reasons to expect nontrivial solutions to this theory. We can formally integrate (4.6) to obtain an expression analogous to (4.1),

$$\frac{1}{4}e^{4\phi_\infty}\frac{(rs')^2}{R^2} = s'(1-f^2)\frac{\mathcal{A}}{R^2} + e^{\delta}s'\left[\int_\infty^r d\tilde{r}\,\tilde{r}^2\frac{\partial V(\phi_\infty, s)}{\partial s}e^{-\delta(\tilde{r})}\right], \quad (4.12)$$

which upon substitution into the alternate form of the a equation (3.19) gives us

$$\frac{d}{dr}\left[\frac{e^{-\delta}}{R^2}e^{-2\phi_\infty}r^2\mathcal{A}\mathcal{E}\right] = 2e^{-\delta}\left[e^{-2\phi_\infty}\left[\frac{1}{2}r^2\mathcal{E}^2 + \frac{f^2\mathcal{A}^2}{R^2}\right] + \frac{1}{2}s'(1-f^2)\frac{\mathcal{A}}{R^2}\right]$$

$$= 2e^{-\delta}\left[e^{-2\phi_\infty}\left[\frac{1}{2}r^2\mathcal{E}^2 + \frac{f^2\mathcal{A}^2}{R^2}\right] + \frac{1}{8}e^{4\phi_\infty}\frac{(rs')^2}{R^2}\right] - s'\left[\int_\infty^r d\tilde{r}\,\tilde{r}^2\frac{\partial V(\phi_\infty, s)}{\partial s}e^{-\delta(\tilde{r})}\right]. \quad (4.13)$$

In the two previous cases considered we used the manifest positivity of the right-hand side of this equation to help establish $\mathcal{A}=\mathcal{E}\equiv 0$ and $s\equiv$const as the only acceptable solution. Now we must contend with the final term in (4.13), the sign and magnitude of which depend on the details of the potential. For the pure axion potential depicted in Fig. 1(b) we can imagine a solution scenario in which s increases from some initial value $4\pi n(b_0/3)<s_0<4\pi(n+1/2)(b_0/3)$, moves over the potential maximum at $s_{\max}=4\pi(n+1/2)(b_0/3)$, and comes to rest at the closest degenerate minimum $s_\infty=4\pi(n+1)(b_0/3)$. With $s'\geq 0$ along the entire solution trajectory, the sign of the final term in (4.13) is completely determined by the integral factor. For $s_{\max}<s<s_\infty$, the integrand is negative definite. For $s_0<s<s_{\max}$, the sign of $\partial V/\partial s$ changes but the integrated contribution from $r>r(s_{\max})$ initially dominates. Since the monotonically increasing factor $r^2e^{-\delta}$ in the integrand weights the $r>r(s_{\max})$ contribution more heavily, and $|\partial V/\partial s|$ is symmetric about $s=s_{\max}$, we can in fact conclude

$$-s'\left[\int_\infty^r d\tilde{r}\,\tilde{r}^2\frac{\partial V(\phi_\infty,s)}{\partial s}e^{-\delta(\tilde{r})}\right]\leq 0 \tag{4.14}$$

for the entire trajectory, and the right-hand side of (4.13) is not positive definite. In order that a be nontrivial, however, the magnitude of the contribution (4.14) must be large enough that the integral of the right-hand side of (4.13) be negative as $r\to\infty$, which from the original equation (3.19) is required for the asymptotic decrease of a^2 toward zero. From (4.12) we must therefore have

$$\frac{1}{4}e^{4\phi_\infty}\frac{(rs')^2}{R^2}-e^{\delta}s'\left[\int_\infty^r d\tilde{r}\,\tilde{r}^2\frac{\partial V(\phi_\infty,s)}{\partial s}e^{-\delta(\tilde{r})}\right]=s'(1-f^2)\frac{\mathcal{A}}{R^2}<0 \tag{4.15}$$

at least as $r\to\infty$, from which we conclude $\mathcal{A}(1-f^2)<0$ asymptotically and $\mathcal{A}<0$ if f^2 approaches $f^2(\infty)=1$ from below. We can infer some general features of f^2 by rewriting the f field equation as

$$\frac{1}{2}e^{\delta}\frac{d}{dr}\left[\frac{e^{-\delta}}{R^2}e^{-2\phi_\infty}(f^2)'\right]=e^{-2\phi_\infty}\frac{f'^2}{R^2}-e^{-2\phi_\infty}\left[(1-f^2)+r^2\frac{\mathcal{A}}{R^2}(\mathcal{A}-e^{2\phi_\infty}s')\right]\frac{f^2}{r^2}\,. \tag{4.16}$$

From this expression we find that solutions must obey the inequality

$$f^2\leq 1+r^2\frac{\mathcal{A}}{R^2}(\mathcal{A}-e^{2\phi_\infty}s')\,, \tag{4.17}$$

when $\mathcal{A}<0$ or $\mathcal{A}>e^{2\phi_\infty}s'$, since outside this region the right-hand side of (4.16) is positive definite and f^2 cannot approach unity asymptotically. For $0<\mathcal{A}<e^{2\phi_\infty}s'$, on the other hand, f^2 can exceed the bound in (4.17) but must then approach $f^2(\infty)$ monotonically, so the inequality simplifies to $f^2\leq 1$. Further implications follow from the original form of the f equation (4.8), which indicates that $ff''<0$ when $f'=0$ and (4.17) is satisfied regardless of the value of $\mathcal{A}$: solutions can exhibit oscillations about $f=0$ at finite radius, but they cannot possess turning points once the inequality bound is exceeded. Combining these observations we can construct a simple picture for the $s'\geq 0$ trajectory that is consistent with the field equations: $\mathcal{A}\leq 0$, with a single extremum and increase toward $\mathcal{A}(\infty)=0$ as the negative contribution (4.14) begins to dominate in (4.13), and $f^2\lesssim 1$, with oscillations and nodes in f possible. Had we begun with $4\pi(n+1/2)(b_0/3)<s_0<4\pi(n+1)(b_0/3)$ and demanded that s decrease monotonically toward $s_\infty=4\pi n(b_0/3)$, we would have found an identical picture with $\mathcal{A}\to-\mathcal{A}$. Thus, in the presence of a sufficiently massive axion field, non-Abelian dyonic solutions for the $\phi\equiv$const theory appear possible.

Unfortunately, the constraint (4.11) that follows from the $\phi\equiv$const *Ansatz* is not consistent with the rest of the field equations. It is not obvious from inspection whether solutions to (4.6)–(4.8), (4.2), and the Einstein equations satisfy (4.11) for some choice of the parameters $\{\mu,\phi_\infty\}$. Because this set of equations without the constraint is sufficient to determine solutions, we should be able to verify (4.11) by utilizing the entire set. When we take the derivative of (4.11) with respect to r and use the other field equations to eliminate higher derivatives, we obtain the relation

$$2e^{-2\phi_\infty}\left[\frac{[(1-f^2)^2]'}{2r^2}-\left[(f^2)'\frac{\mathcal{A}^2}{R^2}+2\mathcal{A}\mathcal{E}f^2\right]\right]-3s'\left[(1-f^2)'\frac{\mathcal{A}}{R^2}+(1-f^2)\mathcal{E}\right]+\frac{2}{r}\left[e^{-2\phi_\infty}f'^2-e^{-2\phi_\infty}f^2\mathcal{A}^2-\frac{1}{4}e^{4\phi_\infty}(rs')^2\right]\left[\frac{1}{R^2}+\frac{r}{R^2}(\ln T)'\right]-2\left[1+\frac{1}{4}A_{\phi_\infty}\right]r^2\frac{\partial V(\phi_\infty,s)}{\partial s}s'+\frac{2}{r}\left[\frac{1}{4}e^{4\phi_\infty}\frac{(rs')^2}{R^2}-A_{\phi_\infty}V(\phi_\infty,s)r^2\right]=0 \tag{4.18}$$

which should be satisfied along with (4.11) for all r. It appears that no choice of $\{\mu,\phi_\infty\}$ or further simplification with the field equations can make (4.18) an identity for nontrivial gauge and axion fields, and we conclude that the $\phi\equiv$const case admits only trivial solutions.

D. General non-Abelian solutions

The failure of the $\phi\equiv$const system to admit nontrivial solutions is closely tied to the constraint created by a nondynamic dilaton field. How does the situation change when we relax the restriction that $\phi\equiv\phi_\infty$?

