Nonsingular cosmology with a time-dependent cosmological term

Arbab I. Arbab and A-M.M. Abdel-Rahman
Department of Physics, Faculty of Science, University of Khartoum, P.O. Box 321, Khartoum, Sudan
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The consequences of the cosmological constant ansatz of Carvalho, Lima, and Waga ($\Lambda = 3\beta H^2 + 3\gamma R^{-2}$) are investigated in an extension of the nonsingular Özer-Taha cosmology. The considered model describes a closed singularity-free universe evolving through successive epochs of pure radiation, matter generation, and radiation and matter. The early phase of the last period is shown to be a concrete realization of the postsingularity radiation era scenario of Freese, Adams, Frieman, and Mottola.

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I. INTRODUCTION

Carvalho, Lima, and Waga [1] have proposed the cosmological constant phenomenological ansatz

$$\Lambda = 3\beta H^2 + \frac{3\gamma}{R^2}, \tag{1}$$

where β and γ are dimensionless numbers of the order of unity (natural units being used), R is the Robertson-Walker scale factor, and $H = \dot{R}/R$ is Hubble's constant (an overdot denotes time differentiation). Equation (1), a consequence of simple dimensional arguments consistent with quantum gravity [1], generalizes an earlier form, $\Lambda \propto R^{-2}$, suggested by Özer and Taha [2] and also by Chen and Wu [3].

Although the Özer-Taha (OT) and Chen-Wu (CW) models postulate the same type of variation for Λ, the resulting cosmological scenarios are not similar. In one case (OT) one has a nonsingular universe with a cold beginning and an early phase transition, in the other (CW) a singular big-bang scenario. These differences are due to the model's different initial conditions and the assignment of opposite signs to a certain integration constant.

Carvalho, Lima, and Waga [1] studied the modifications introduced by the β term in Eq. (1) on the cosmology of Chen and Wu. In this work we investigate the effect of this term in an extended Özer-Taha cosmology.

II. NONSINGULAR MODEL

In a Robertson-Walker universe with a perfect-fluid energy-momentum tensor, Einstein's equations with a variable Λ give ($\alpha = 3/8\pi G$)

$$\alpha^{-1}\rho = \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \frac{1}{3}\Lambda(t) , \tag{2}$$

$$\tfrac{3}{2}\alpha^{-1}(\rho + p) = \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \frac{\ddot{R}}{R}, \tag{3}$$

where ρ and p are the cosmic energy density and pressure, respectively, and k the curvature index.

Combining Eq. (3) and the differentiated form of Eq. (2) one has the energy ($= E \equiv \rho R^3$) equation

$$\frac{dE}{dR} + 3pR^2 = -\frac{\alpha}{3}R^3\frac{d\Lambda}{dR} . \tag{4}$$

Also from Eqs. (1) and (2),

$$\Lambda = \frac{3}{1-\beta}\left(\alpha^{-1}\beta\rho + \frac{\gamma - \beta k}{R^2}\right) . \tag{5}$$

In the radiation- ($p = \rho/3$) dominated (RD) universe Eqs. (1)–(3) yield ($\beta \neq \frac{1}{2}$, $\rho_v \equiv \frac{1}{3}\alpha\Lambda$, and A_0 a constant) [1]

$$\dot{R}^2 = \frac{2\gamma - k}{1-2\beta} + A_0 R^{-2+4\beta}, \tag{6}$$

$$\rho = \frac{\alpha(\gamma - \beta k)}{1-2\beta}R^{-2} + \alpha(1-\beta)A_0 R^{-4+4\beta}, \tag{7}$$

$$\rho_v = \frac{\alpha(\gamma - \beta k)}{1-2\beta}R^{-2} + \alpha\beta A_0 R^{-4+4\beta} . \tag{8}$$

For $A_0 > 0$, $\beta < 1$, the *singular* cosmological model based on these equations was studied by Carvalho, Lima, and Waga [1]. Here we investigate a scenario obtained by requiring $A_0 < 0$ and for which the model is *nonsingular*. For simplicity and physical relevance [4] we take $\beta > 0$ (taking $\beta = 0$, $A_0 < 0$, $\gamma = k = 1$ reproduces the nonsingular OT [2] model).

