Gauge-independent chiral symmetry breaking in quenched QED

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(Received 25 July 1994)

In quenched QED we construct a nonperturbative fermion-boson vertex that ensures the fermion propagator satisfies the Ward-Takahashi identity, is multiplicatively renormalizable, agrees with perturbation theory for weak couplings, and has a critical coupling for dynamical mass generation that is strictly gauge independent. This is in marked contrast to the *rainbow* approximation in which the critical coupling changes by 50% just between the Landau and Feynman gauges. The use of such a vertex should lead to a more believable study of mass generation.

PACS number(s): 11.15.Tk, 11.10.Gh, 12.20.Ds

I. INTRODUCTION

The standard model is highly successful in collating experimental information on the basic forces. Yet, its key parameters, the masses of the quarks and leptons, are theoretically undetermined. In the simplest version of the model, these masses are specified by the couplings of the Higgs boson, couplings that are in turn undetermined. However, it could be that it is the dynamics of the fundamental gauge theories themselves that generate the masses of all the matter fields. To explore this possibility, the favorite starting point is to consider quenched QED [1–9] as the simplest example of a gauge theory and study the behavior of the fermion propagator. Then in the *rainbow* approximation, it is well known that the fermion field can have a dynamically generated mass if the interaction is strong enough; i.e., the coupling α is larger than some critical value α_c. This critical coupling marks a change of phase and so its value should be gauge independent. Unfortunately, the rainbow approximation allows a far from gauge-invariant treatment [10,11]. The purpose of this paper is to construct a nonperturbative fermion-boson interaction that respects the Ward-Takahashi identity, ensures the fermion propagator is multiplicatively renormalizable, agrees with perturbation theory when $\alpha \ll 1$, and possesses a gauge-independent critical coupling.

The *gauge technique* of Delbourgo and Salam, and later collaborators [12] was developed to solve essentially such constraints. However, despite formal results on the first two of these [13,14], their expression in terms of the spectral representation for the fermion propagator has proved difficult in practical calculations of the fermion propagator, for example, Ref. [7]. Consequently, we develop an explicit construction procedure amenable to straightforward computation.

We start with the Schwinger-Dyson equation for the fermion propagator. This nonlinear integral equation encodes all we can know about the fermion propagator. To be able to consider this equation alone of the infinite set of Schwinger-Dyson equations, one for each Green's function, we must make an ansatz for the full fermion-boson vertex. Quite generally, this vertex can be regarded as the sum of two components: the longitudinal and transverse parts. The well-known Ward-Takahashi identity constrains the longitudinal part. How to satisfy this constraint in a manner free of kinematic singularities has been solved some time ago by Ball and Chiu [15]. That multiplicative renormalizability constrains the transverse vertex has also been known for some time [16,14]. However, it is more recently that Curtis and one of the present authors [17] explicitly constructed a simple form (perhaps the simplest possible form) to ensure the multiplicative renormalizability of the fermion propagator. This ansatz is called the CP vertex.

Subsequent study has shown that with this vertex the fermion propagator still has the possibility of a chiral-symmetry-breaking phase [18]. Moreover, in dramatic contrast to the rainbow approximation, the critical coupling required is only very weakly gauge dependent in the neighborhood of the Landau gauge. However weak this variation, any gauge dependence shows that the CP vertex cannot be the exact choice. Here, we determine the constraints on the full fermion-boson vertex that ensures gauge covariance for the fermion propagator and exact gauge independence for the critical coupling. The resulting vertex involves two unknown functions W_1 and W_2, which each satisfy a sum rule and a constraint on their derivatives. Any choice of these satisfies our fundamental constraints as long as it correctly matches onto perturbation theory. This construction builds on the CP vertex, extending the work of Dong, Munczek, and Roberts [19]. Though the discussion in Sec. II of how to ensure the gauge covariance of the wave-function renormalization of the fermion propagator is very close to that of Dong, Munczek, and Roberts [19], to make the extension to the gauge independence of the critical coupling clear, we have given all the details of our formulation making our construction in Sec. III self-contained.

In general, only the position of the pole in a propagator has to be gauge independent. At that value of the momentum, when $p^2 = m^2$ in Minkowski space (or equivalently at $p^2 = -m^2$ in the Euclidean space in which we work) the fermion mass function has to be independent of the gauge. Atkinson and Fry [20] proved this independence follows from the Ward-Takahashi identities. However, at the critical coupling for dynamical mass generation, multiplicative renormalizability imposes such a simple form of the mass function that this whole function becomes gauge independent. This is embodied in our construction.

Our results have to be compared with earlier work. For example, Rembiesa [21] and Haeri [8], using the previously mentioned gauge technique, construct fermion-boson vertices that make the fermion propagator itself gauge independent. This is, of course, at variance with its behavior in perturbation theory and consequently with the renormalization group in the weak-coupling limit. Rembiesa [21] then went on to find that the critical coupling for mass generation with such a vertex is strongly gauge dependent, being given by $\alpha_c = \pi/(3+\xi)$. In complete contrast, Kondo [22] finds a gauge-independent coupling as here, but at the expense of using a vertex that has singularities. The construction presented here aims to overcome these deficiencies.