Exciting the dilaton degree of freedom restores the original form of the potential (2.15), but the expansion of the potential in powers of $\hat{\phi}\equiv\phi-\phi_\infty$ can be written in the form

$$V(\hat{\phi},s)=V(\phi_\infty,s)\left[1+\frac{\alpha_\infty^2+\alpha_\infty+2}{\alpha_\infty+1}\hat{\phi}+\frac{\alpha_\infty^4+\alpha_\infty^2+6\alpha_\infty+4}{2(\alpha_\infty+1)^2}\hat{\phi}^2+O(\hat{\phi}^3)\right]+\mu^4\alpha_\infty^2\left[\frac{\alpha_\infty-1}{\alpha_\infty+1}\right]^2\hat{\phi}^2+O(\hat{\phi}^3)\ . \tag{4.19}$$

Thus $\partial V/\partial s$ deviates from the purely axionic form discussed above, but only by a $\hat{\phi}$-dependent scale factor. This scale factor breaks the symmetry of $|\partial V/\partial s|$ about $s=s_{\max}$, but it is of order unity when ϕ is near the minimum at ϕ_∞, so the reasoning behind (4.14) for the $s'\geq 0$ solution scenario remains intact when the expansion (4.19) is valid. Relaxing $\phi\equiv\phi_\infty$ also gives dynamical exponential couplings in the field equations, but neither these changes nor the scale factor seem to qualitatively alter our solution discussion in this $\hat{\phi}$ regime. The dilaton kinetic term is restored in δ',

$$\delta'=-\frac{2}{r}\left[e^{-2\phi}[f'^2+f^2\mathcal{A}^2]+\frac{1}{8}e^{4\phi}(rs')^2+\frac{1}{2}(r\phi')^2\right]\ , \tag{4.20}$$

and similarly in the expressions (3.14) and (3.15) for m' and T', while the axion and gauge field equations acquire additional terms linear in ϕ':

$$\frac{r^2}{R^2}s''+\left[\frac{2r+4r^2\phi'}{R^2}+r^2e^{\delta}\frac{d}{dr}\left[\frac{e^{-\delta}}{R^2}\right]\right]s'-4e^{-4\phi}r^2\left[\frac{\partial V(\phi,s)}{\partial s}\right]-4e^{-4\phi}\left[\mathcal{E}(1-f^2)-2ff'\frac{\mathcal{A}}{R^2}\right]=0\ , \tag{4.21}$$

$$r^2\mathcal{E}'+(2r-2r^2\phi')\mathcal{E}-2f^2\mathcal{A}-e^{2\phi}s'(1-f^2)=0\ , \tag{4.22}$$

$$\frac{r^2}{R^2}f''+\left[-\frac{2r^2\phi'}{R^2}+r^2e^{\delta}\frac{d}{dr}\left[\frac{e^{-\delta}}{R^2}\right]\right]f'+\left[(1-f^2)+r^2\frac{\mathcal{A}}{R^2}(\mathcal{A}-e^{2\phi}s')\right]f=0\ . \tag{4.23}$$

Equations (4.12), (4.13), and (4.16), which were derived from the axion and gauge field equations, retain their form, and the inequalities (4.14), (4.15), and (4.17) which followed from them are still valid if we take $\phi_\infty\to\phi(r)$.

Although the solution scenario explored in connection with $\phi\equiv\phi_\infty$ appears promising when the scale factor in (4.19) is convergent, the general non-Abelian case still requires an analysis of the dilaton equation

$$\frac{1}{2}e^{\delta}\frac{d}{dr}\left[\frac{e^{-\delta}}{R^2}r^2(\hat{\phi}^2)'\right]=\frac{(r\hat{\phi}')^2}{R^2}+\hat{\phi}\frac{\partial V(\phi,s)}{\partial\phi}r^2+\hat{\phi}\left[\frac{1}{2}e^{4\phi}\frac{(rs')^2}{R^2}\right]-2\hat{\phi}e^{-2\phi}\left[\frac{f'^2}{R^2}+\frac{(1-f^2)^2}{2r^2}-\frac{1}{2}r^2\mathcal{E}^2-\frac{f^2\mathcal{A}^2}{R^2}\right]\ , \tag{4.24}$$

which we have rewritten in a form analogous to (4.16). Since the inequalities derived above provide no information about the relative magnitudes of the terms on the right-hand side of (4.24), it is difficult to extract an inequality restricting the behavior of $\hat{\phi}$, but we can make some observations. Since $\partial V/\partial\phi<0$ for $\hat{\phi}<0$ and near the minimum, the potential contribution and possibly the gauge field contribution to the right-hand side of (4.24) are positive, which tends to drive $\hat{\phi}$ away from the vacuum. For $\hat{\phi}>0$ near the minimum, these signs reverse and make a monotonic decrease of $\hat{\phi}$ toward $\hat{\phi}=0$ more likely. Although our preceding analysis does not require such monotonic behavior for the dilaton, it is certainly consistent with the solution scenario and provides a potentially viable alternative to the $\phi\equiv\phi_\infty$ *Ansatz.*

The complexity of the full string-inspired theory is well reflected in our inability to proceed any further analytically. It appears that the question of existence of non-Abelian solutions with nontrivial dilaton and axion fields can only be settled by further numerical study. Although we have not actually obtained numerical solutions in the general case, we have considered a strategy for simplifying this four-parameter problem. In past (two-parameter) shooting problems, we have found the method of "shooting to a fitting point" a convincing way to confirm our numerical results. In this method, shooting parameters at both fixed points ($r=0$; r_h and $r=\infty$) are chosen so that the field equations can be integrated toward a common midpoint, at which the two trajectories join smoothly if the parameters correspond to a solution. Though this procedure requires more than double the original number of shooting parameters, the deviation of the trajectories at the midpoint can be used to choose new shooting parameters via a multidimensional Newton-

Raphson algorithm. If the initial shooting parameters are reasonably close to a solution, this procedure is moderately successful at converging on the solution, but it is difficult to match the trajectories with an error comparable to the global tolerance of the integration routine. Once the neighborhood of a solution is determined, this appears to be a more promising approach to finding the solution than the method employed in the next section, at least for three or more shooting parameters. We hope to utilize this procedure in the future to find solutions to the full string-inspired theory.

By closely examining some simplifying *Ansätze* we have been able to narrow the possible solution classes to two: a massive dilaton coupled to a single magnetic Yang-Mills degree of freedom, which we denote EYMD theory, and the gauge field coupled to both massive dilaton and massive axion fields. Although we could only speculate about solution scenarios for the latter theory, in the next section we present and analyze numerical solutions to the former.

V. EYMD THEORY: REGULAR AND BLACK HOLE SOLUTIONS

A. EYMD equations

As we observed above, the *Ansätze* $s'\equiv 0$ or $a'\equiv 0$ in the non-Abelian sector give $a\equiv 0$ and $s\equiv s_\infty$ for both regular and black hole solutions. To obtain numerical solutions to the resulting EYMD system, it is instructive to reexpress the dilaton as $\phi(r)\equiv h(r)/r+\phi_\infty$, where h/r, which we also denote $\hat{\phi}$, is the deviation of ϕ from its vacuum value. With this change and the simplifications $a\equiv 0$ and $s=s_\infty=b_0(4\pi n)/3$, the remaining gauge field equation becomes

$$\frac{d}{dr}\left[\frac{f'}{RT}\right]+\frac{f(1-f^2)}{r^2}\frac{R}{T}-2\frac{f'(h/r)'}{RT}=0\;, \tag{5.1}$$

and the dilaton equation assumes the form

$$\frac{d}{dr}\left[\frac{r^2(h/r)'}{RT}\right]-V'(\phi,s_\infty)r^2\frac{R}{T}+2e^{-2(h/r+\phi_\infty)}\left[\frac{f'^2}{R^2}+\frac{[1-f^2]^2}{2r^2}\right]\frac{R}{T}=0\;, \tag{5.2}$$

where $V'\equiv\partial V/\partial\phi|_{s=s_\infty}$. The Einstein equations (3.14) and (3.15) for EYMD theory are

$$m'=e^{-2(h/r+\phi_\infty)}\left[\left[1-\frac{2m}{r}\right]f'^2+\frac{1}{2}\frac{(1-f^2)^2}{r^2}\right]+\frac{1}{2}\left[1-\frac{2m}{r}\right]\left[h'-\frac{h}{r}\right]^2+V(\phi,s_\infty)r^2\;, \tag{5.3}$$

$$r\left[1-\frac{2m}{r}\right]\frac{T'}{T}=e^{-2(h/r+\phi_\infty)}\left[-\left[1-\frac{2m}{r}\right]f'^2+\frac{1}{2}\frac{(1-f^2)^2}{r^2}\right]-\frac{m}{r}-\frac{1}{2}\left[1-\frac{2m}{r}\right]\left[h'-\frac{h}{r}\right]^2+V(\phi,s_\infty)r^2\;, \tag{5.4}$$

with the auxiliary equation

$$\delta'=-\frac{2}{r}\left[e^{-2(h/r+\phi_\infty)}f'^2+\frac{1}{2}\left[h'-\frac{h}{r}\right]^2\right] \tag{5.5}$$

replacing the T' equation for black hole solutions. To simplify the integration of the equations of motion, we use the metric derivative relation (3.18) to express the gauge field and dilaton equations in a form independent of T or δ:

$$r^2\left[1-\frac{2m}{r}\right]f''+\left[2m-e^{-2(h/r+\phi_\infty)}\frac{(1-f^2)^2}{r}-2V(\phi,s_\infty)r^3-2r\left[1-\frac{2m}{r}\right]\left[h'-\frac{h}{r}\right]\right]f'+(1-f^2)f=0\;, \tag{5.6}$$

$$r^2\left[1-\frac{2m}{r}\right]h''+\left[2m-e^{-2(h/r+\phi_\infty)}\frac{(1-f^2)^2}{r}-2V(\phi,s_\infty)r^3\right]\left[h'-\frac{h}{r}\right]-V'(\phi,s_\infty)r^3$$
$$+2e^{-2(h/r+\phi_\infty)}\left[r\left[1-\frac{2m}{r}\right]f'^2+\frac{(1-f^2)^2}{2r}\right]=0\;. \tag{5.7}$$

Although it is the h-dependent form of the field equations we choose to integrate we pass freely between ϕ, $\hat{\phi}$, and h in our analysis when describing the dilaton.

