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Self-adjoint extension approach to the spin-1/2 Aharonov-Bohm-Coulomb problem

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The spin-1/2 Aharonov-Bohm problem is examined in the Galilean limit for the case in which a Coulomb potential is included. It is found that the application of the self-adjoint extension method to this system yields singular solutions only for one-half the full range of the flux parameter, which is allowed in the limit of a vanishing Coulomb potential. Thus one has a remarkable example of a case in which the condition of normalizability is necessary but not sufficient for the occurrence of singular solutions. Expressions for the bound state energies are derived. Also the conditions for the occurrence of singular solutions are obtained when the nongauge potential is $\xi/r^p (0 \leq p < 2)$.

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I. INTRODUCTION

In classical electrodynamics, the scalar and vector potentials are merely a convenient tool for the calculation of fields. In quantum mechanics, however, Aharonov and Bohm [1] (AB) gave physical significance to the vector potential, a result that has led to many theoretical and experimental attempts [2] to establish the AB effect.

Recently the spin-1/2 AB problem has been of interest in various branches of theoretical physics. For example it appears in the interaction between matter and cosmic strings [3,4]. It also appears in the anyonic theory [5].

The magnetic flux tube, which shows up in the AB problem, has been treated in different ways. Hagen [6,7] chose the physically motivated expression

$$H \propto \lim_{R\to 0} \frac{1}{R}\delta(r-R), \tag{1.1}$$

where H is the magnetic field, and used it to examine the validity of requiring solutions to be regular at the origin as done by Aharonoz and Bohm [1]. He applied his method [8] to the spin-1/2 AB scattering problem and concluded that the elimination of the singular solution *ab initio* is not valid. Using the expansion of the upper component of the Dirac field

$$\psi 1(r,\theta) = \sum_{m=-\infty}^{\infty} fm(r)e^{im\theta} , \tag{1.2}$$

he calculated the solutions of the radial Schrödinger equation in the $r < R$ and $r > R$ regions separately. Upon applying the boundary conditions at $r = R$ in the $R \to 0$ limit, he derived the fact that, for the case

$$|m| + |m+\alpha| = -\alpha s , \tag{1.3}$$

$$|m+\alpha| < 1 , \tag{1.4}$$

where

$$\alpha = -e\int_0^\infty dr\, rH(r) ,$$
$$s = \begin{cases} 1 & \text{for spin up} , \\ -1 & \text{for spin down} , \end{cases} \tag{1.5}$$

only the solution singular at the origin contributes to the radial wave function. This contribution of the singular solution gives a nontrivial scattering amplitude.

There was another more mathematical approach, which was carried out by Gerbert [9]. He chose the expression of the magnetic flux tube to be

$$H \propto \frac{1}{r}\delta(r) \tag{1.6}$$

and imposed (1.4) as a normalizability condition. He then applied the self-adjoint extension [10,11] method to the partial wave, satisfying (1.4). Since the self-adjoint extension method gives a one-parameter family of solutions, his scattering amplitude contains a self-adjoint extension parameter. His results coincide with those of Hagen when the self-adjoint extension parameter, say θ, equals $\pi/4$. Thus for $\theta = \pi/4$ only singular solutions contribute to the radial wave function. Subsequently Jackiw [12] gave some additional insight into the self-adjoint extension formalism. He proved that the self-adjoint extension formalism gives a result identical to that of the renormalization method when the potential is a δ function if a certain relation between the self-adjoint extension parameter and the renormalized coupling constant is satisfied. Although Gerbert obtained the condition (1.4) by invoking normalizability it will be shown that the condition of normalizability is only necessary but not sufficient for the occurrence of singular solutions. This statement can be proved by applying the self-adjoint extension method to the spin-1/2 AB problem but including also a Coulomb potential $V(r) = \xi/r$.

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Recently the spin-1/2 Aharonov-Bohm-Coulomb (ABC) problem was analyzed by Hagen [13]. He showed that this system yields singular solutions only for one-half the full range, which is allowed in the limit of a vanishing Coulomb potential, namely,

$$|m+\alpha| < \frac{1}{2} \tag{1.7}$$

and also obtained the bound state energies

$$\varepsilon_n = -\frac{1}{2}\frac{M\xi^2}{[n-\frac{1}{2}\pm|m+\alpha|]^2}\ , \quad n=1,2,\dots\ , \tag{1.8}$$

where the upper and lower signs refer, respectively, to the case of regular and irregular solutions.