II. THE FERMION EQUATION

The Schwinger-Dyson equation for the fermion propagator $S_F(p)$ in QED with a bare coupling e is displayed in Fig. 1, and is given by

$$iS_F^{-1}(p) = iS_F^{0^{-1}}(p) - e^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\mu S_F(k) \times\Gamma^\nu(k,p)\Delta_{\mu\nu}(q) , \quad (1)$$

where $q = k - p$ and $S_F(p)$ can be expressed in terms of two Lorentz scalar functions, $F(p^2)$ the wave-function renormalization and $\mathcal{M}(p^2)$ the mass function, so that

$$S_F(p) = \frac{F(p^2)}{\not p - \mathcal{M}(p^2)} .$$

The bare propagator $S_F^0(p) = 1/(\not p - m_0)$, where m_0 is the constant (bare) mass. In quenched QED, the photon propagator is unrenormalized and so is given by its bare form

$$\Delta_{\mu\nu} \equiv \Delta^0_{\mu\nu}(q) = \frac{1}{q^2}\left(g_{\mu\nu} + (\xi-1)\frac{q_\mu q_\nu}{q^2}\right) \equiv \Delta^T_{\mu\nu}(q) + \xi\frac{q_\mu q_\nu}{q^4} ,$$

where the transverse part $\Delta^T_{\mu\nu}(q)$ is defined by this equation and ξ is the standard covariant gauge parameter. $\Gamma^\mu(k,p)$ is the full fermion-boson vertex that must satisfy the Ward-Takahashi identity

$$q^\mu\Gamma_\mu(k,p) = S_F^{-1}(k) - S_F^{-1}(p) . \quad (2)$$

We can simplify Eq. (1) by making use of the Ward-Takahashi identity, Eq. (2),

$$S_F^{-1}(p) = S_F^{0^{-1}}(p) + ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{q^2}\gamma^\mu S_F(k)\Gamma^\nu(k,p)\Delta^T_{\mu\nu}(q) + ie^2\xi\int\frac{d^4k}{(2\pi)^4}\frac{\not q}{q^4} - ie^2\xi\int\frac{d^4k}{(2\pi)^4}\frac{\not q}{q^4}S_F(k)S_F^{-1}(p) . \quad (3)$$

The third term on the right vanishes,[1] as it is an odd integral, and we are left with

$$S_F^{-1}(p) = S_F^{0^{-1}}(p) + ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{q^2}\gamma^\mu S_F(k)\Gamma^\nu(k,p)\Delta^T_{\mu\nu}(q) - ie^2\xi\int\frac{d^4k}{(2\pi)^4}\frac{\not q}{q^4}S_F(k)S_F^{-1}(p) . \quad (4)$$

To solve this equation we must make an ansatz for the full vertex $\Gamma^\mu(k,p)$. Our aim is to construct a vertex that automatically embodies as much of the physics of the interaction as possible. Following Ball and Chiu [15], we first write the vertex as a sum of longitudinal and transverse components:

$$\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p) . \quad (5)$$

To satisfy Eq. (2) in a manner free of kinematic singularities, which in turn ensures the Ward identity is satisfied, we have (following Ball and Chiu)

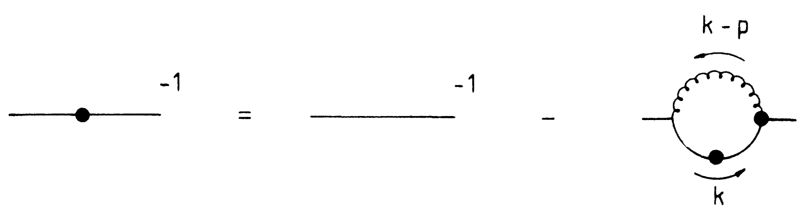

FIG. 1. Schwinger-Dyson equation for the fermion propagator. The straight lines represent fermions and the wavy line the photon. The solid dots indicate full, as opposed to bare, quantities.

[1]This was not noted in Ref. [18] as pointed out in [19], who better remembered Ref. [20] of [23] than the authors.

$$\Gamma_L^\mu(k,p) = a(k^2,p^2)\gamma^\mu + b(k^2,p^2)(\not k + \not p)(k+p)^\mu - c(k^2,p^2)(k+p)^\mu , \tag{6}$$

where

$$a(k^2,p^2) = \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) ,$$
$$b(k^2,p^2) = \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{1}{k^2-p^2} , \tag{7}$$
$$c(k^2,p^2) = \left(\frac{\mathcal{M}(k^2)}{F(k^2)} - \frac{\mathcal{M}(p^2)}{F(p^2)}\right)\frac{1}{k^2-p^2} ,$$

and

$$q_\mu\Gamma_T^\mu(k,p) = 0, \quad \Gamma_T^\mu(p,p) = 0 . \tag{8}$$

Ball and Chiu wrote down a set of eight basis vectors $T_i^\mu(k,p)$ for the transverse part [15], that ensures these conditions, Eq. (8), are satisfied:

$$\Gamma_T^\mu(k,p) = \sum_i^8 \tau_i(k^2,p^2,q^2)T_i^\mu(k,p) , \tag{9}$$

provided that in the limit $k \to p$, $\tau_i(p^2,p^2,0)$ are finite. Our aim is to determine the full vertex by requiring the multiplicative renormalizability of the fermion propagator and the gauge independence of the chiral symmetry breaking phase transition. Since the longitudinal part of this vertex is specified, Eq. (6), this amounts to determining the transverse part and hence the τ_i of Eq. (9).