B. Analytical features and boundary conditions

We can anticipate the general features of the solutions and boundary conditions from the field equations. The gauge field equation may be rewritten

$$\frac{1}{2}\frac{d}{dr}\left[\frac{e^{-2\phi}(f^2)'}{RT}\right]=e^{-2\phi}\left[\frac{(f')^2}{RT}+\frac{R}{r^2T}(f^2-1)f^2\right]\;, \tag{5.8}$$

Since the right-hand side of this equation is manifestly positive for $f^2>1$, the only nontrivial solutions having finite f^2 must satisfy $f^2 \leq 1$. By expanding the left-hand side of the equation we can see that solutions satisfy $ff''<0$ for $f'=0$ and $f^2<1$, which is characteristic of oscillations about $f=0$. As in past studies with non-Abelian gauge fields coupled to gravity (see, e.g., [7–10]) we expect that solutions will be classifiable by the number of nodes which occur at f oscillates.

From the f field equation we note that the boundary conditions for the gauge field include $|f|=1$ and $f=0$ as $r\to\infty$. The asymptotic behavior of the field equations reveal that the latter condition implies $f\equiv 0$, which for black hole solutions corresponds to an Abelian gauge field carrying purely magnetic charge. Without the dilaton, this case just reduces to the Reissner-Nordström solution, while with the dilaton we recover the type of theory studied in [44,45]. The $f\equiv 0$ case for regular solutions is forbidden by boundary conditions at the origin: regularity of the metric only allows the possibility $|f(0)|=1$. If we attempt to set $|f|\equiv 1$, corresponding to the theory of a massive dilaton coupled to Einstein gravity, we find that both regular and black hole solutions are forbidden: according to (3.17), T must be monotonically increasing, but boundary conditions provide the incompatible restrictions $T(0)>T(\infty)$ and $T(r_h)>T(\infty)$. We conclude that for fundamentally non-Abelian solutions to EYMD theory, we must have $|f(\infty)|=|f(0)|=1$ for regular solutions, $|f(\infty)|=1$ and $f(r_h)$ unspecified for black hole solutions, and $f^2(r)\leq 1$ for all solutions.

To better understand the expected behavior of ϕ we rewrite the dilaton equation

$$\frac{1}{2}\frac{d}{dr}\left[\frac{r^2(\hat{\phi}^2)'}{RT}\right]=\frac{(r\hat{\phi}')^2}{RT}-\hat{\phi}\left[2e^{-2\phi}\left[\frac{f'^2}{R^2}+\frac{(1-f^2)^2}{2r^2}\right]-V'(\phi,s_\infty)r^2\right]\frac{R}{T}\,, \quad (5.9)$$

where $\hat{\phi}\equiv(h/r)$ is the deviation of ϕ from the asymptotic value fixed by the minimum of V. For $\hat{\phi}<0$ near the minimum, $V'<0$ and the right-hand side of (5.9) is positive definite, which implies that $\hat{\phi}^2$ is strictly increasing. Although V' changes sign before it vanishes as $\hat{\phi}\to-\infty$ (cf. Fig. 1), the gauge field contribution to (5.9) is exponentially amplified in the same limit, so the details of the potential should not alter the conclusion that the region $\hat{\phi}<0$ is forbidden. We can also predict that solutions exhibit a monotonic decrease from $\hat{\phi}>0$ to the vacuum $\hat{\phi}=0$. This is unambiguous in the $\mu^2=0$ case, for which (5.2) indicates $\hat{\phi}'$ is strictly negative, but nonzero V' introduces the possibility of turning points. Carrying out the derivatives on the left-hand side of (5.9) demonstrates that for $\hat{\phi}'=0$, $\hat{\phi}''<0$ if the gauge field contribution dominates V', while $\hat{\phi}''>0$ if V' is dominant. It follows that $\hat{\phi}'=0$ can only occur at a plateau in the former case, but for V' dominant a turning point in $\hat{\phi}$ is possible. Since the gauge field contribution is exponentially suppressed and V' grows even larger as $\hat{\phi}$ increases, it appears that a turning point is a precursor to diverging $\hat{\phi}$ and cannot be a solution feature. To summarize, we expect all solutions for which $\phi(\infty)$ is finite to be characterized by $\hat{\phi}$ rolling monotonically to zero from above, and $|f|$ approaching unity after crossing at least once through $f=0$.

In addition to providing a sketch of the behavior of the fields for regular and black hole solutions, the preceding analysis provides insight into the behavior of the fields when we are close to a solution. To qualify what we mean by "close," we introduce some details of the approach to solving this system numerically. For regular solutions, finite energy density $(\kappa^2/\hat{g})T_{00}$ and regularity of the metric at the origin give the following behavior as $r\to 0$:

$$f(r)=-1+\tfrac{1}{2}f''(0)r^2+O(r^4)\,, \quad (5.10)$$

$$h(r)=h'(0)r+O(r^3)\,, \quad (5.11)$$

$$2m(r)=O(r^3)\,, \quad (5.12)$$

$$\ln T(r)=O(r^2)\,, \quad (5.13)$$

where $f''(0)$ and $h'(0)=\hat{\phi}(0)$ are parameters we must adjust to "shoot" a solution to match the asymptotic conditions $|f(\infty)|=1$ and $\hat{\phi}(\infty)=0$, and we have taken $f(0)=-1$ and used the rescaled initial condition $T(0)=1$ introduced in (2.18). All the terms not shown explicitly in (5.10)–(5.13) depend only on the two shooting parameters and the free parameters μ^2 and ϕ_∞ associated with V. Similarly, for black hole solutions we can use the metric condition $1/R^2(r_h)=0$ for $m(r_h)=r_h/2$ to expand near the horizon:

$$f(r)=f(r_h)+f'(r_h)(r-r_h)+O((r-r_h)^2)\,, \quad (5.14)$$

$$h(r)=h(r_h)+h'(r_h)(r-r_h)+O((r-r_h)^2)\,, \quad (5.15)$$

$$m(r)=r_h/2+m'(r_h)(r-r_h)+O((r-r_h)^2)\,, \quad (5.16)$$

$$\delta(r)=0+\delta'(r_h)(r-r_h)+O((r-r_h)^2)\,, \quad (5.17)$$

where $f(r_h)$ and $h(r_h)$ are now the shooting parameters, r_h is a free parameter, and the field equations (5.3)–(5.7) give the derivatives $f'(r_h)$, $h'(r_h)$, $m'(r_h)$, and $\delta'(r_h)$ as functions of these parameters.

On the basis of past investigations [7–10] we expect solutions (for a particular choice of free parameters) to exist only for discrete values of the shooting parameters. Using the above analysis we can anticipate the asymptotic behavior of the fields for a small neighborhood surrounding those discrete values in shooting parameter space. Since f exhibits oscillatory behavior for $f^2<0$, and $f^2>0$ leads to diverging f, we should look for a parameter range in which $|f|$ approaches unity at large r and either exhibits a turning point or exceeds $|f|=1$ and diverges. If for such a parameter range $\hat{\phi}$ approaches 0 and (for μ^2 nonzero) then undergoes a turning point and diverges, or becomes negative and diverges, then our neighborhood should contain a point which gives the correct asymptotic behavior. We explain the shooting procedure in more detail below.