In the present paper, the ABC problem is analyzed in the context of the self-adjoint extension method. In order to avoid complications associated with Klein's paradox [14] the analysis is performed within the framework of the Galilean [15] spin-1/2 wave equation. It will be shown that the ABC system is a striking example of a case in which the condition of normalizability is a necessary, but not a sufficient, condition for the occurrence of singular solutions. It will also be proved that the condition for the occurrence of singular solutions does not coincide with that of normalizability in the more general case $V(r)=\xi/r^p$ $(0<p<2)$. The sufficiency conditions are derived when this general potential is included in the AB system. Before proceeding, it is important to state somewhat more explicitly the reasoning, which motivates the use of the Galilean limit. Insofar as numerical calculations are concerned, one clearly expects that limit to be useful only in a domain in which the binding energy of a state is much less than the resting energy. For scattering states a realistic theory must be capable of dealing with particle creation processes as well. Thus, the reliability of any wave equation (whether Galilean or special relativistic) is again limited in energy. The essential problem is that the unbounded nature of the $1/r$ potential gives rise to difficulties related to the Klein's paradox even when calculating at relatively low energies. The Galilean equation has no such defect and is thus an ideal tool for the study of the ABC problem.

This paper is organized as follows. In Sec. II the ABC problem is solved in the $r\neq 0$ region. The necessary conditions for the occurrence of a singular solution are obtained. In Sec. III the self-adjoint extension method is applied to the restricted subspace obtained in Sec. II. It is shown that the self-adjoint extension of the Hamiltonian gives the boundary conditions at the origin and that these boundary conditions reduce the possible range of α for the occurrence of singular solutions. Also the expressions for the bound state energies are derived. In Sec. IV the condition for solutions singular at the origin is examined when the potential ξ/r^p $(0\le p<2)$ is included in the AB problem. In Sec. V a brief conclusion is given.

II. SOLUTION OF THE ABC PROBLEM IN THE $r\neq 0$ REGION

In this section the ABC problem is solved in the $r\neq 0$ region within the framework of the Galilean limit. We start with the Dirac equation

$$[m\beta+\beta\gamma_i\Pi_i]\psi = E\psi\ , \tag{2.1}$$

where

$$\begin{aligned}\Pi_i &= -i\partial_i - eA_i\ ,\\ \beta &= \sigma_3\ ,\\ \beta\gamma_i &= (\sigma_1, s\sigma_2)\ ,\end{aligned} \tag{2.2}$$

and s is given in Eq. (1.5). If one takes the Galilean limit

$$\begin{aligned}E &= M+\mathcal{E}\ ,\\ M &\gg \mathcal{E}\ ,\end{aligned} \tag{2.3}$$

and includes the Coulomb potential by letting $\mathcal{E}\to\mathcal{E}-\frac{\xi}{r}$, Eq. (2.1) becomes

$$\begin{pmatrix}\mathcal{E}-\frac{\xi}{r}, & -(\Pi_1-is\Pi_2)\\ -(\Pi_1+is\Pi_2), & 2M\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=0\ . \tag{2.4}$$

It may be worth noting that there is no contradiction between (2.3) and the use of an unbounded Coulomb potential. In fact, once the free Galilean wave equation has been derived, any potential consistent with general Galilean invariance can be considered. The magnetic flux tube is specified by

$$\begin{aligned}eA_i &= \alpha\epsilon_{ij}\frac{r_j}{r^2}\ ,\\ eH &= -\alpha\delta(r)/r\ ,\end{aligned} \tag{2.5}$$

where $\epsilon_{ji}=-\epsilon_{ji}$ and $\epsilon_{12}=+1$. By using Eq. (2.5) the Schrödinger equation of ψ_1 is easily derived:

$$H\psi_1 = \mathcal{E}\psi_1\ , \tag{2.6}$$

where

$$H = H_0 + \frac{\alpha s}{2Mr}\delta(r)\ , \tag{2.7}$$

and

$$H_0 = \frac{1}{2M}\left[\left(\frac{1}{i}\vec{\nabla}-e\vec{\mathbf{A}}\right)^2+\frac{2M\xi}{r}\right]\ .$$