Of the eight basis vectors T_i^μ, four have even numbers of γ matrices and four have odd numbers.

It is here that we make three simplifying assumptions. First, we demand that a chirally symmetric solution should be possible when the bare mass is zero, just as in perturbation theory. This is most easily accomplished if only those transverse vectors with odd numbers of γ matrices contribute to $\Gamma_T^\mu(k,p)$. Then the sum in Eq. (9) involves just i =2, 3, 6, and 8. The corresponding vectors are

$$T_2^\mu(k,p) = (p^\mu k\cdot q - k^\mu p\cdot q)(\not k + \not p) ,$$
$$T_3^\mu(k,p) = q^2\gamma^\mu - q^\mu \not q ,$$
$$T_6^\mu(k,p) = \gamma^\mu(k^2-p^2) - (k+p)^\mu(\not k - \not p) ,$$
$$T_8^\mu(k,p) = -\gamma^\mu p^\nu k^\rho \sigma_{\nu\rho} + p^\mu \not k - k^\mu \not p , \tag{10}$$

where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu,\gamma_\nu]$.

The second assumption is that the functions τ_i multiplying the transverse vectors, Eq. (9), only depend on k^2 and p^2 but *not* q^2. This allows the angular integrations in Eqs. (1) and (4) to be performed. Third, we assume that, in the Landau gauge, the transverse component of the vertex vanishes. This is motivated by its large momentum behavior in perturbation theory. There for $k^2 \gg p^2$, the leading logarithmic behavior is [17]

$$\Gamma_T^\mu(k,p) \simeq -\frac{\alpha\xi}{8\pi}\ln\frac{k^2}{p^2}\left[\gamma^\mu - \frac{k^\mu \not k}{k^2}\right] , \tag{11}$$

where as usual $\alpha = e^2/4\pi$.

The fermion propagator is determined by the two functions $F(p^2)$ and $\mathcal{M}(p^2)$. We can project out equations for these by taking the trace of Eq. (4) having multiplied by $\not p$ and 1 in turn.

On Wick rotating to Euclidean space,

$$\frac{1}{F(p^2)} = 1 - \frac{\alpha}{4\pi^3}\frac{1}{p^2}\int d^4k\frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)}\frac{1}{q^2}\Bigg(a(k^2,p^2)\left\{-2k\cdot p - \frac{1}{q^2}\left[-2k^2p^2 + (k^2+p^2)k\cdot p\right]\right\}$$
$$+b(k^2,p^2)\left[2k^2p^2 + (k^2+p^2)k\cdot p - \frac{1}{q^2}(k^2-p^2)^2k\cdot p\right] + \mathcal{M}(k^2)c(k^2,p^2)\left[p^2 + k\cdot p - \frac{1}{q^2}(k^2-p^2)(k\cdot p - p^2)\right]$$
$$-\frac{\xi}{q^2F(p^2)}[p^2(k^2-k\cdot p) + \mathcal{M}(k^2)\mathcal{M}(p^2)(k\cdot p - p^2)] + \tau_2(k^2,p^2)\{(k^2+p^2)[k^2p^2 - (k\cdot p)^2]\}$$
$$+\tau_3(k^2,p^2)[-2k^2p^2 + 3(k^2+p^2)k\cdot p - 4(k\cdot p)^2] + \tau_6(k^2,p^2)[(k^2-p^2)3k\cdot p] + \tau_8(k^2,p^2)[-2k^2p^2 + 2(k\cdot p)^2]\Bigg) \tag{12}$$

and

$$\frac{\mathcal{M}(p^2)}{F(p^2)} = m_0 - \frac{\alpha}{4\pi^3}\int d^4k\frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)}\frac{1}{q^2}\Bigg\{-3a(k^2,p^2)\mathcal{M}(k^2) - b(k^2,p^2)\mathcal{M}(k^2)\left[(k+p)^2 - \frac{1}{q^2}(k^2-p^2)^2\right]$$
$$+c(k^2,p^2)\left[(k^2+k\cdot p) - \frac{1}{q^2}(k^2-p^2)(k^2-k\cdot p)\right] - \frac{\xi}{q^2F(p^2)}[\mathcal{M}(p^2)(k^2-k\cdot p) - \mathcal{M}(k^2)(p\cdot k - p^2)]$$
$$+\tau_2(k^2,p^2)\mathcal{M}(k^2)[-2k^2p^2 + 2(k\cdot p)^2] + 3\tau_3(k^2,p^2)\mathcal{M}(k^2) + \tau_6(k^2,p^2)\mathcal{M}(k^2)[3(k^2-p^2)]\Bigg\} . \tag{13}$$

We are only interested in solving this equation when the bare mass m_0 is zero. One solution of the mass equation, Eq. (13), is, as anticipated, $\mathcal{M}(p^2) = 0$. We first consider the wave-function renormalization $F(p^2)$ in this case.