Although the vacuum values $|f(\infty)|=1$ and $\hat{\phi}(\infty)=0$ are shared by all solutions, the behavior of the field equations or $r\to\infty$ provides interesting distinctions between massless and massive dilatons. For massive dilatons, the leading-order asymptotic expansions of the fields and metric functions are

$$f(r) \sim \pm \left[-1 + \frac{c}{r} \right] , \tag{5.18}$$

$$h(r) \sim a e^{-m_\phi r} , \tag{5.19}$$

$$m(r) \sim M - \frac{c^2 e^{-2\phi_\infty}}{r^3} , \tag{5.20}$$

$$\ln T(r) \sim \ln \left[\frac{1}{T_0} \right] + \frac{M}{r} , \tag{5.21}$$

$$\delta(r) \sim -\delta_0 + \frac{1}{2} \frac{c^2 e^{-2\phi_\infty}}{r^4} , \tag{5.22}$$

where c and a are positive constants, T_0 and δ_0 are the rescaled metric constants introduced in the discussion of (2.18), and the $+(-)$ sign in (5.18) corresponds to an even(odd) number of nodes in the function $f(r)$. Note that the presence of m_ϕ, defined by

$$m_\phi^2 \equiv \left[\frac{\partial^2 V}{\partial \phi^2} \right]_{\phi_\infty} = 2\alpha_\infty^2 \left[\frac{\alpha_\infty - 1}{\alpha_\infty + 1} \right]^2 \mu^4 , \tag{5.23}$$

forces the dilaton to approach its asymptotic value exponentially as we expect. For $\alpha_\infty \neq 1$ we find it convenient to use m_ϕ in place of μ^2 as a free parameter. In the massless case $\mu^2 = 0$, the expansions are

$$f(r) \sim \pm \left[-1 + \frac{c}{r} + \frac{1}{8} \frac{c(6M + 2h_\infty - c)}{r^2} \right] , \tag{5.24}$$

$$h(r) \sim h_\infty + \frac{M h_\infty}{r} + \frac{1}{6} \frac{h_\infty (8M^2 - h_\infty^2)}{r^2} , \tag{5.25}$$

$$m(r) \sim M - \frac{1}{2} \frac{h_\infty^2}{r} - \frac{1}{2} \frac{M h_\infty^2}{r^2} , \tag{5.26}$$

$$\ln T(r) \sim \ln \left[\frac{1}{T_0} \right] + \frac{M}{r} + \frac{M^2}{r^2} , \tag{5.27}$$

$$\delta(r) \sim -\delta_0 + \frac{1}{2} \frac{h_\infty^2}{r^2} + \frac{4}{3} \frac{M h_\infty^2}{r^3} , \tag{5.28}$$

where we have included the first two nontrivial orders in $1/r$ to help demonstrate that the constant h_∞ plays the role of a Coulombic charge [49] in one metric function

$$R^2(r) = \left[1 - \frac{2M}{r} + \frac{h_\infty^2}{r^2} + O\left[\frac{1}{r^3} \right] \right]^{-1} , \tag{5.29}$$

which has the form of the Reissner-Nordström solution, but is absent from the unrescaled form of the other:

$$\frac{1}{T^2(r)} = \left[1 - \frac{2M}{r} + O\left[\frac{1}{r^3} \right] \right] . \tag{5.30}$$

Noting that $h_\infty = \lim_{r\to\infty}(-e^{-\delta} r^2 \phi'/R^2)$ we can obtain an integral expression for h_∞ [49] from (5.2):

$$\begin{aligned} h_\infty &= \int_{r_0}^{\infty} dr \frac{d}{dr} \left[-e^{-\delta} \frac{r^2 \phi'}{R^2} \right] \\ &= \int_{r_0}^{\infty} dr\, e^{-\delta} \left[2e^{-2(h/r + \phi_\infty)} \left[\frac{f'^2}{R^2} + \frac{[1-f^2]^2}{2r^2} \right] \right. \\ &\qquad \left. - r^2 V' \right] . \end{aligned} \tag{5.31}$$

We have included V' in this expression for the sake of generality, and have denoted the lower limit r_0 to emphasize that (5.31) applies to both regular and black hole solutions, since $(-e^{-\delta} r^2 \phi'/R^2)$ vanishes at $r_0 = r_h$ as well as $r_0 = 0$. It is difficult to extract any further information about h_∞ from this expression alone, but we can derive a simpler expression for h_∞ and other useful relations by applying the scaling argument techniques of [53] and utilizing the metric relation (3.17). Before doing so in the next subsection we address an ambiguity which arises from the definition of m_ϕ.

In the above discussion we are careful to quality "massless dilaton" by the condition $\mu^2 = V = 0$, because (5.23) indicates that there are two ways to achieve massless dilatons in this theory: m_ϕ^2 vanishes if either $\mu^2 = 0$ or $\alpha_\infty = 1$. The latter case, corresponding to $\phi_\infty = \frac{1}{2}\ln(3/b_0) \approx 0.830$, is also special in another respect. If we consider a power series expansion of V about $\alpha = \alpha_\infty$, we find that the coefficient of the nth order term varies as $(1/\alpha_\infty)^{(n-1)}$, so $\alpha_\infty = 1$ marks the radius of convergence of the potential expansion. This would present a restriction if we chose to work with the representation $Y \equiv e^{-2\phi}$ for the dilaton preferred by some authors, but for our field choice the power series expansion is well behaved for all $\alpha_\infty \geq 0$:

$$\begin{aligned} V(\hat{\phi}, s_\infty) &= \mu^4 \alpha_\infty^2 \left[\frac{\alpha_\infty - 1}{\alpha_\infty + 1} \right]^2 \hat{\phi}^2 + \mu^4 \alpha_\infty^3 \frac{(\alpha_\infty - 1)(\alpha_\infty - 3)}{(\alpha_\infty + 1)} \hat{\phi}^3 \\ &\quad + \mu^4 \alpha_\infty^2 \frac{(4[1 - 11\alpha_\infty] + \alpha_\infty^2 [\alpha_\infty - 3][7\alpha_\infty - 33])}{12(\alpha_\infty + 1)^2} \hat{\phi}^4 + O(\hat{\phi}^5) . \end{aligned} \tag{5.32}$$

From this expression we find

$$V(\phi, s_\infty) = \tfrac{1}{4}\mu^4 (\phi - \phi_\infty)^4 + O((\phi - \phi_\infty)^5) \tag{5.33}$$

for $\alpha_\infty = 1$ and ϕ near $\phi_\infty = \frac{1}{2}\ln(3/b_0)$, so the potential resembles that of a $\lambda\phi^4$ theory with $\lambda \equiv 3!\mu^4$ in the vicinity of its minimum, while for all other positive values of α_∞ the potential has $\hat{\phi}^2$ and $\hat{\phi}^3$ contributions. When we examine the asymptotic behavior of the field equations for this special case, we find the same leading order behavior as in (5.18)–(5.22) with the exception of the expansion for $h(r)$:

$$h(r) \sim -\frac{1}{2}\frac{c^2 e^{-2\phi_\infty}}{r^3} - \frac{1}{5}\frac{c^2 e^{-2\phi_\infty}(2M+c)}{r^4} . \quad (5.34)$$

Note that the behavior of the field is completely determined by the gauge field to leading order: although the presence of a cubic V' in (5.7) eliminates the possibility of nonzero h_∞ which arises in the $\mu^2=0$ case, it does not otherwise influence the asymptotic behavior of $h(r)$ until several orders beyond those shown in (5.34). Most significantly, in the $\alpha_\infty = 1$ case the dilaton field can only match the boundary conditions if $\hat{\phi}<0$ near $r=\infty$, which from our discussion of (5.9) is inconsistent with the behavior dictated by the full dilaton equation for $\hat{\phi}_0>0$ or $\hat{\phi}(r_h)>0$. The same arguments also apply to the range $\hat{\phi}_{\max} \le \hat{\phi}_0, \hat{\phi}(r_h) \le 0$, where $\hat{\phi}_{\max}$ is the location of the local maximum of V (see Fig. 1). For the range $\hat{\phi} < \hat{\phi}_{\max}$, where $V'>0$, the dilaton field is not strictly forbidden to increase, but the exponential amplification of the gauge field term relative to V' in (5.7) and (5.9) makes a solution with $\hat{\phi}$ increasing toward $\hat{\phi}=0$ highly unlikely. After checking this possibility for a wide range of initial conditions, we believe that no solutions to EYMD theory exist for $\alpha_\infty = 1$.

C. Further analysis

To further explore the EYMD system analytically we make use of a scaling technique developed in [53] for regular solutions and extend it to study black hole solutions. We find that the same results may be obtained from the metric relation (3.17), but we investigate the equivalence of the two approaches elsewhere [72].