A universe with a nonvanishing minimum scale factor R_0 at $t = 0$ arises from Eq. (6) if $A_0 < 0$, $\beta < \frac{1}{2}$, and $k < 2\gamma$. Then from Eq. (7), $\rho_0 = \alpha(k-\gamma)/R_0^2$ so that $\rho_0 \geq 0$ implies $k \geq \gamma$ also. Hence $\forall \beta < \frac{1}{2}$, one has $\frac{1}{2} < \gamma \leq k = 1$, implying a closed universe as in Refs. [2,5].

From Eqs. (4) and (8), the rate of change of the entropy S at temperature T is given by ($k = 1$, $R \geq R_0$)

$$T\frac{dS}{dR} = -R^3\frac{d\rho_v}{dR} = \frac{2\alpha(\gamma-\beta)}{1-2\beta}\left(1 - \frac{2\beta(2\gamma-1)(1-\beta)R_0^{2-4\beta}}{(\gamma-\beta)R^{2-4\beta}}\right). \quad (9)$$

Choosing $\gamma = 1$ guarantees that $dS/dR > 0$, thereby solving the entropy problem of standard cosmology. In this case Eqs. (1) and (2) give the density parameter $\Omega \equiv \rho/\alpha H^2 = 1-\beta < 1\,\forall t$.

Thus the equations of the model in the pure radiation early cosmic phase can be written as

$$\dot{R}^2 = \frac{\rho R^2}{\alpha(1-\beta)} = \frac{1}{1-2\beta}\left(1 - \frac{R_0^{2-4\beta}}{R^{2-4\beta}}\right), \quad (10)$$

$$\rho_v = \frac{\alpha(1-\beta)}{(1-2\beta)R^2}\left(1 - \frac{\beta R_0^{2-4\beta}}{(1-\beta)R^{2-4\beta}}\right), \quad (11)$$

with $\rho_0 = 0$ and $\rho = \rho(\text{maximum}) = \alpha/2R_{mx}^2$ at

$$R = R_{mx} = [2(1-\beta)]^{1/(2-4\beta)}R_0.$$

To estimate R_0 note from Eq. (10) that $0 < \ddot{R} \leq R_0^{-1}$, implying a cosmic acceleration limit in the early Universe. Such a limit of the order of the Planck mass $M_{\rm Pl} = G^{-1/2}$ has been discussed before [6]. Taking $R_0^{-1} \sim M_{\rm Pl}$ yields $R_0 \sim 10^{-33}$ cm and hence $\rho(\text{maximum}) \lesssim 10^{95}$ kg m^{-3}.

III. RADIATION AND MATTER

In the wake of the pure radiation era, the rest mass is generated during the period $R_1 \leq R \leq R_2$, say [2]. Thereafter ($R \geq R_2$), we take the cosmic matter fluid to be a noninteracting mixture of radiation of density ρ_r and pressureless nonrelativistic matter of density ρ_m ($\rho_r + \rho_m = \rho$). We also assume that the background vacuum couples to radiation only [2,5]. Hence $p = \rho_r/3$ and $E_m = \rho_m R^3 = \rho_{mp}R_p^3 = E_{mp}$, where subscript p denotes present-day quantities.

Thus for $R \geq R_2$, Eq. (4), with Λ given by Eq. (5) ($\gamma = k = 1$), integrates to

$$\rho_r = \frac{3\beta\rho_{mp}R_p^3}{(1-4\beta)R^3} + \frac{\alpha(1-\beta)}{(1-2\beta)R^2}\left(1 + \frac{\omega R_p^{2-4\beta}}{R^{2-4\beta}}\right), \quad (12)$$

where

$$1+\omega = \frac{(1-2\beta)R_p^2\rho_{mp}}{\alpha(1-\beta)}\left(\delta_{rp} - \frac{3\beta}{1-4\beta}\right), \quad (13)$$