Carrying out the angular integrations in Euclidean space gives

$$\frac{1}{F(p^2)} = 1 + \frac{\alpha\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2)}{F(p^2)} - \frac{3\alpha}{16\pi}\int_0^{p^2}\frac{dk^2}{p^2}\frac{k^2}{p^2}\frac{k^2+p^2}{k^2-p^2}\left(1-\frac{F(k^2)}{F(p^2)}\right) - \frac{3\alpha}{16\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{k^2+p^2}{k^2-p^2}\left(1-\frac{F(k^2)}{F(p^2)}\right)$$
$$-\frac{\alpha}{8\pi}\int_0^{p^2}\frac{dk^2}{p^2}\frac{k^2}{p^2}F(k^2)K_1(k^2,p^2) - \frac{\alpha}{8\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}F(k^2)K_2(k^2,p^2)\ , \tag{14}$$

where

$$K_1(k^2,p^2) = (k^2-3p^2)\left[\tau_3(k^2,p^2)+\tau_8(k^2,p^2) - \tfrac{1}{2}(k^2+p^2)\tau_2(k^2,p^2)\right] + 3(k^2-p^2)\tau_6(k^2,p^2)\ , \tag{15}$$

$$K_2(k^2,p^2) = (p^2-3k^2)\left[\tau_3(k^2,p^2)+\tau_8(k^2,p^2) - \tfrac{1}{2}(k^2+p^2)\tau_2(k^2,p^2)\right] + 3(k^2-p^2)\tau_6(k^2,p^2)\ . \tag{16}$$

The following treatment turns out to be very close to that of Dong, Munczek, and Roberts [19], in a more suitable form for our extension to dynamical mass generation. It is convenient to define the combination $\bar\tau$ of τ_2, τ_3, and τ_8:

$$\bar\tau(k^2,p^2) = \tau_3(k^2,p^2)+\tau_8(k^2,p^2) - \tfrac{1}{2}(k^2+p^2)\tau_2(k^2,p^2). \tag{17}$$

Then

$$K_1(k^2,p^2) = (k^2-3p^2)\bar\tau(k^2,p^2) + 3(k^2-p^2)\tau_6(k^2,p^2)\ , \tag{18}$$

$$K_2(k^2,p^2) = (p^2-3k^2)\bar\tau(k^2,p^2) + 3(k^2-p^2)\tau_6(k^2,p^2)\ , \tag{19}$$

which can be reexpressed in terms of functions with definite symmetry properties when $k \leftrightarrow p$. Thus

$$K_1(k^2,p^2) = h_s(k^2,p^2) + h_a(k^2,p^2)\ , \tag{20}$$

$$K_2(k^2,p^2) = h_s(k^2,p^2) - h_a(k^2,p^2)\ , \tag{21}$$

where $h_s(k^2,p^2)$ and $h_a(k^2,p^2)$ are symmetric and antisymmetric, respectively, under the interchange of k and p:

$$h_s(k^2,p^2) = -(k^2+p^2)\bar\tau(k^2,p^2) + 3(k^2-p^2)\tau_6(k^2,p^2)\ , \tag{22}$$

$$h_a(k^2,p^2) = 2(k^2-p^2)\bar\tau(k^2,p^2)\ . \tag{23}$$

As discussed in [24,17,18], multiplicative renormalizability requires that the solution of this integral equation for the wave-function renormalization $F(p^2)$ must be of the form

$$F(p^2) = A(p^2)^\nu\ . \tag{24}$$

As shown in [19], gauge covariance requires $\nu = \alpha\xi/4\pi$. Burden and Roberts [25] noted numerically that the fermion equation with the simple CP vertex correctly generates this behavior, even though the authors of Refs. [26,18] found $\nu = 2\alpha\xi/(8\pi+\alpha\xi)$ as a result of not imposing translational invariance on their loop integrations, Eqs. (3) and (4), as discussed earlier.

This simple power behavior is generated by the 1 and the first integral on the right-hand side of Eq. (12). This requires, as noted in Refs. [26,19], a cancellation among the remaining integrals. Thus multiplicative renormalizability imposes the constraint

$$\frac{3}{2}\int_0^{p^2}\frac{dk^2}{p^2}\frac{k^2}{p^2}\frac{k^2+p^2}{k^2-p^2}\left(1-\frac{F(k^2)}{F(p^2)}\right) + \frac{3}{2}\int_{p^2}^{\Lambda^2}\frac{dk_2}{k^2}\frac{k^2+p^2}{k^2-p^2}\left(1-\frac{F(k^2)}{F(p^2)}\right) + \int_0^{p^2}\frac{dk^2}{p^2}\frac{k^2}{p^2}F(k^2)[h_s(k^2,p^2)+h_a(k^2,p^2)]$$
$$+\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}F(k^2)[h_s(k^2,p^2)-h_a(k^2,p^2)] = 0\ , \tag{25}$$

where $F(p^2) = A(p^2)^\nu$ and the artificial cutoff Λ can be taken to infinity with impunity. The scale invariance of the integrals makes it convenient to introduce the variable x, where for $0 \le k^2 < p^2$, $x = k^2/p^2$, and for $p^2 \le k^2 < \infty$,