If one decomposes the fermion field as

$$\psi(r,\theta)=\sum_{m=-\infty}^{\infty}\begin{pmatrix}\chi_{1,m}(r)\\ \chi_{2,m}(r)\end{pmatrix}e^{im\theta}\ , \tag{2.8}$$

the Schrödinger equation for $\chi_{1,m}(r)$ becomes

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k^2-\frac{(m+\alpha)^2}{r^2}-\frac{2M\xi}{r}-\alpha s\delta(r)/r\right]\chi_{1,m}(r)=0\ , \quad (2.9)$$

where $k^2 = 2M\mathcal{E}$, and $\chi_{2,m}(r)$ is derived from $\chi_{1,m}(r)$ by

$$\chi_{2,m}(r)=-\frac{i}{2M}\left(\frac{d}{dr}-\frac{(m+\alpha)s}{r}\right)\chi_{1,m}(r)\ . \quad (2.10)$$

By directly solving Eq. (2.9) the solutions for $\chi_{1,m}(r)$ in the $r\neq 0$ region is derived

$$\begin{aligned}\chi_{1,m}(r)=&A_m e^{ikr}(-2ikr)^{m+\alpha}F\left(m+\alpha+\frac{1}{2}+\frac{iM\xi}{k}|2(m+\alpha)+1|-2ikr\right)\\&+B_m e^{ikr}(-2ikr)^{-(m+\alpha)}F\left(-(m+\alpha)+\frac{1}{2}+\frac{iM\xi}{k}|1-2(m+\alpha)|-2ikr\right)\ ,\end{aligned} \quad (2.11)$$

where $F(a|c|Z)$ is the usual confluent hypergeometric function. By inserting Eq. (2.11) into Eq. (2.10) one obtains

$$\begin{aligned}\chi_{2,m}(r)=&-A_m e^{ikr}\left[\frac{(m+\alpha+\frac{1}{2})^2+(M\xi/k)^2}{4M(m+\alpha+1)(m+\alpha+\frac{1}{2})^2}(-2ikr)^{m+\alpha+1}F\left(m+\alpha+\frac{3}{2}+\frac{iM\xi}{k}|2(m+\alpha)+3|-2ikr\right)\right.\\&\left.+\frac{i\xi}{2(m+\alpha)+1}(-2ikr)^{m+\alpha}F\left(m+\alpha+\frac{1}{2}+\frac{iM\xi}{k}|2(m+\alpha)+1|-2ikr\right)\right]\\&+B_m e^{ikr}\left[\frac{2k(m+\alpha)}{M}(-2ikr)^{-(m+\alpha)-1}F\left(-(m+\alpha)-\frac{1}{2}+\frac{iM\xi}{k}|-2(m+\alpha)-1|-2ikr\right)\right.\\&\left.-\frac{i\xi}{2(m+\alpha)+1}(-2ikr)^{-(m+\alpha)}F\left(-(m+\alpha)+\frac{1}{2}+\frac{iM\xi}{k}|1-2(m+\alpha)|-2ikr\right)\right]\ \ (\text{for } s=1)\ ,\end{aligned}$$

$$\begin{aligned}\chi_{2,m}(r)=&A_m e^{ikr}\left[-\frac{2k(m+\alpha)}{M}(-2ikr)^{m+\alpha-1}F\left(m+\alpha-\frac{1}{2}+\frac{iM\xi}{k}|2(m+\alpha)-1|-2ikr\right)\right.\\&\left.+\frac{i\xi}{2(m+\alpha)-1}(-2ikr)^{m+\alpha}F\left(m+\alpha+\frac{1}{2}+\frac{iM\xi}{k}|2(m+\alpha)+1|-2ikr\right)\right]\\&+B_m e^{ikr}\left[\frac{(m+\alpha-\frac{1}{2})^2+(M\xi/k)^2}{4M(m+\alpha-1)(m+\alpha-\frac{1}{2})^2}(-2ikr)^{-(m+\alpha)+1}F\left(-(m+\alpha)+\frac{3}{2}+\frac{iM\xi}{k}|-2(m+\alpha)+3|-2ikr\right)\right.\\&\left.+\frac{i\xi}{2(m+\alpha)-1}(-2ikr)^{-(m+\alpha)}F\left(-(m+\alpha)+\frac{1}{2}+\frac{iM\xi}{k}|-2(m+\alpha)+1|-2ikr\right)\right](\text{for } s=-1)\ .\end{aligned} \quad (2.12)$$

Either A_m or B_m must be zero by the condition of normalizability except in the subspace

$$\begin{aligned}&s=1, \quad m=-N-1,\\&s=-1, \quad m=-N,\end{aligned} \quad (2.13)$$

where

$$\alpha = N+\beta\ ,$$

$$N \text{ is an integer}\ , \quad (2.14)$$

$$0<\beta<1\ .$$

It is easy to see that Eq. (2.13) is merely a different description of Eqs. (1.3) and (1.4). From Eq. (2.11) and (2.12) it follows that $\chi_{1,m}(r)$ and $\chi_{2,m}(r)$ cannot both be chosen to be regular solutions when the condition (2.13) is satisfied. The next section will analyze the ABC problem by the self-adjoint extension method and obtain the result that the condition (2.13) is necessary but not sufficient for the occurrence of a singular solution.