$\delta_{rp} \equiv \rho_{rp}/\rho_{mp}$ being the present ratio of radiation-to-matter energy density. Hence

$$\rho_v = \frac{\beta\rho_{mp}R_p^3}{(1-4\beta)R^3} + \frac{\alpha(1-\beta)}{(1-2\beta)R^2}\left(1 + \frac{\beta\omega R_p^{2-4\beta}}{(1-\beta)R^{2-4\beta}}\right), \quad (14)$$

and

$$\dot{R}^2 = \frac{\alpha^{-1}\rho R^2}{1-\beta} = \frac{\alpha^{-1}\rho_{mp}R_p^3}{(1-4\beta)R} + \frac{\omega R_p^{2-4\beta}}{(1-2\beta)R^{2-4\beta}} + (1-2\beta)^{-1}. \quad (15)$$

Let $R = R_{\rm eq} > R_2$ at $t = t_{\rm eq}$, the time when radiation and matter were equal, i.e., $E_r(R_{\rm eq}) = E_m(R_{\rm eq}) = E_{mp}$, where $E_r(R) = \rho_r R^3$ is the radiation energy. Then the condition that E_r was decreasing as R approached $R_{\rm eq}$ leads to

$$(1-4\beta)\omega > \frac{R_{\rm eq}^{2-4\beta}}{R_p^{2-4\beta}}. \quad (16)$$

Thus, either $\omega > 0$ and $\beta < \frac{1}{4}$ or $\omega < 0$ and $\frac{1}{4} < \beta < \frac{1}{2}$. However, the latter case implies $\dot{R}^2 < 0$, $\rho < 0$ for $R_2 \leq R \leq R_{\rm eq}$ [see Eq. (15)] and must therefore be excluded.

In the model of Ref. [2] one approximately has $\rho_r R^4$ constant or RT constant for $R \leq R_{\rm eq}$ in the radiation and matter universe. An important consequence of this feature is that primordial nucleosynthesis in cosmologies of the OT type [5] proceeds as in the standard model.

Here the radiation density in the radiation and matter universe is given by Eq. (12). Clearly if the third ω-dependent term in this equation is dominant it might lead to approximate $1/R^4$ dominance of ρ_r. We therefore investigate when, if at all, this term dominates over the other two terms.

We first consider Eq. (13) with $\omega > 0$ and $\beta < \frac{1}{4}$. Assuming the present Universe is matter dominated (MD) leads immediately to a strong constraint on β: viz.,

$$0 < \beta < \delta_{rp} \ll 1\ . \quad (17)$$

Now we consider Eq. (12) when $\delta_r \equiv \rho_r/\rho_m \gg 1$, i.e., in the early radiation and matter phase. The first term contributes negligibly to δ_r because, from Eq. (13), $3\beta(1-4\beta)^{-1} < \delta_{rp} \ll 1$. Also, by condition (16), $\omega R_p^{2-4\beta}/R^{2-4\beta} \gg 1$ for $R \ll R_{\rm eq}$. Hence

$$\rho_r \approx \frac{\alpha\omega(1-\beta)R_p^{2-4\beta}}{(1-2\beta)R^{4-4\beta}}, \quad (18)$$

demonstrating the dominance, when $R \ll R_{\rm eq}$, of the ω-dependent term in Eq. (12).

It is interesting to note that provided β is not negligibly small (more precisely not much smaller than δ_{rp}), and $\rho_v/\rho_m \gg 1$, which would expectedly hold when $R \ll R_{\rm eq}$, then one can similarly show that the ω-dependent term in Eq. (14) for ρ_v is also dominant when $R \ll R_{\rm eq}$. Thus, in the early radiation and matter phase ρ_v and ρ_r redshift at the same rate, which is the basic postulate that underlies the decaying vacuum cosmologies of Freese *et al.* [7]. Specifically one finds

$$\rho_v \approx \frac{\alpha\beta\omega R_p^{2-4\beta}}{(1-2\beta)R^{4-4\beta}} \approx \frac{\beta\rho_r}{1-\beta},\quad R \ll R_{\rm eq}\ , \quad (19)$$

comprising the same relation between ρ_v and ρ_r as Eq.

(7) of Freese *et al.* [7], where a parameter x replaces β. However, in contrast with Ref. [7], one cannot take $\beta \to 0$ in Eq. (19) because its derivation assumed that β is not vanishingly small. [Letting naively $\beta \to 0$ in Eq. (19) one concludes erroneously that $\rho_v = 0$ in the OT model. One notes that the result (19) has no counterpart in the OT cosmology where $\beta = 0$ and the first and third terms on the right-hand side (RHS) of Eq. (14) do not contribute to ρ_v.]