$x = p^2/k^2$ [27,18]. Then

$$\frac{3}{2}\int_0^1 dx \frac{x+1}{x-1} r_1(x) + \int_0^1 dx\, x^{\nu+1} F(p^2)\left[h_s(xp^2,p^2) + h_a(xp^2,p^2)\right] + \int_0^1 dx\, x^{-\nu-1} F(p^2)\left[h_s(p^2/x,p^2) - h_a(p^2/x,p^2)\right] = 0\ , \quad (26)$$

where

$$r_1(x) = x(1-x^\nu) - x^{-1}(1-x^{-\nu})\ ,$$
$$r_1(1/x) = -r_1(x)\ .$$

Since this equation must hold true at all p^2, the integrands cannot be functions of p^2 but only of x. Thus

$$F(p^2)h_s(xp^2,p^2) \equiv h_1(x)\ ,$$
$$F(p^2)h_a(xp^2,p^2) \equiv h_2(x)\ ,$$

define h_1, h_2. Then Eq. (26) becomes

$$\frac{3}{2}\int_0^1 dx \frac{x+1}{x-1} r_1(x) + \int_0^1 dx\, x^{\nu+1}\left[h_1(x) + h_2(x)\right] + \int_0^1 dx\, x^{-\nu-1}\left[h_1(1/x) - h_2(1/x)\right] = 0\ . \quad (27)$$

The original symmetry of the τ's under the exchange of k^2 and p^2 translates as follows in terms of the x variable [19]:

$$h_1(1/x) = x^\nu h_1(x)\ ,$$
$$h_2(1/x) = -x^\nu h_2(x)\ .$$

In the most compact way, Eq. (27) can be written as

$$\int_0^1 dx\, W_1(x) = 0\ , \quad (28)$$

where

$$W_1(x) = \frac{3}{2}\frac{x+1}{x-1} r_1(x) + \left(x^{\nu+1} + x^{-1}\right)\left[h_1(x) + h_2(x)\right]\ . \quad (29)$$

Thus this function $W_1(x)$ fixes $\tau_6(k^2,p^2)$ and the combination $\bar{\tau}(k^2,p^2)$, so that

$$\bar{\tau}(k^2,p^2) = \frac{1}{4}\frac{1}{k^2-p^2}\frac{1}{s_1(k^2,p^2)} \times \left[W_1\left(\frac{k^2}{p^2}\right) - W_1\left(\frac{p^2}{k^2}\right)\right]\ , \quad (30)$$

$$\tau_6(k^2,p^2) = -\frac{1}{2}\frac{k^2+p^2}{(k^2-p^2)^2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) + \frac{1}{3}\frac{k^2+p^2}{k^2-p^2}\bar{\tau}(k^2,p^2) + \frac{1}{6}\frac{1}{k^2-p^2}\frac{1}{s_1(k^2,p^2)} \times \left[W_1\left(\frac{k^2}{p^2}\right) + W_1\left(\frac{p^2}{k^2}\right)\right]\ , \quad (31)$$

where

$$s_1(k^2,p^2) = \frac{k^2}{p^2}F(k^2) + \frac{p^2}{k^2}F(p^2)\ .$$

It is the first term in Eq. (31) that is essentially the CP vertex in the massless theory. Note the automatic appearance of the difference $[F(k^2)^{-1} - F(p^2)^{-1}]$, which Curtis and Pennington [17] conjectured was the nonperturbative generalization of the leading logarithm behavior in lowest-order perturbation theory, Eq. (11). Indeed, agreement with this behavior is naturally achieved if $W_1 \to 0$ in this limit.

The vertex can only have singularities for good dynamical reasons. It cannot have kinematic singularities. A sufficient condition for this is to assume that each of the τ_i ($i = 1,8$) is free of kinematic singularities. Ball and Chiu [15] found that with their choice of basis vectors T_i^μ this is indeed true at one-loop order in perturbation theory in the Feynman gauge and Kizilersü, Reenders, and Pennington [28] have now shown this in any covariant gauge at this order, too. In the present nonperturbative analysis that this continues to hold with the Ball-Chiu basis vectors is a plausible simplifying assumption. Thus

$$\lim_{k^2\to p^2}(k^2-p^2)\tau_6(k^2,p^2) = 0\ , \quad (32)$$

which requires

$$W_1(1) + W_1'(1) = -6\nu\ , \quad (33)$$

as found by [19]. Perturbation theory demands $W_1(x)$ be $\mathcal{O}(\alpha)$. While the form of the coefficient function τ_6 is determined by the constrained function $W_1(x)$, it is only the combination $\bar{\tau}$ of τ_2, τ_3, τ_8 that is so specified. By imposing the gauge independence of the critical coupling for mass generation, we will be able to separate these functions as we now show in Sec. III.

III. THE MASS FUNCTION

While for $\alpha < \alpha_c$, there is only one solution $\mathcal{M}(p^2) = 0$, as $\alpha \to \alpha_c$ a second nonzero solution becomes possible. This solution bifurcates away from the other solution. Bifurcation analysis allows us to investigate precisely when this happens. In the neighborhood of the critical coupling, terms quadratic in the mass function can be rigorously neglected. Thus the wave-function renormalization $F(p^2)$ is that of the massless theory, Sec. II, and the equation for the mass function $\mathcal{M}(p^2)$, Eq. (13) with $m_0 \equiv 0$, linearizes

$$\begin{aligned}\frac{\mathcal{M}(p^2)}{F(p^2)} &= \frac{\alpha\xi}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} + \frac{\alpha\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\\
&+\frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\left[\mathcal{M}(k^2)+\frac{p^2}{2(k^2-p^2)}\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)-\frac{k^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right]\\
&+\frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\left[\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)}+\frac{k^2}{2(k^2-p^2)}\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)-\frac{p^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right]\\
&-\frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)F(k^2)\left[\frac{k^2}{6}(k^2-3p^2)\tau_2(k^2,p^2)+p^2\tau_3(k^2,p^2)+(k^2-p^2)\tau_6(k^2,p^2)\right]\\
&-\frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)F(k^2)\left[\frac{p^2}{6}(p^2-3k^2)\tau_2(k^2,p^2)+k^2\tau_3(k^2,p^2)+(k^2-p^2)\tau_6(k^2,p^2)\right]\,. \end{aligned} \tag{34}$$