III. SELF-ADJOINT EXTENSION

In Sec. II it was shown that $\chi_{1,m}(r)$ and $\chi_{2,m}(r)$ cannot both be chosen as regular solutions when the condition (2.13) is satisfied. This means that the Hamiltonian of $\chi_{1,m}(r)$,

$$H = H_0 + \frac{\alpha s}{2Mr}\delta(r) \,,$$
$$H_0 = -\frac{1}{2M}\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+\alpha)^2}{r^2} - \frac{2M\xi}{r}\right] \,, \tag{3.1}$$

is not a self-adjoint operator. In order for the Hamiltonian to be a self-adjoint operator, the domain of definition of H_0 has to be extended to the deficiency subspace of $H_0|_{m=-N-1 \text{ or } m=-N}$, which is spanned by the solutions of the differential equations:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+\alpha)^2}{r^2} \pm ik^2 - \frac{2M\xi}{r}\right]\chi_{\mp}(r) = 0 \,. \tag{3.2}$$

This means that the boundary condition at the origin

$$\lim_{r\to 0} r^{|m+\alpha|}\chi(r) = \lambda \lim_{r\to 0}\frac{1}{r^{|m+\alpha|}}\left[\chi(r) - [\lim_{r'\to 0} r'^{|m+\alpha|}\chi(r')]\frac{1}{r^{|m+\alpha|}}\right] \,, \tag{3.3}$$

where λ is the self-adjoint extension parameter, must be required for $\chi_{1,m}(r)$. To treat both cases of Eq. (2.13) simultaneously an expression for $\chi_{1,m}(r)$, which is different from Eq. (2.11) is more convenient; namely,

$$\chi_{1,m}(r) = A_m e^{ikr}(-2ikr)^{|m+\alpha|}F\left(|m+\alpha| + \frac{1}{2} + \frac{iM\xi}{k}|2|m+\alpha|+1| - 2ikr\right)$$
$$+B_m e^{ikr}(-2ikr)^{-|m+\alpha|}F\left(-|m+\alpha| + \frac{1}{2} + \frac{iM\xi}{k}|1-2|m+\alpha|| - 2ikr\right) \,. \tag{3.4}$$

Note that A_m and B_m are the coefficients of the regular and irregular solutions, respectively. By inserting Eq. (3.4) into the boundary condition (3.3), the following relation between the coefficients A_m and B_m is derived:

$$\lambda(-2ik)^{|m+\alpha|}A_m = (-2ik)^{-|m+\alpha|}B_m\left[1 - \frac{2\lambda M\xi}{1-2|m+\alpha|}(\lim_{r\to 0} r^{1-2|m+\alpha|})\right] \,. \tag{3.5}$$

Note that the coefficient of B_m diverges as $\lim_{r\to 0} r^{1-2|m+\alpha|}$, if $|m+\alpha| > \frac{1}{2}$. Thus B_m is zero if $|m+\alpha| > \frac{1}{2}$, and the condition for the occurrence of a singular solution is [16]

$$|m+\alpha| < \tfrac{1}{2} \,. \tag{3.6}$$

Since this is a subspace of Eq. (1.3) and (1.4), or equivalently Eq. (2.13), it is seen that the condition of normalizability is only a necessary condition for the occurrence of singular solution. So let us restrict ourselves to the subspace (3.6).

Since the bound states are obtained in the positive imaginary region k, one can derive the bound state from Eq. (3.4) by changing $k \to i\sqrt{2MB}$, where $B = -\varepsilon$ is the bound state energy:

$$\chi^B_{1,m}(r) = A_m e^{-\sqrt{2MB}r}(2\sqrt{2MB}r)^{|m+\alpha|}F(|m+\alpha| + \tfrac{1}{2} + \sqrt{M/2B}\xi|2|m+\alpha|+1|2\sqrt{2MB}r)$$
$$+B_m e^{-\sqrt{2MB}r}(2\sqrt{2MB}r)^{-|m+\alpha|}F(-|m+\alpha| + \tfrac{1}{2} + \sqrt{M/2B}\xi|1-2|m+\alpha||2\sqrt{2MB}r) \,. \tag{3.7}$$