From Eqs. (15) (with $\rho \approx \rho_r$), (18), and (19) now follow the Freese *et al.* [7] results:

$$R \sim t^{1/2(1-\beta)} \tag{20}$$

and

$$\rho_v \approx \frac{\beta \rho_r}{1-\beta} \approx \frac{\alpha\beta}{4(1-\beta)^2 t^2}, \tag{21}$$

valid in the early radiation and matter universe. Observe that Eq. (21) is independent of the parameter ω. We also note that the dependence of ρ_v on t^{-2} in Eq. (21) has been widely discussed [7,8], with β being model dependent. In the present model $0 < \beta \lesssim \delta_{rp}$.

Equations (20) and (21) reproduce the baryon-to-photon ratio and nucleosynthesis constraints of Ref. [7]. Specifically, it was shown there [7] that nucleosynthesis requires $\beta \leq 0.1$. This is satisfied in the present model where $\beta \lesssim \delta_{rp} \ll 1$.

Other predictions of the present work that can be readily obtained are the bounds $t_p < H_p^{-1}$ and $0 < q_p \lesssim \frac{1}{2}$ for the age of the Universe and the deceleration parameter respectively. In the singular cosmology of Carvalho, Lima, and Waga [1] β is a free parameter that can be adjusted to produce $t_p > H_p^{-1}$ and $q_p < 0$.

We have also examined the consequences of the proposed cosmology for the classical low redshift cosmological tests. The results agree with the Einstein–de Sitter model.

IV. PHASE TRANSITION

We consider, finally, the matter generation period $R_1 \leq R \leq R_2$. For $R \geq R_2$ Eq. (15) implies $\ddot{R} < 0$ if $\omega > 0$ and $\beta < \frac{1}{4}$, in contrast with $\ddot{R} > 0$ for $R < R_1$ as previously noted. Thus the appearance of rest-mass ushers in decelerated expansion.

In addition, from Eqs. (4) and (5) with $\gamma = k = 1$,

$$\frac{dE}{dR} = 2\alpha(1-\beta) + 3\beta R^2 \rho - 3(1-\beta)^2 p, \tag{22}$$

so that on using Eq. (12) one has

$$3\int_{R_0}^{R_2} R^2\left[(1-\beta)p - \beta\rho\right] dR = 2\alpha(1-\beta)(R_2 - R_0) - E_2$$
$$= -2\alpha(1-\beta)R_0 - \frac{(1-\beta)E_{mp}}{1-4\beta} + \frac{\alpha(1-\beta)(1-4\beta)R_2 F(\omega)}{1-2\beta}, \tag{23}$$

where, by condition (16),

$$F(\omega) \equiv 1 - \frac{\omega R_p^{2-4\beta}}{(1-4\beta)R_2^{2-4\beta}} < 0\ . \tag{24}$$

Hence the integral in Eq. (23) is negative, implying that $3p/\rho < 3\beta(1-\beta)^{-1} \ll 1$ [9,10] for some values of R in the interval (R_1, R_2). This period, during part of which the pressure becomes small or negative, is then a phase transition era separating the pure radiation and the radiation and matter epochs.

V. CONCLUSION

We have introduced in this paper a nonsingular cosmological scenario that exhibits features of both the Özer-Taha [2] and the Freese *et al.* [7] models. Since these two models reflect, as already remarked in Ref. [7], "different points of view," e.g., they differ markedly in their initial conditions, their present synthesis is interesting. A particularly noteworthy aspect of the proposed cosmology is that it satisfies the nucleosynthesis constraint of Freese *et al.* provided the Universe today is matter dominated.

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[9] In more detail, $3p/\rho < 3\beta(1-\beta)^{-1} < 3\beta(1-4\beta)^{-1} < \delta_{rp} \ll 1$, where the third inequality follows from Eq. (13).
[10] $3p/\rho \ll 1$ is incompatible with the equation $3p/\rho = 1 - \rho_m/\rho$ in the early radiation and matter phase, where $\rho_m/\rho \ll 1$.