The general procedure of [53] involves defining a nonlocal energy functional

$$M = \int_0^\infty dr (e^{-\delta} m)' = \int_0^\infty dr\, e^{-\delta} \mathcal{L}^{(0)} , \quad (5.35)$$

where

$$\delta(\bar{r}) = 2\int_{\bar{r}}^\infty dr \left[\sum_k \frac{U_k(r)}{r} \right] \quad (5.36)$$

is a sum of (non-negative) contributions from the kinetic terms U_k of the matter fields. Demanding that M be stationary with respect to variations of the independent matter fields then yields the Euler-Lagrange equations when we treat $\mathcal{L}^{(0)}$ and δ as functionals of those fields, and we restrict our attention to theories for which the action may be written

$$S \propto \int dt\, dr \left[\frac{U}{R^2} + V \right] e^{-\delta} , \quad (5.37)$$

where U and V are independent of the metric functions. The combination $\mathcal{L}^{(0)} \equiv m' - \delta' m$ in (5.35) is just m' with the metric function $R(r)$ set to unity; with natural units restored, $\mathcal{L}^{(0)}$ is equivalent to an effective matter Lagrangian with $G=0$. By introducing radial scaling transformations for the independent matter fields ψ of the type

$$\psi_\lambda(r) = \lambda^{n_\psi} \psi(\lambda r) , \quad (5.38)$$

which lead to the decompositions

$$U_\lambda(r) = \sum_k \lambda^{p_k} U_k(\lambda r) ,$$
$$\mathcal{L}_\lambda^{(0)} = \sum_j \lambda^{l_j} \mathcal{L}_j^{(0)}(\lambda r) , \quad (5.39)$$

we can find the λ-dependent energy functional $M(\lambda)$ and obtain the constraint

$$\left[\frac{dM}{d\lambda} \right]_{\lambda=1} = \int_0^\infty dr \sum_j \left[\mathcal{L}_j^{(0)} \left[l_j - 1 - \sum_k p_k \delta_k \right] \right] e^{-\delta}$$
$$= 0 . \quad (5.40)$$

Depending on the positivity properties of $\mathcal{L}_j^{(0)}$ and the various scaling constants $\{l_j, p_k\}$, the constraint can be used to establish nonexistence theorems or useful virial-type relationships for the kinetic and potential contributions of the matter fields. In the present case,

$$U_f = e^{-2\phi}(f')^2, \quad V_f = e^{-2\phi}\frac{(1-f^2)^2}{2r^2} ,$$
$$U_\phi = \tfrac{1}{2}(r\phi')^2, \quad V_\phi = r^2 V(\phi, s_\infty) , \quad (5.41)$$

and the constraint $dM/d\lambda|_{\lambda=1}=0$ assumes the form

$$\int_0^\infty dr (y_f' + y_\phi')$$
$$= 2\int_0^\infty dr\, y_f' - 2\int_0^\infty dr\, \delta_f y' - 2\int_0^\infty dr\, e^{-\delta} V_\phi , \quad (5.42)$$

where δ_f is the contribution to δ from U_f, $y \equiv e^{-\delta} m$, and

$$y_f' \equiv e^{-\delta}(U_f + V_f), \quad y_\phi' \equiv e^{-\delta}(U_\phi + V_\phi) , \quad (5.43)$$

are the gauge field and dilaton contributions to y'.

The approach to black holes in [53] involves defining an effective Lagrangian $\mathcal{L}^{(B)}$ on a fixed black hole background and a different scaling for the matter fields, $\psi_\lambda(r/r_h) = \psi((r/r_h)^\lambda)$. Such an approach yields a complicated integral relation which generically involves $\ln(r/r_h)$ in the integrand. Although still quite useful in establishing nonexistence theorems, the result of the procedure is not very useful as a pseudovirial relation. Since we are interested in a means of simplifying integral relations such as (5.31), a relation analogous to (5.40) for black holes is more appropriate for our purposes. To extend the regular solution analysis, we write the energy functional as a sum of horizon and nonhorizon contributions:

$$M = \int_{r_0}^\infty dr\, e^{-\delta} \mathcal{L}^{(0)} + e^{-\delta(r_0)} m(r_0) . \quad (5.44)$$

By treating $\mathcal{L}^{(0)}$, δ, and $\delta(r_0)$ as functionals of the matter fields, we again recover the correct equations of motion from the variation of M. The corresponding constraint equation is

$$\left[\frac{dM}{d\lambda}\right]_{\lambda=1}=\int_{r_0}^{\infty}dr\sum_j\left[\mathcal{L}_j^{(0)}\left[l_j-1-\sum_k p_k\delta_k\right]\right]e^{-\delta}-e^{-\delta(r_0)}m'(r_0)r_0-e^{-\delta(r_0)}m(r_0)\left[\sum_k p_k\delta_k(r_0)\right]$$
$$=0\ . \tag{5.45}$$

In the notation introduced above, this constraint can be written for EYMD theory as

$$\int_{r_0}^{\infty}dr(y_f'+y_\phi')=2\int_{r_0}^{\infty}dr\,y_f'-2\int_{r_0}^{\infty}dr\,\delta_f y'-2\int_{r_0}^{\infty}dr\,e^{-\delta}V_\phi-2e^{-\delta(r_0)}m(r_0)\delta_f(r_0)-r_0e^{-\delta(r_0)}[V_f(r_0)+V_\phi(r_0)]\ , \tag{5.46}$$

the left-hand side of which is equivalent to $M-e^{-\delta(r_0)}m(r_0)$.

With the addition of the definition $x\equiv -r^2\phi'$, the integrated dilaton equation (5.31) in this notation becomes

$$h_\infty=\int_{r_0}^{\infty}dr\frac{d}{dr}\left[\left[e^{-\delta}-\frac{2y}{r}\right]x\right]=2\int_{r_0}^{\infty}dr[y_f'-\delta_f y']-\int_{r_0}^{\infty}dr\,r^2e^{-\delta}V'+2\int_0^{\infty}dr(\delta_f y)'\ . \tag{5.47}$$

Substituting the constraint (5.46) then gives

$$h_\infty=\int_{r_0}^{\infty}dr(y_f'+y_\phi')-\int_{r_0}^{\infty}dr\,r^2e^{-\delta}(V'-2V)+2[\delta_f y]_0^{\infty}+r_0e^{-\delta(r_0)}[V_f(r_0)+V_\phi(r_0)]+2y(r_0)\delta_f(r_0)$$
$$=M-e^{-\delta(r_0)}m(r_0)-\int_{r_0}^{\infty}dr\,r^2e^{-\delta}(V'-2V)+r_0e^{-\delta(r_0)}[V_f(r_0)+V_\phi(r_0)]\ . \tag{5.48}$$

With $\mu^2=0$, this establishes the simple result $h_\infty=M$ for regular solutions and

$$h_\infty=M+r_he^{-\delta(r_h)}\left[e^{-2\phi(r_h)}\frac{[1-f^2(r_h)]^2}{2r_h^2}-\frac{1}{2}\right] \tag{5.49}$$

for black holes with $r_0=r_h=2m(r_h)$.

For nonvanishing potential, Eq. (5.48) also gives the interesting regular solution relation

$$M=\int_0^{\infty}dr\,r^2e^{-\delta}(V'-2V)=\frac{1}{4\pi}\int d^3x\sqrt{-g_{\rm str}}\,e^{-2\phi}\left[\frac{\partial V_{\rm str}}{\partial\phi}\right]\ , \tag{5.50}$$

where $e^{2\phi}V_{\rm str}\equiv V$ and "str" describes quantities in the string frame, which is related to our Einstein frame by the conformal transformation $g_{\mu\nu}=e^{-2\phi}g_{\mu\nu}^{\rm str}$:

$$\int d^4x\sqrt{-g_{\rm str}}e^{-2\phi}(R_{\rm str}+4\partial_\mu\phi\partial^\mu\phi-4V_{\rm str}(\phi,s)+\cdots)\mapsto\int d^4x\sqrt{-g}\,(R-2\partial_\mu\phi\partial^\mu\phi-4V(\phi,s)+\cdots)\ . \tag{5.51}$$

If there is any significance to the suggestive form of (5.50), it is not readily apparent.

The regular solution scaling analysis of [53] and its extension to black hole solutions provide an analytical relation among the solution parameters and the integrals of the dilaton potential which can be verified numerically. Since this nontrivial information about the system must be realized by the field equations, we expect that it might be obtained directly from them. A closer examination of the metric relation (3.17) reveals this to be the case. In the notation of the scaling discussion, (3.17) may be rewritten

$$\frac{d}{dr}\left[\frac{r^2}{R}\left[\frac{1}{T}\right]'\right]=-\frac{d}{dr}\left[\frac{r^2e^{-\delta}}{R^2}(\ln T)'\right]=2y_f'+2y\delta_f'-2e^{-\delta}V_\phi\ . \tag{5.52}$$

Using the T equation

$$\frac{r}{R^2}(\ln T)'=-\frac{1}{R^2}(U_f+U_\phi)+(V_f+V_\phi)-\frac{m}{r} \tag{5.53}$$

to express the boundary terms we obtain the relation

$$M+\left[\frac{r^2e^{-\delta}}{R^2}(\ln T)'\right]_{r=r_0}=M-e^{-\delta(r_0)}m(r_0)+r_0e^{-\delta(r_0)}[V_f(r_0)+V_\phi(r_0)]$$
$$=2\int_{r_0}^{\infty}dr\,y_f'+2\int_{r_0}^{\infty}dr\,\delta_f'y-2\int_{r_0}^{\infty}dr\,e^{-\delta}V_\phi\ , \tag{5.54}$$

which is identical to the scaling result (5.46) with the $\delta_f y'$ contribution integrated by parts. This result provides the alternate expressions

$$h_\infty = M + \left[\frac{r^2 e^{-\delta}}{R^2} (\ln T)' \right]_{r=r_h} , \tag{5.55}$$

$$M = \frac{1}{4\pi} \int d^3x \sqrt{-g_{\rm str}}\, e^{-2\phi} \left[\frac{\partial V_{\rm str}}{\partial \phi} \right] - \left[\frac{r^2 e^{-\delta}}{R^2} (\ln T)' \right]_{r=r_h} , \tag{5.56}$$

for Eqs. (5.49) and (5.50) generalized to black hole solutions. Note that the different signs of the kinetic contributions in m' and $r(\ln T)'/R^2$ indicates that the result from the functional approach (5.45) agrees with (5.54) only when r_0 is the radius of an event horizon, not some arbitrary radius. It is also interesting to note that the regular solution parameter values correspond to the $r_h \to 0$ limit of the black hole values, which is consistent with solution properties observed in previous studies of non-Abelian gauge fields coupled to Einstein gravity [7–9].