However, Eq. (3.7) is still not guaranteed to be a bound state. In order to be a bound state, $\chi^B_{i,m}(r)$ must be normalizable at large r. This condition gives the relation

$$A_m\frac{\Gamma(2|m+\alpha|+1)}{\Gamma(|m+\alpha| + \frac{1}{2} + \sqrt{M/2B}\xi)}$$
$$+B_m\frac{\Gamma(1-2|m+\alpha|)}{\Gamma(\frac{1}{2} - |m+\alpha| + \sqrt{M/2B}\xi)} = 0 \,. \tag{3.8}$$

By inserting Eq. (3.8) into Eq. (3.7), the bound state is then obtained

$$\chi^B_{1,m}(r) = \mathcal{N}_m\frac{1}{\sqrt{r}}W_{-\sqrt{M/2B}\xi,|m+\alpha|}(2\sqrt{2MB}r) \,, \tag{3.9}$$

where $\mathcal{N}_m$ is a normalization constant and $W_{a,b}(z)$ is the usual Whittaker function.

Another relation between A_m and B_m is derived by inserting Eq. (3.7) into the boundary condition (3.3):

$$\lambda(2\sqrt{2MB})^{|m+\alpha|}A_m - (2\sqrt{2MB})^{-|m+\alpha|}B_m = 0 \,. \tag{3.10}$$

Thus the bound state energy in implicitly determined from Eqs. (3.8) and (3.10) by the secular equation

$$(2\sqrt{2MB})^{-|m+\alpha|}\frac{\Gamma(2|m+\alpha|+1)}{\Gamma(\frac{1}{2}+|m+\alpha|+\sqrt{M/2B}\xi)}$$

$$+\lambda(2\sqrt{2MB})^{|m+\alpha|}\frac{\Gamma(1-2|m+\alpha|)}{\Gamma(\frac{1}{2}-|m+\alpha|+\sqrt{M/2B}\xi)}=0 \ . \tag{3.11}$$

Although Eq. (3.11) is too complicated to evaluate the bound state energies, its limiting feature is interesting. First, in the $\lambda \to 0$ or ∞ limit, bound state energies are explicitly determined as the poles of the Γ function:

$$\lim_{\lambda\to 0} B = \frac{M}{2}\frac{\xi^2}{[n-\frac{1}{2}+|m+\alpha|]^2}, \quad n=1,2,\ldots \ . \tag{3.12}$$

$$\lim_{\lambda\to\infty} B = \frac{M}{2}\frac{\xi^2}{[n-\frac{1}{2}-|m+\alpha|]^2} \ .$$

These coincide (as expected) with Eq. (1.7) of Hagen [13]. Another interesting limit is the case of vanishing Coulomb potential. In the $\xi \to 0$ limit a bound state energy is explicitly determined from Eq. (3.11):

$$\lim_{\xi\to 0} B = \frac{2}{M}\left[-\frac{\Gamma(1+|m+\alpha|)}{\lambda\Gamma(1-|m+\alpha|)}\right]^{1/|m+\alpha|} , \tag{3.13}$$

which is the Galilean limit of Gerbert's [9] bound state energy. In fact Eq. (3.13) coincides with Gerbert's result when

$$\lambda = -2\left(\frac{E-M}{E+M}\right)^{1/2}|m+\alpha|\frac{M^{1-2|m+\alpha|}}{k}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) , \tag{3.14}$$

and the relativistic relation $k^2 = E^2 - M^2$ is used.

IV. CONDITIONS FOR SINGULAR SOLUTIONS WHEN ξ/r^p POTENTIAL IS INCLUDED

In this section yet another example is given, which illustrates that the conditions for singular solutions does not coincide with those of normalizability. Instead of the Coulomb potential one includes a more general potential ξ/r^p in Eq. (2.9).

Having already shown that the self-adjoint extension method gives the same conditions for singular solutions as the method used in Ref. [13], here one uses the latter approach. To this end, Eq. (2.9) with a ξ/r^p potential is divided into

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k_0^2\right]\chi_{1,m}(r)=0 \quad (r<R) \ , \tag{4.1a}$$

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k^2-\frac{(m+\alpha)^2}{r^2}-\frac{2M\xi}{r^p}\right]\chi_{1,m}(r)=0 \quad (r>R) \ , \tag{4.1b}$$

where

$$k_0^2 = 2M(\varepsilon - U_R) \ , \qquad k^2 = 2M\varepsilon \ , \tag{4.2}$$

and U_R is the constant potential ξ/R^p.