If this equation is to be multiplicatively renormalizable with a gauge-independent bifurcation, then this imposes further constraints on the transverse vertex, $\tau_i (i = 2, 3, 6)$. We first work in the Landau gauge, where we continue to assume the transverse vertex vanishes. This is motivated by the perturbative result of Eq. (11).

Then we have simply

$$\begin{aligned}\mathcal{M}(p^2) &= \frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\left[\mathcal{M}(k^2)-\frac{k^2}{2(k^2-p^2)}[\mathcal{M}(k^2)-\mathcal{M}(p^2)]\right]\\
&+\frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\left[\mathcal{M}(k^2)-\frac{p^2}{2(k^2-p^2)}[\mathcal{M}(k^2)-\mathcal{M}(p^2)]\right]\,.\end{aligned} \tag{35}$$

This equation has the multiplicatively renormalizable solution

$$\mathcal{M}(k^2) = B(k^2)^{-s}\,, \tag{36}$$

where Eq. (35) requires

$$\frac{8\pi}{3\alpha} = 1 + \frac{3}{s} + \frac{1}{1-s} - \pi\cot\pi s \equiv f(s)\,. \tag{37}$$

There are two roots for s between 0 and 1. Bifurcation occurs when the two roots for s merge at $s = s_c$, specified by $f'(s_c) = 0$. This point defines the critical coupling [4,27,18], $\alpha_c = 8\pi/3f(s_c)$. Numerically, $\alpha_c = 0.933\,667$ and $s_c = 0.470\,966$. A little away from this critical point the exponent s in Eq. (36) is given by

$$s = s_c \pm \left(\frac{2f(s_c)}{f''(s_c)}\right)^{1/2}\left(1-\frac{\alpha}{\alpha_c}\right)^{1/2}\,. \tag{38}$$

It is only at the bifurcation point that the simple behavior of Eq. (36) holds at all momenta. There, only when the mass is still effectively zero is there just one scale Λ for the momentum dependence of $\mathcal{M}(k^2)$. Multiplicative renormalizability then forces a simple power behavior. Such a multiplicatively renormalizable mass function must exist in all gauges. Consequently, the exponent s_c must be gauge independent. Moreover, dynamical mass generation marks a physical phase change and so the critical coupling α_c must also be gauge independent. Thus the critical values α_c, s_c found in the Landau gauge must hold in all gauges. This is achieved as follows. We recall Eqs. (14) and (25):

$$\frac{1}{F(p^2)} = 1 + \frac{\alpha\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2)}{F(p^2)}\,. \tag{39}$$

Multiplying this equation by $\mathcal{M}(p^2)$ and subtracting it from Eq. (34), we obtain

$$\begin{aligned}\mathcal{M}(p^2) = &\frac{\alpha\xi}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)}\\ &+\frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\left[\mathcal{M}(k^2)+\frac{p^2}{2(k^2-p^2)}\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)-\frac{k^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right]\\ &+\frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\left[\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)}+\frac{k^2}{2(k^2-p^2)}\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)-\frac{p^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right]\\ &-\frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)F(k^2)\left[\frac{k^2}{6}(k^2-3p^2)\tau_2(k^2,p^2)+p^2\tau_3(k^2,p^2)+(k^2-p^2)\tau_6(k^2,p^2)\right]\\ &-\frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)F(k^2)\left[\frac{p^2}{6}(p^2-3k^2)\tau_2(k^2,p^2)+k^2\tau_3(k^2,p^2)+(k^2-p^2)\tau_6(k^2,p^2)\right]\ . \end{aligned}\tag{40}$$

In order for the above equation to reduce to Eq. (35), it must be true that

$$\begin{aligned}\frac{\xi}{3}\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} = &-\int_0^{p^2}dk^2\frac{\mathcal{M}(k^2)}{2(k^2-p^2)}\left(1-\frac{F(k^2)}{F(p^2)}\right)-\int_{p^2}^{\Lambda^2}dk^2\frac{\mathcal{M}(k^2)}{2(k^2-p^2)}\left(1-\frac{F(k^2)}{F(p^2)}\right)\\ &+\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)F(k^2)\left[\frac{k^2}{6}(k^2-3p^2)\tau_2(k^2,p^2)+p^2\tau_3(k^2,p^2)+(k^2-p^2)\tau_6(k^2,p^2)\right]\\ &+\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)F(k^2)\left[\frac{p^2}{6}(p^2-3k^2)\tau_2(k^2,p^2)+k^2\tau_3(k^2,p^2)+(k^2-p^2)\tau_6(k^2,p^2)\right]\end{aligned}\tag{41}$$

at all momentum p and in all gauges ξ.