We can use the combination of the metric relation (3.17) and the dilaton field equation to determine an additional analytical relation. The combination

$$\frac{d}{dr} \left[\frac{r^2 e^{-\delta}}{R^2} [\phi' - (\ln T)'] \right] = r^2 e^{-\delta} (V' - 2V) \tag{5.57}$$

can be integrated with the help of (5.48) to yield a useful relation between ϕ and T:

$$[\phi(r) - \phi_\infty] = \ln T(r) + (M - h_\infty) \int_\infty^r d\tilde{r} \frac{R^2(\tilde{r}) e^{\delta(\tilde{r})}}{\tilde{r}^2} + \int_\infty^r d\tilde{r} \frac{R^2(\tilde{r}) e^{\delta(\tilde{r})}}{\tilde{r}^2} \int_\infty^{\tilde{r}} d\hat{r}\, \hat{r}^2 e^{-\delta(\hat{r})} (V' - 2V) . \tag{5.58}$$

In the $\mu^2 = 0$ case we have the simple regular solution feature $\phi(r) - \phi_\infty = \ln T(r)$, which can be easily verified numerically, while for black holes the above equation gives

$$[\phi(r) - \phi_\infty] = \ln T(r) + (M - h_\infty) \int_\infty^r d\tilde{r} \frac{R^2(\tilde{r}) e^{\delta(\tilde{r})}}{\tilde{r}^2} . \tag{5.59}$$

The generic relation for a nontrivial potential is obtained from (5.58) by setting h_∞ to zero.

D. Numerical regular solutions

As we observed above, solving the EYMD equations numerically is a two-parameter shooting problem. In terms of the shooting parameters $b \equiv f''(0)/2$ and $\phi_0 \equiv \phi(0) = [h'(0) + \phi_\infty]$ the boundary conditions (5.10)–(5.13) become

$$f(r) = -1 + br^2 + \tfrac{1}{30} b(-24b^2 e^{-2\phi_0} - 9b + 12V_0 + 4V_0')r^4 + O(r^6) , \tag{5.60}$$

$$\begin{aligned} h(r) = (\phi_0 - \phi_\infty) r &+ \tfrac{1}{6}(V_0' - 12b^2 e^{-2\phi_0}) r^3 + \tfrac{1}{360} V_0 (20V_0' + 3V_0'') r^5 \\ &+ \tfrac{1}{360} b^2 e^{-2\phi_0} (-576b^2 e^{-2\phi_0} + 288b - 432V_0 + 48V_0' - 36V_0'') r^5 + O(r^7) , \end{aligned} \tag{5.61}$$

$$2m(r) = \left[4b^2 e^{-2\phi_0} + \tfrac{2}{3} V_0 \right] r^3 + \tfrac{1}{45}(b^2 e^{-2\phi_0}[-144b + 96V_0 - 48V_0'] + 4V_0'^2) r^5 + O(r^7) , \tag{5.62}$$

$$\begin{aligned} \ln T(r) = &-(2b^2 e^{-2\phi_0} - \tfrac{1}{3} V_0) r^2 + \tfrac{1}{180}(20V_0^2 + 3V_0'^2) r^4 \\ &- \tfrac{1}{180} b^2 e^{-2\phi_0} (288b^2 e^{-2\phi_0} - 144b + 144V_0 + 48V_0') r^4 + O(r^6) , \end{aligned} \tag{5.63}$$

where V_0, V_0', and V_0'' are the potential and its derivatives with respect to ϕ at $\phi = \phi_0$, and the shooting parameters satisfy $b > 0$ and $\phi_0 > \phi_\infty$. We evaluate the initial conditions at $r = 10^{-3}$ and use global error tolerance 10^{-12} in an adaptive fifth-order Runge-Kutta ordinary differential equation solver, adjusting (b, ϕ_0) for fixed ϕ_∞ and m_ϕ^2 and integrating toward $r = \infty$. For a range of b and ϕ_0 in the vicinity of a solution, the fields behave much as we anticipated: the gauge field either undergoes a turning point at $|f| \lesssim 1$ or diverges, and $\hat{\phi}$ either undergoes a turning point at $\hat{\phi} \gtrsim 0$ or becomes negative and diverges. In the $\mu^2 = 0$ case, the property $\hat{\phi}' \le 0$ excludes $\hat{\phi}$ turning points, and we instead observe a transition between monotonic decrease to $\hat{\phi}(\infty) = 0$ and the divergence $\hat{\phi} \to -\infty$ over a small parameter range. In terms of h, the finite dilaton behavior is characterized by a turning point after exponential decrease toward $h(\infty) = 0$, or the approach toward a positive constant value h_∞ when $\mu^2 = 0$.

In the massive case, finding a neighborhood of some discrete point (b, ϕ_0) which exhibits these properties does not guarantee that one has found a legitimate solution to the EYMD system. The key to determining whether the results of the shooting procedure constitute valid solutions lies in the exponential behavior of h. For h to decay exponentially to zero, the gauge field coupling term in

(5.7) must be insignificant. Since this contribution is positive definite at finite radius, we must rely on its algebraic approach to zero via (5.18) to satisfy this condition. As μ^2 is increased from zero, the radius at which h exponentially decays behaves roughly as $r \sim 1/m_\phi$, and eventually encroaches upon the fixed region where the gauge contribution algebraically decays. In other words, the screening of the Coulombic dilaton charge h_∞ occurs at a radius $r \sim 1/m_\phi$ which approaches the zone where the local magnetic charge density vanishes. According to (5.7), once m_ϕ is large enough that these regions overlap, the gauge field source drives the dilaton away from its vacuum value and solutions are not possible. For a range of m_ϕ near this overlap, h appears to decay exponentially and satisfy the numerical solution criteria, but close examination reveals deviations from exponential behavior which prohibit extrapolation to $h(\infty)$. To determine a maximum allowable mass $(m_\phi)_{\max}$ in practice, we must set a limit on the deviation of h from the behavior required to match the boundary conditions. By considering next-to-leading order terms in (5.7) we find that

$$h \sim a r^{-m_\phi M} e^{-m_\phi r} \tag{5.64}$$

describes the asymptotic behavior of the dilaton more precisely than (5.19), so that

$$\frac{\delta h''}{h''} \equiv \frac{1}{h''}\left[h'' - m_\phi^2\left[1+\frac{2M}{r}\right]h\right] \tag{5.65}$$

gives a fair measure of the deviation from ideal behavior. Another useful quantity is

$$\frac{\Delta h''}{h''} \equiv -\frac{1}{h''}\left[\frac{2R^2}{r}e^{-2(h/r+\phi_\infty)}\left[\frac{f'^2}{R^2}+\frac{(1-f^2)^2}{2r^2}\right]\right], \tag{5.66}$$

which directly measures the contribution of the gauge field coupling term to h''. In the asymptotic regime of a valid solution, we expect $|\Delta h''/h''| \ll h'' \ll 1$ and (5.66) to be comparable to (5.65), but the maximum acceptable value of either quantity is somewhat ambiguous. Since the size of the contributions to the dilaton equation which are not accounted for by the deviation formula (5.65) are roughly of order h''/r^2, and $\bar{r} \sim 10^3$ is the characteristic radius at which f obeys the asymptotic form (5.18), we adopt the criterion $|\delta h''/h''| \lesssim 1/\bar{r}^2 \approx 10^{-6}$. We find that criterion gives the consistent result $(m_\phi)_{\max} \sim 1/\bar{r} \sim 10^{-3}$ for the classes of solutions we investigate.

When actually obtaining solutions, we adjust the shooting parameters until the solution bracketing conditions (the turning points and divergent behavior of f and h which characterize the solution neighborhood) indicate that the intervals containing the discrete solution values are smaller than our machine accuracy. To achieve this precision as m_ϕ is increased, we truncate h by taking $h \to 0$ or by attaching an exponential tail $h \to a e^{-m_\phi r}$ at the turning point; this allows the integration to proceed so that the f bracketing condition can be determined. To justify this procedure, we apply the deviation criterion (5.65) at the turning point and verify that h behaves according to (5.64) to better than one part in 10^6. Since h tends algebraically to h_∞ rather than experiencing a turning point when the dilaton is massless, the truncation procedure is unnecessary and the numerical pitfalls posed by the final term in (5.7) disappear.

