Evaluation of two-loop self-energy diagram with three propagators

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(Received 23 August 1994)

A small momentum expansion of the "sunset" diagram with three different masses is obtained. Coefficients at powers of p^2 are evaluated explicitly in terms of dilogarithms and elementary functions. Also some power expansions of the "sunset" diagram in terms of different sets of variables are given.

PACS number(s): 11.10.Gh, 11.15.Bt

The "sunset" diagram (see Fig. 1) was the object of investigation in several recent works [1–5]. Such activity was initiated, certainly, by a rather high precision of experimental testing of the standard model that sometimes needs evaluation of two-loop Feynman diagrams for comparison of the theory and experiments.

Most likely, the "sunset" diagram cannot be expressed through known special functions of one variable. So for the investigation of this diagram numerical methods [4,5] and some approximations schemes, such as expansions in powers of p^2 (and possibly logarithms) [1–3], were used.[1]

In particular, in paper [3] a small momentum expansion of the "sunset" diagram in terms of the Lauricella series was obtained. These series are convergent in the region

$$\sqrt{p^2} + m_1 + m_2 < m_3 \tag{1}$$

where m_i, $i = 1,2,3$, are masses in propagators. But, for the majority of applications, $m_1 + m_2 > m_3$ for any number of masses. Such a situation takes place, for instance, in the course of the evaluation of the self-energy of the Higgs boson (when $m_1 = m_2 = m_3 = m_H$), and in the course of the evaluation of the contribution to the Z-boson self-energy due to the vertex Z^2WW^* (when $m_1 = m_2 = m_W$, $m_3 = m_Z$).

The aim of the present paper is to complete the results of [3] and to obtain a small momentum expansion of the "sunset" diagram that can be used for all values of masses, and to evaluate in closed form coefficients at powers of p^2. In addition, we will give some power expansions of the "sunset" diagram that are convergent, at least, in the region

$$|p^2| + m_3^2 < (m_1 + m_2)^2 \ . \tag{2}$$

Obviously, these expansions can be used for all values of masses if one defines m_3 as $\min\{m_1, m_2, m_3\}$.

Let $I = I(p^2, m_1^2, m_2^2, m_3^2)$ be the "sunset" diagram. Our main tool will be the following onefold integral representation for the function I:

$$I = \int_{(m_1+m_2)^2}^{\infty} d\sigma^2 \rho(\sigma^2, m_1, m_2) \Bigg\{ J(p^2, m_3^2, \sigma^2) - \pi^2 \left[\frac{m_3^2}{\sigma^2} \ln \frac{m_3^2}{\sigma^2} + m_3^2 f_1(\sigma^2) + p^2 f_2(\sigma^2) + f_3(\sigma^2) \right] \Bigg\} , \tag{3}$$

where J is a one-loop "bubble" diagram (see Fig. 2), ρ is a spectral function of J,

$$\rho = \pi^2 \sqrt{\left(1 - \frac{(m_1+m_2)^2}{\sigma^2}\right)\left(1 - \frac{(m_1-m_2)^2}{\sigma^2}\right)} \ , \tag{4}$$

and functions f_i, $i = 1,2,3$, must be defined in such a way that integral (3) converges.

Representation (3) was obtained in a previous work [7].[2] A very similar integral representation was obtained independently in the above-mentioned papers [3,4]. Another onefold integral representation was given in [5]. An analogous onefold integral representation for the five propogator self-energy diagram was derived in [8,9].

One notes that, due to renormalization freedom, the function I is defined up to a polynomial of the first degree with respect to p^2. So it is sufficient to evaluate the function

$$\bar{I}(p^2, ...) = I(p^2, ...) - I(0, ...) - p^2 \frac{dI}{dp^2}(0, ...) \ . \tag{5}$$

Comparing (3) and (5), one obtains the following integral representation for $\bar{I}$:

$$\bar{I} = \int_{(m_1+m_2)^2}^{\infty} d\sigma^2 \rho(\sigma^2, m_1, m_2) \bar{J}(p^2, \sigma^2, m_3^2) \ , \tag{6}$$

where

$$\bar{J}(p^2, ...) = J(p^2, ...) - J(0, ...) - p^2 \frac{dJ}{dp^2}(0, ...) \ . \tag{7}$$

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[1]See also an old paper by Mendels [6] where the expansion of the "sunset" diagram in the equal mass case in powers of $p^2/(p^2 + m^2)$ was obtained.

[2]In [7] representation (3) was derived only for the case $m_1 = m_2$, but, in fact, this condition was never used and, for the general case, representation (3) can be obtained in the same way.