This equation can be written as

$$\begin{aligned}\xi\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} = &-\int_0^{p^2}dk^2\frac{3\mathcal{M}(k^2)}{2(k^2-p^2)}\left(1-\frac{F(k^2)}{F(p^2)}\right)-\int_{p^2}^{\Lambda^2}dk^2\frac{3\mathcal{M}(k^2}{2(k^2-p^2)}\left(1-\frac{F(k^2)}{F(p^2)}\right)\\ &+\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)F(k^2)K_3(k^2,p^2)+\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)F(k^2)K_4(k^2,p^2)\ ,\end{aligned}\tag{42}$$

where $K_3(k^2,p^2)$ and $K_4(k^2,p^2)$ can, like $K_1(k^2,p^2)$ and $K_2(k^2,p^2)$, be expressed in terms of functions with definite symmetry properties under the interchange of k and p,

$$\begin{aligned}g_s(k^2,p^2) &= \tfrac{1}{4}[(k^2-p^2)^2-4k^2p^2]\tau_2(k^2,p^2)+\tfrac{3}{2}(k^2+p^2)\tau_3(k^2,p^2)+3(k^2-p^2)\tau_6(k^2,p^2)\ ,\\ g_a(k^2,p^2) &= \tfrac{1}{4}(k^2-p^2)[(k^2+p^2)\tau_2(k^2,p^2)-6\tau_3(k^2,p^2)]\ ,\end{aligned}\tag{43}$$

so that

$$\begin{aligned}K_3(k^2,p^2) &= g_s(k^2,p^2)+g_a(k^2,p^2)\ ,\\ K_4(k^2,p^2) &= g_s(k^2,p^2)-g_a(k^2,p^2)\ .\end{aligned}$$

Introducing the variable x as before and knowing that $\mathcal{M}(k^2)\sim(k^2)^{-s_c}$ and $F(k^2)\sim(k^2)^{\nu}$, Eq. (42) becomes

$$\begin{aligned}\xi\int_0^1 dx\,x^{\nu-s_c}+\frac{3}{2}\int_0^1\frac{dx}{x-1}[x^{-s_c}-x^{\nu-s_c}-x^{s_c-1}+x^{s_c-\nu-1}]-\int_0^1 dx\,x^{\nu-s_c}F(p^2)[g_s(xp^2,p^2)+g_a(xp^2,p^2)]\\ -\int_0^1 dx\,x^{s_c-\nu-1}F(p^2)[g_s(p^2/x,p^2)-g_a(p^2/x,p^2)]=0\ .\end{aligned}\tag{44}$$

Once again, this equation must hold true for all p^2, and so the integrands cannot be functions of p^2 but solely of x.

Thus, we conveniently define

$$F(p^2)g_s(xp^2,p^2) \equiv g_1(x) \ , \quad F(p^2)g_a(xp^2,p^2) \equiv g_2(x) \ .$$

Then we have

$$\xi \int_0^1 dx\, x^{\nu-s_c} + \frac{3}{2}\int_0^1 \frac{dx}{x-1}[x^{-s_c} - x^{\nu-s_c} - x^{s_c-1} + x^{s_c-\nu-1}]$$

$$- \int_0^1 dx\, x^{\nu-s_c}[g_1(x)+g_2(x)] - \int_0^1 dx\, x^{s_c-\nu-1}[g_1(1/x) - g_2(/x)] = 0 \ . \quad (45)$$

The symmetry of the vertex [19] under $k \leftrightarrow p$ means that

$$g_1(1/x) = x^\nu g_1(x) \ , \quad g_2(1/x) = -x^\nu g_2(x) \ .$$

In contrast with our discussion in Sec. II when the equations for the wave-function renormalization $F(p^2)$ apply for all values of the coupling, Eqs. (44) and (45) only hold when $\alpha = \alpha_c$.

Equation (45) can be written in a compact way as

$$\int_0^1 \frac{dx}{\sqrt{x}} W_2(x) = 0 \ , \quad (46)$$

where

$$W_2(x) = \xi x^{\nu-s_c+1/2} + \frac{3}{2}\frac{r_2(x)}{x-1} - x^{\nu-s_c+1/2}[g_1(x)+g_2(x)] - x^{-\nu+s_c-1/2}[g_1(1/x) - g_2(1/x)] \quad (47)$$

with

$$r_2(x) = x^{1/2-s_c}(1-x^\nu) - x^{s_c-1/2}(1-x^{-\nu}) \ , \quad (48)$$

which has the property, $r_2(1/x) = -r_2(x)$. Conveniently defining the combination

$$s_2(k^2,p^2) = \frac{k}{p}\frac{\mathcal{M}(k^2)}{\mathcal{M}(p^2)}F(k^2) + \frac{p}{k}\frac{\mathcal{M}(p^2)}{\mathcal{M}(k^2)}F(p^2) \ , \quad (49)$$

where $k = (k\cdot k)^{1/2}$, $p = (p\cdot p)^{1/2}$ we have

$$g_s(k^2,p^2) = \frac{\xi}{2s_2(k^2,p^2)}\left[\frac{k}{p}\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} + \frac{p}{k}\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right] + \frac{3}{4}\frac{k^2+p^2}{k^2-p^2}\frac{1}{s_2(k^2,p^2)}r_2\left(\frac{k^2}{p^2}\right)$$
$$-\frac{1}{2}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) + W_2\left(\frac{p^2}{k^2}\right)\right] \ , \quad (50)$$

$$g_a(k^2,p^2) = \frac{\xi}{2s_2(k^2,p^2)}\left[\frac{k}{p}\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} - \frac{p}{k}\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right] - \frac{3}{4}\frac{1}{s_2(k^2,p^2)}r_2\left(\frac{k^2}{p^2}\right)$$
$$-\frac{1}{2}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) - W_2\left(\frac{p^2}{k^2}\right)\right] \ . \quad (51)$$