The results of the shooting procedure for the choice $\phi_\infty=0$ are displayed in Fig. 2, with the massless and $(m_\phi)_{\max}$ solution properties summarized in Table I. Like the results of previous studies [7–10], solutions can be classified by the number of nodes k exhibited by the non-Abelian gauge field function f. Although an infinite number of solution classes exist, we focus our attention on the lowest odd- and even-k classes. For both of these classes we performed an identical shooting procedure with the toy potential $V(\phi)=m_\phi^2\phi^2/2$ for comparison. As Table I indicates, the results agree to better than one part in 10^4 for the narrow range of allowable dilaton masses, which indicates that the higher-order $\hat{\phi}$ terms in the potential expansion (5.32) are negligible for this choice of ϕ_∞. Although the total mass M measurably increases as k increases and m_ϕ varies over the allowed range, the only substantial change in the function plots occurs for $h(r)$. It exhibits a maximum at small radius and approaches h_∞ at large r, where the Coulombic dilaton charge is exponentially screened progressively closer to the decay zone of the gauge field. The plot of the actual dilaton field $h(r)/r$, along with the gauge function $f(r)$ and the mass-energy $m(r)$, exhibits nontrivial variation only in the decades surrounding $r=1$: the characteristic radius of the solution is fixed by the string coupling, which in our dimensionless variables mimics the choice $\hat{g} \equiv 2/\sqrt{\alpha'}=1$. Note that the maximum dilaton mass, which we only determine to one decimal place using our imprecise criterion, decreases as the characteristic radius

TABLE I. EYMD regular solutions $\phi_\infty=0$.

k	m_ϕ	b	ϕ_0	M	h_∞	$\ln T(0)$
1	0	1.075 524 3	0.932 283 9	0.576 985 6	0.576 985 6	0.932 283 9
1	3×10^{-3}	1.071 677 2	0.930 484 2	0.577 504 9	0	0.932 277 1
1[a]	3×10^{-3}	1.071 778 4	0.930 534 7	0.577 492 9	0	0.932 276 2
2	0	8.362 081 5	1.792 793 5	0.684 833 2	0.684 833 2	1.792 793 5
2	2×10^{-3}	8.339 161 1	1.791 417 6	0.685 314 6	0	1.792 824 5
2[a]	2×10^{-3}	8.339 517 1	1.791 440 0	0.685 306 5	0	1.792 814 5

[a]Run with $V(\phi)=\frac{1}{2}m_\phi^2\phi^2$ for comparison.

of the gauge field decay zone increases with increasing k.

The solution parameters M, h_∞, and $\ln T(0)$ in Table I are determined with the aid of the asymptotic expansions (5.18)–(5.21) and (5.24)–(5.27). In the massless case they provide a good check of some of the analytical relations determined above. In particular, $h_\infty = M$ to better than seven figures in accordance with (5.49), while the prediction of (5.59) that $\hat{\phi}(r) = \ln T(r)$ is confirmed at $r=0$ (and other points) to an accuracy exceeding the global error tolerance 10^{-12}. The agreement of numerical and analytical results is compelling evidence for the general accuracy of our shooting method; the numerical results of other authors who recently considered the massless case [49] do not clearly exhibit these relations, though they are in general agreement with Table I.

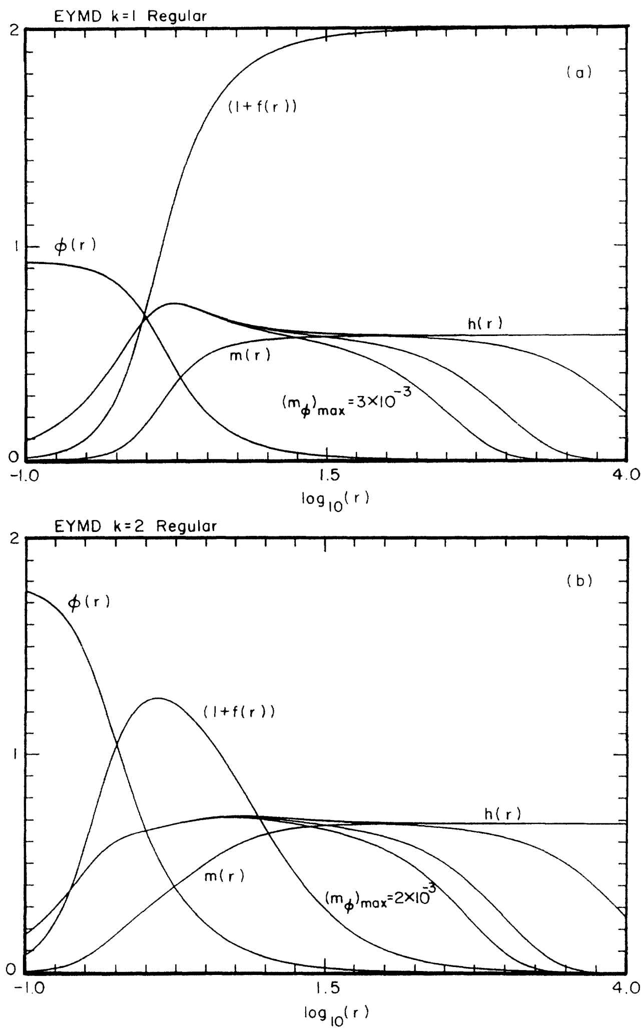

FIG. 2. One- and two-node regular solutions to Einstein-Yang-Mills-dilaton theory for the dilaton potential of Fig. 1(a). The connection function $(1+f(r))$, total mass-energy $m(r)$, dilaton $\phi(r)$, and $h(r)=r\phi(r)$ are plotted as functions of radius for a range of dilaton masses $0 \le m_\phi < (m_\phi)_{\max}$. The maximum mass $(m_\phi)_{\max}$ is reached as the exponential screening of $h(r)$, which asymptotically resembles a Coulombic charge $h_\infty \equiv h(\infty)$ in the $m_\phi = 0$ case, occurs near the zone where the local magnetic gauge charge vanishes. The other functions do not vary appreciably in this mass range. The exact correspondence between h_∞ and $M \equiv m(\infty)$, as well as $\phi(r) - \phi_\infty$ and the metric function $\ln T(r)$, is proven analytically in Sec. V for the massless dilaton case.

As values of ϕ_∞ in the range $-0.8 \lesssim \phi_\infty \lesssim 0.8$ are used in our procedure, solutions appear to be related to the $\phi_\infty = 0$ solutions by the scaling of the radius and some physical parameters. To understand and quantify the scaling, we consider the simplified case of the massless dilaton. When we ignore the axion and the dilaton-axion potential in the action (2.14), the dilaton explicitly appears only in the exponential coupling to the non-Abelian field strength. If we absorb the constant $e^{-2\phi_\infty}$ into the gauge coupling g^2 and then rewrite the theory in terms of dimensionless variables and parameters, we recover the $\phi_\infty = 0$ theory but with $\hat{g} \to e^{-\phi_\infty}\hat{g}$. From the definitions of the dimensionless quantities (3.2)–(3.3), we therefore expect the radial structure of the $\phi_\infty \ne 0$ solutions to scale according to $r \to e^{-\phi_\infty} r$. The mass energy should similarly scale as $m \to e^{-\phi_\infty} m$, but the amplitude of the dilaton deviation field $\hat{\phi}$ and the gauge field should remain unchanged. It follows that the dilaton shooting parameter shifts such that $\hat{\phi}(0) = \phi_0 - \phi_\infty$ is unaffected, while the gauge field parameter scales as $b \to b e^{2\phi_\infty}$ to compensate for the scaling of the initial value of r^2. Though the introduction of the potential and its complicated dependence on ϕ_∞ changes this picture, solutions approximately obey the same scaling relations for the range of ϕ_∞ examined, with $(m_\phi)_{\max} \to e^{\phi_\infty}(m_\phi)_{\max}$ as expected from (3.2) and the relationship $(m_\phi)_{\max} \sim 1/\bar{r}$ discussed above. As ϕ_∞ approaches the critical value $\frac{1}{2}\ln(3/b_0) \approx 0.830$, at which the potential (5.32) reduces to the $\lambda\hat{\phi}^4$ form (5.33) to leading order, solutions become harder to obtain numerically and appear to be forbidden at $\frac{1}{2}\ln(3/b_0)$ for reasons examined above. For ϕ_∞ above this critical value, the leading order $m_\phi^2 \hat{\phi}^2$ term is restored in the potential expansion and solutions are again possible, though we do not explore the scaling properties of this solution region in depth.