One easily obtains $\chi_{1,m}(r)$ for $r<R$ region, since it must be regular at the origin. Thus,

$$\chi_{1,m}(r) = C_m J_{|m|}(k_0 r) \ . \tag{4.3}$$

Although the analytic solution of Eq. (4.1b) cannot be found easily, we can evaluate the solution by an expansion about the origin. The first few terms are

$$\chi_{1,m}(r)=\begin{cases} A_m e^{ikr}r^{|m+\alpha|}(1-ikr+\cdots)+B_m e^{ikr}r^{-|m+\alpha|}\left(1-ikr+\frac{2M\xi}{(2-p)(2-p-2|m+\alpha|)}r^{2-p}+\cdots\right) & (0\le p<1), \\ A_m e^{ikr}r^{|m+\alpha|}\left(1+\frac{2M\xi}{(2-p)(2-p+2|m+\alpha|)}r^{2-p}+\cdots\right)+B_m e^{ikr}r^{-|m+\alpha|}\left(1+\frac{2M\xi}{(2-p)(2-p-2|m+\alpha|)}r^{2-p}+\cdots\right) & (1<p<2) \ , \end{cases} \tag{4.4}$$

where A_m and B_m are the coefficients of the regular and irregular solutions, respectively.

By using the usual matching conditions

$$\lim_{\varepsilon\to 0+}[\chi_{1,m}(R+\varepsilon)-\chi_{1,m}(R-\varepsilon)]=0 \ ,$$
$$\lim_{\varepsilon\to 0+}\frac{d}{dr}[\chi_{1,m}(R+\varepsilon)-\chi_{1,m}(R-\varepsilon)]=\frac{\alpha s}{R}\chi_{1,m}(R) \ , \tag{4.5}$$

one gets the relation between A_m and B_m. In the limit of small R it becomes

$$\frac{A_m}{B_m} = -\frac{|m|+|m+\alpha|+\alpha s}{|m|-|m+\alpha|+\alpha s}R^{-2|m+\alpha|} +O(R^{-2|m+\alpha|+2-p}) \tag{4.6}$$

for $0 \le p < 2$.

From (4.6) one gets the conditions for the singular solutions:

$$|m| + |m+\alpha| = -\alpha s \, ,$$
$$|m+\alpha| > 1 - \tfrac{p}{2} \, . \tag{4.7}$$

Thus, except for the trivial case $p = 0$ the conditions for singular solutions do not coincide with the normalizability condition $|m+\alpha| < 1$.

V. CONCLUSION

The self-adjoint extension method has been applied here to the spin-1/2 ABC problem. It was shown that the self-adjoint extension method yields $|m+\alpha| < \frac{1}{2}$ as the condition for the occurrence of singular solution. This is in agreement with the results of Ref. [13], which were obtained by a different method. It has also been shown that the condition of normalizability is necessary but not sufficient for the occurrence of singular solutions. Expressions for the bound state wave function and bound state energy have been derived as well. For the $\lambda \to 0$ and ∞ limits, these bound states have been seen to coincide with those of the regular and irregular cases, respectively, of Ref. [13]. Also the $\xi \to 0$ limit of the Galilean version of the bound state energy of Ref. [9] was obtained. Finally it was found that the condition for the existence of singular solutions is $|m+\alpha| > 1 - p/2$ when the general nongauge potential ξ/r^p is included in the AB problem. This result implies that for a gas of such particles the discontinuities in the second virial coefficient [17] are shifted from integer values of α for the Coulomb potential to $n \pm /2$ values (n is an integer). As in the Coulomb case the transition point has no dependence on the strength of the potential.

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[16] Note that it is possible, in principle, to obtain a solution for the domain $1/2 < |m+\alpha| < 1$ if (3.3) is replaced by

$$\lim_{r\to 0} r^{|m+\alpha|}\chi(r) = \lambda \lim_{r\to 0} \frac{1}{r^{-1|m+\alpha|}} \times \left[\chi(r) - [\lim_{r'\to 0} r'^{|m+\alpha|}\chi(r')]\frac{1}{r^{|m+\alpha|}}\right]$$

and if one simultaneously performs a fine-tuning on the extension parameter λ. Since the latter is contrary to the spirit of the self-adjoint extension method [which places no conditions on the extension parameter(s)] this approach is not considered in the present work.

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