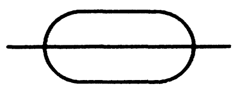

FIG. 1. The "sunset" diagram.

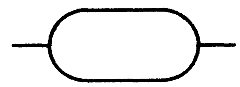

FIG. 2. The "bubble" diagram.

The function $\bar{J}$ can be represented by the power series

$$\bar{J} = \pi^2 \sum_{k\geq 2,\, l\geq 0} \frac{(k+l-1)!(k+l)!}{l!(2k+l+1)!} \left(-\frac{p^2}{\sigma^2}\right)^k \left(1-\frac{m_3^2}{\sigma^2}\right)^l . \tag{8}$$

The proof of (8) is very simple.[3] One considers the representation of J through Feynman parameters:

$$J = -\pi^2 \int_0^1 d\xi \ln\left[\frac{p^2\xi(1-\xi)+\xi m_3^2+(1-\xi)\sigma^2}{\sigma^2}\right] . \tag{9}$$

Expanding the integrand in a powers series with respect to

$$\frac{p^2\xi(1-\xi)+\xi(m_3^2-\sigma^2)}{\sigma^2}$$

and integrating, one obtains (8). These operations are correct if

$$\max_{0\leq\xi\leq 1}\left|\frac{p^2\xi(1-\xi)+\xi(m_3^2-\sigma^2)}{\sigma^2}\right| \leq 1 . \tag{10}$$

From (10) one can derive that series (8) converges, at least, if

$$|p^2| + m_3^2 < \sigma^2 . \tag{11}$$

One introduces the dimensionless variables

$$x = \frac{(m_1+m_2)^2}{\sigma^2}, \quad y = \frac{m_3^2}{(m_1+m_2)^2} < 1,$$

$$z = -\frac{p^2}{(m_1+m_2)^2}, \quad \lambda = \frac{(m_1-m_2)^2}{(m_1+m_2)^2} . \tag{12}$$

In these variables, using the definition of the Gauss hypergeometric function ${}_2F_1$, one can rewrite (8) as

$$\bar{J} = \pi^2 \sum_{k\geq 2} x^k z^k \frac{(k-1)!k!}{(2k+1)!} \times {}_2F_1(k,\ k+1;\ 2k+2;\ 1-xy) . \tag{13}$$

However,

$$\frac{(k-1)!k!}{(2k+1)!}\ {}_2F_1(k,\ k+1;\ 2k+2;\ u)$$

$$= \frac{1}{k!(k+1)!}\frac{d^k}{du^k}(1-u)^{k+1}\frac{d^k}{du^k}\frac{\ln(1-u)}{u} \tag{14}$$

(see, for instance, [11]). From the last formula, after some elementary transformations, one can obtain

$$\frac{(k-1)!k!}{(2k+1)!}\ {}_2F_1(k,\ k+1;\ 2k+2;\ 1-xy)$$

$$= \frac{1}{k!(k+1)!}\frac{1}{x^{k-2}}\frac{d}{dx}\frac{d^k}{dy^k}y^{k+1}\frac{d^{k-1}}{dy^{k-1}}\frac{\ln(xy)}{y(1-xy)} . \tag{15}$$

Substituting (13) and (15) in (6) and integrating by parts with respect to x, one finds

$$\bar{I} = \pi^4(m_1+m_2)^2 \sum_{k\geq 2}\frac{z^k}{k!(k+1)!}f_k(y,\lambda) , \tag{16}$$

where functions $f_k(y,\lambda)$ are defined by

$$f_k(y,\lambda) = -\frac{d^k}{dy^k}y^{k+1}\frac{d^{k-1}}{dy^{k-1}}\frac{f(y,\lambda)}{y} , \tag{17}$$

$$f(y,\lambda) = \ln y + \int_0^1 dx \frac{\ln(xy)}{1-xy}\frac{d}{dx}\sqrt{(1-x)(1-\lambda y)} . \tag{18}$$

The integral in (18) can be easily evaluated and we obtain for $f(y,\lambda)$ the explicit expression