E. Numerical black hole solutions

To find numerical black hole solutions we use the conditions

$$f'(r_h) = \frac{-[1-f^2(r_h)]f(r_h)}{r_h - e^{-2\phi(r_h)}[1-f^2(r_h)]^2/r_h - 2V(\phi(r_h))r_h^3}, \tag{5.67}$$

$$h'(r_h) = \frac{V'(\phi(r_h))r_h^3 - e^{-2\phi(r_h)}[1-f^2(r_h)]^2/r_h}{r_h - e^{-2\phi(r_h)}[1-f^2(r_h)]^2/r_h - 2V(\phi(r_h))r_h^3} + \frac{h(r_h)}{r_h}, \tag{5.68}$$

$$m'(r_h) = e^{-2\phi(r_h)}\frac{[1-f^2(r_h)]^2}{2r_h^2} + V(\phi(r_h))r_h^2, \tag{5.69}$$

TABLE II. EYMD black hole solutions $\phi_\infty = 0$.

k	m_ϕ	$f(r_h)$	$\phi(r_h)$	M	h_∞	$\delta(r_h)$
1	0	−0.593 548 2	0.442 271 3	0.836 706 3	0.512 133 3	0.241 881 8
1	2×10^{-3}	−0.593 942 6	0.441 637 5	0.836 975 9	0	0.242 325 4
1[a]	2×10^{-3}	−0.593 938 3	0.441 650 7	0.836 972 1	0	0.242 318 5
2	0	−0.132 085 1	0.544 546 4	0.865 072 7	0.574 832 8	0.151 032 4
2	3×10^{-4}	−0.132 112 9	0.544 485 7	0.865 122 5	0	0.151 093 5
2[a]	3×10^{-4}	−0.132 112 8	0.544 486 0	0.865 122 3	0	0.151 093 2

[a]Run with $V(\phi)=\frac{1}{2}m_\phi^2\phi^2$ for comparison.

$$\delta'(r_h) = -\frac{2}{r_h}\left[f'^2(r_h)+\frac{1}{2}\left[h'(r_h)-\frac{h(r_h)}{r_h}\right]^2\right], \qquad (5.70)$$

on the horizon, and use $f(r_h)$ and $h(r_h)=r_h[\phi(r_h)-\phi_\infty]$ as shooting parameters for $r_h=1$. The asymptotic properties of the fields, which we use to locate the neighborhood of a solution in shooting parameter space, are identical to the regular solution properties, so we follow precisely the same shooting procedure detailed above. In particular, the same truncation of h and maximum dilaton mass criterion are used to determine solutions in the massive case.

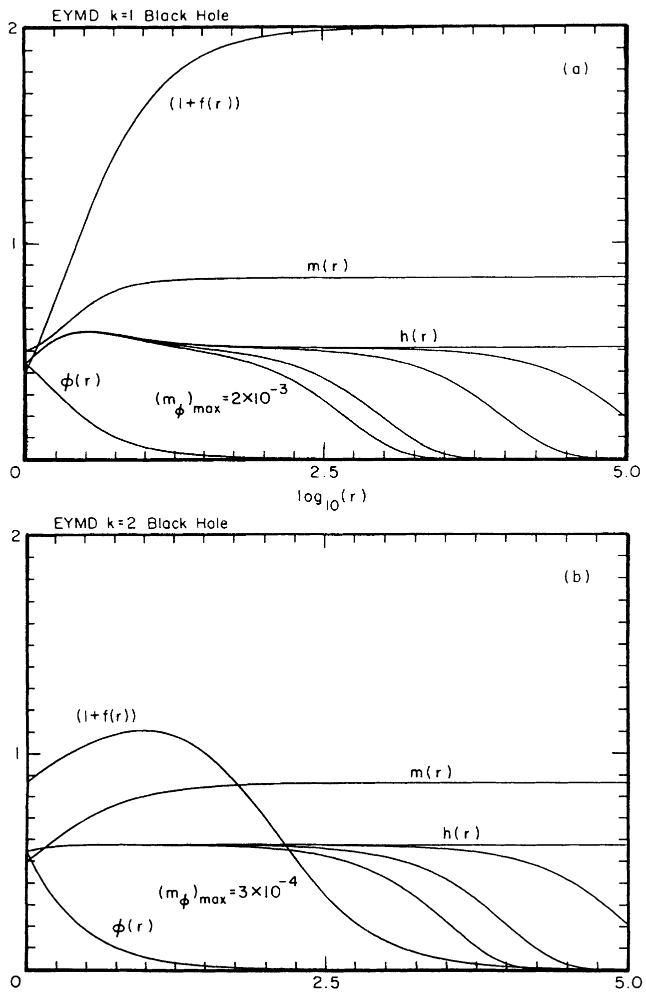

FIG. 3. One- and two-node regular solutions to Einstein-Yang-Mills-dilaton theory for horizon radius $r_h=1$ and the dilaton potential of Fig. 1(a). As in the regular case, only the dilaton function $h(r)=r\phi(r)$ varies appreciably as the dilaton mass approaches a maximum value. Although h_∞ and M are not equal for $m_\phi=0$, an analytic expression is derived in Sec. V which relates these quantities to other global solution parameters.

We again examine only the $k=1$- and $k=2$-node solution classes for the choice $\phi_\infty=0$ and compare results for the full potential (2.15) and the toy potential $V(\phi)=m_\phi^2\phi^2/2$; the results are shown in Fig. 3 and Table II. Again only the function h varies significantly as m_ϕ increases over the small allowed interval, with h approaching h_∞ and then vanishing exponentially at progressively smaller radius. The solution structure is nearly identical to the $r\geq1$ portion of the regular solutions, which in part reflects the occurrence of the event horizon at the characteristic radius $r=1$ of the dimensionless system. A closer examination of the regular solution function reveals a sharp peak in the metric function $R(r)$ near $r=1$, where $2m(r)/r$ closely approaches unity, thus demonstrating that even the regular solutions are strongly gravitating. As we might expect from previous work with black hole solutions to theories with non-Abelian gauge fields [7–9], the black hole solutions reduce to the regular solutions for the same choice of $\{\phi_\infty, m_\phi\}$ in the limit $r_h\to0$.

Although the total mass energy and dilaton charge do not obey the simple relationship $h_\infty=M$ enjoyed by the massless dilaton regular solutions, the analytical result (5.49) does relate h_∞ to M, $\delta_0=\delta(r_h)$ and the integration parameters r_h, $f(r_h)$, and $\hat{\phi}(r_h)$. Again, the results of our numerical procedure verify a nontrivial relationship between the physical parameters of solutions to better than seven significant figures. Black hole solutions for the massive dilaton also exhibit the approximate scaling properties explored above for nonzero ϕ_∞, but the specific relations for r, $m(r)$, and $(m_\phi)_{max}$ only hold when we scale the horizon radius according to $r_h\to e^{-\phi_\infty}r_h$.

VI. CONCLUSIONS

In this paper we have studied static, spherically symmetric regular, and black hole solutions to SU(2) gauge theory coupled to a massive dilaton, massive axion, and Einstein gravity. Our intentions have been twofold: to explore solutions in the physically relevant context of low-energy string theory with massive scalar fields, and to determine whether "stringy" scalar fields lead to non-Abelian solutions with primary hair and good prospects for stability. After analyzing all the possibilities for fundamentally non-Abelian solutions, we found strong numerical evidence for regular and black hole solutions of a massive dilaton coupled to the Yang-Mills field (EYMD theory), and established a deeper understanding of certain solution existence techniques [53] in the course of exploring the solutions analytically. Although the case of a massive axion coupled to the gauge field appeared promising we found that the full theory, which describes a massive dilaton and massive axion coupled to a dyonic non-Abelian configuration, is the only other situation which can admit solutions. We presented no numerical evidence for such solutions, but we were able to construct a consistent solution scenario.

An important issue that we have not addressed in depth is the stability of our solutions. As we noted above, the primary hair solutions to EYMD theory are structurally very similar to the solutions of EYM [10,7–9] and EYMH [23] theories, which have been interpreted as generalized sphalerons and are generically unstable. Although examples of stable solutions with non-Abelian structure have been found, such solutions typically possess a net gauge or topological charge (e.g., "black holes inside magnetic monopoles" [20–22]) or, at the very least, occur in theories which admit stable flat-space soliton solutions (e.g., black hole solutions in Einstein-Skyrme theory [27,28]). Neither of these characteristics is shared by our solutions. These observations, and the linear analysis of [46] which established the instability of EYMD solutions for a massless dilaton, make the stability of our massive dilaton solutions very unlikely.

In light of this conclusion one might question the relevance of pursuing numerical solutions to the full theory. As the only examples of gravitating SU(2) solutions with both magnetic and electric fields, such solutions would be interesting in their own right, but the lack of a net electric charge (which follows here from the asymptotic behavior of the field equations) would appear *not* to improve the chances of stability. It is conceivable, however, that the structure arising from the coupled electric and magnetic charge densities substantially modifies the sphaleron character of EYMD solutions, even in the absence of a net charge. Only in such circumstances, it seems, could we reasonably hope for stable solutions. Since it requires a four-parameter shooting procedure, the task of *finding* such solutions could present enough obstacles that these questions might remain unanswered. Though we have not yet attempted to obtain such solutions, we have described a strategy which might simplify this formidable task. We hope to test the efficacy of this strategy in future investigations of string-inspired non-Abelian dyonic solutions.

Note added. While this paper was being completed, I received a preprint by Donets and Gal'tsov [49], which overlaps with some of this work, and I recently became aware of related papers by Lavrelashvili and Maison [46–47] and Bizon [48].

ACKNOWLEDGMENTS

I would like to thank Brian Greene for his guidance and suggesting the problem, and Alfred Shapere for several helpful discussions. This work was supported in part by the National Science Foundation.

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