$$\begin{aligned} f(y,\lambda) = \ln y - \frac{\sqrt{\lambda}}{y}&\left[\ln\left(\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}\right)\ln\left(\frac{4y}{(1-\sqrt{\lambda})^2}\right) - \mathrm{Sp}\left(-\frac{2\sqrt{\lambda}}{1+\sqrt{\lambda}}\right) + \mathrm{Sp}\left(-\frac{4\sqrt{\lambda}}{(1-\sqrt{\lambda})^2}\right) - \mathrm{Sp}\left(-\frac{2\sqrt{\lambda}}{1-\sqrt{\lambda}}\right)\right] \\ &+\frac{(2\lambda-y\lambda+y)}{2y\sqrt{(1-y)(\lambda-y)}}\left[\ln\left(\frac{4y}{(1-\sqrt{\lambda})}\right)\ln\left(\frac{t-c}{1-ct}\right) + \mathrm{Sp}\left(\frac{t-1}{t-\frac{1}{c}}\right) - \mathrm{Sp}\left(\frac{t-1}{t-c}\right)\right. \\ &\left.+\mathrm{Sp}\left(\frac{1-t}{1-\frac{t}{c}}\right) - \mathrm{Sp}\left(\frac{1-t}{1-ct}\right) + \mathrm{Sp}\left(\frac{1-t^2}{1-ct}\right) - \mathrm{Sp}\left(\frac{1-t^2}{1-\frac{t}{c}}\right)\right] , \end{aligned} \tag{19}$$

[3]Another proof was given in [10].

where Sp(x) is the Spence function (or dilogarithm),

$$t = \frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}},$$

$$c = \frac{1}{y(1-\lambda)}[2\lambda - y(\lambda+1) + 2\sqrt{\lambda(1-y)(\lambda-y)}]. \tag{20}$$

Formulas (16), (17), and (19) give explicit expressions for coefficients at powers of p^2 in a Taylor expansion of "sunset" diagram. But formula (19) seems too complicated. Fortunately, in almost all applications $m_1 = m_2$ (that is, $\lambda = 0$), and, just in this case, formula (19) simplifies considerably:

$$f(y,0) = \ln y + \frac{1}{\sqrt{y(1-y)}}\mathrm{Cl}_2(2\arcsin\sqrt{y}) \tag{21}$$

where $\mathrm{Cl}_2(\theta)$ is Clausen integral.

In conclusion, we will derive two series expansions for the "sunset" diagram for the case of three different masses, when direct use of explicit formulas (16), (17), and (19) is difficult due to the complexity of formula (19).

First, one substitutes the decomposition (8) in (6), expands the integrand with respect to λ, and performs the integration. Then, after some elementary transformations, one obtains

$$\bar{I} = \frac{\pi^4(p^2)^2}{(m_1+m_2)^2}\sum_{j,k,l\geq 0}\left(\sum_{n\geq 0}\frac{(k+l+n+1)!\,(k+l+n+2)!\,\Gamma(n+\frac{3}{2})}{n!\,(2k+l+n+5)!\,\Gamma(j+k+l+n+\frac{5}{2})}\right)\frac{(j+k+l)!\,\Gamma(j-\frac{1}{2})}{l!\,j!\,\Gamma(-\frac{1}{2})}\lambda^j(1-y)^l z^k\,. \tag{22}$$

The above-mentioned operations are correct if condition (13) is valid for all $\sigma^2 \geq (m_1+m_2)^2$. This means that series (22) converges, at least, at region (2).

The sum over n can be evaluated in terms of finite sums. But the corresponding expression is too cumbersome to be useful. So, if it is desirably to have a closed expression for coefficients at $\lambda^j y^l z^k$, it is preferable to use another series expansion for $\bar{I}$: namely,

$$\bar{I} = \pi^4(m_1+m_2)^2\sum_{j\geq 0,k\geq 2}\frac{z^k\lambda^j}{2j!}\Gamma(j+\frac{1}{2})\left\{\frac{(j+k-2)!}{k(k+1)\,\Gamma(j+k+\frac{1}{2})}\right.$$
$$\left.+\sum_{l\geq 0}\frac{(k+l)!\,(k+l+1)!\,(j+k+l-1)!}{k!\,(k+1)!\,l!\,(l+1)!\,\Gamma(j+k+l+\frac{3}{2})}[h_{jkl}+\ln y]\,y^{l+1}\right\}, \tag{23}$$

where

$$h_{jkl} = \psi(j+k+l) + \psi(k+l+1) + \psi(k+l+2) - \psi(j+k+l+\tfrac{3}{2}) - \psi(l+1) - \psi(l+2)\,.$$

In order to prove (23), it is sufficient to write the function ${}_2F_1$ in (13) as a power series with respect to y and $\ln y$, to substitute this series in (6), and to perform the integration.

These operations again are correct if condition (11) is valid for all $\sigma^2 \geq (m_1+m_2)^2$. So the region of convergence of series (23) is not less than one defines by formula (2).

A large momentum expansion for the "sunset" diagram also may be derived from integral representation (3). But in this case the results of [3] seem quite exhaustive and so further investigations of this problem are unnecessary.

The author is indebted to A.I. Davydychev and B.A. Smirnov for interesting discussions and comments.

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