Solving the last two equations for τ_2 and τ_3 in terms of τ_6 and W_2, we obtain

$$\tau_2(k^2,p^2) = \frac{2\xi}{(k^2-p^2)^2}\frac{q_2(k^2,p^2)}{s_2(k^2,p^2)} - 6\frac{\tau_6(k^2,p^2)}{k^2-p^2} - \frac{1}{(k^2-p^2)^2}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) + W_2\left(\frac{p^2}{k^2}\right)\right]$$
$$-\frac{k^2+p^2}{(k^2-p^2)^3}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) - W_2\left(\frac{p^2}{k^2}\right)\right] \ , \quad (52)$$

where

$$q_2(k^2,p^2) = \frac{1}{k^2-p^2}\left[\frac{k^3}{p}\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} - \frac{p^3}{k}\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right] , \tag{53}$$

where $q_2(k^2,p^2)$ is obviously a symmetric function of k and p, and

$$\begin{aligned}\tau_3(k^2,p^2) = &-\frac{k^2+p^2}{k^2-p^2}\tau_6(k^2,p^2) + \frac{1}{k^2-p^2}\frac{1}{s_2(k^2,p^2)}\left[\frac{1}{2}r_2\left(\frac{k^2}{p^2}\right) - \frac{\xi}{3}q_3(k^2,p^2)\right]\\ &-\frac{1}{6}\frac{k^2+p^2}{(k^2-p^2)^2}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) + W_2\left(\frac{p^2}{k^2}\right)\right]\\ &+\frac{1}{6}\frac{k^4+p^4-6k^2p^2}{(k^2-p^2)^3}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) - W_2\left(\frac{p^2}{k^2}\right)\right] ,\end{aligned} \tag{54}$$

where

$$q_3(k^2,p^2) = \frac{kp}{(k^2-p^2)^2}\left[(p^2-3k^2)\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} - (k^2-3p^2)\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right] , \tag{55}$$

where $q_3(k^2,p^2)$ is antisymmetric in k and p. The relation, Eq. (17),

$$\bar{\tau}(k^2,p^2) = \tau_3(k^2,p^2) + \tau_8(k^2,p^2) - \tfrac{1}{2}(k^2+p^2)\tau_2(k^2,p^2)$$

then fixes $\tau_8(k^2,p^2)$:

$$\tau_8(k^2,p^2) = -2\frac{k^2+p^2}{k^2-p^2}\tau_6(k^2,p^2) + \bar{\tau}(k^2,p^2) - \frac{1}{k^2-p^2}\frac{1}{s_2(k^2,p^2)}\left[\frac{1}{2}r_2\left(\frac{k^2}{p^2}\right) - \frac{\xi}{3}q_8(k^2,p^2)\right] \tag{56}$$

$$\begin{aligned}&-\frac{1}{3}\frac{k^2+p^2}{(k^2-p^2)^2}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) + W_2\left(\frac{p^2}{k^2}\right)\right]\\ &-\frac{2}{3}\frac{k^4+p^4}{(k^2-p^2)^3}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) - W_2\left(\frac{p^2}{k^2}\right)\right] ,\end{aligned} \tag{57}$$

where

$$q_8(k^2,p^2) = \frac{1}{(k^2-p^2)^2}\left[\frac{k}{p}(3k^4+p^4)\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} - \frac{p}{k}(k^4+3p^4)\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right] , \tag{58}$$

which is clearly antisymmetric in k and p. Imposing the condition that the vertex and its components should be free of kinematic singularities means that

$$\lim_{k^2\to p^2}(k^2-p^2)\tau_i(k^2,p^2) = 0, \quad i = 2,3,8 ,$$

noting that the antisymmetry of τ_6 means $\tau_6(p^2,p^2) = 0$. Thus

$$W_2(1) + 2W_2'(1) = 2\xi(\nu - s + 1) , \tag{59}$$

where $s = s_c$ at the critical point. The transverse vertex has the correct lowest-order perturbative limit, viz., $\Gamma_T^\mu = \mathcal{O}(\alpha)$, provided

$$W_2(k^2/p^2) = \xi\frac{k}{p}\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} + \mathcal{O}(\alpha) . \tag{60}$$

Since at large momenta we expect the power behavior of Eqs. (24) and (36) even away from criticality, Eq. (59) will hold for all values of the coupling α. In contrast, Eq. (46) is only true at the bifurcation point. Its exact form for all α is not known, but Eq. (38) might suggest

$$\int_0^1 \frac{dx}{\sqrt{x}}W_2(x) \approx \xi\left(1 - \frac{\alpha}{\alpha_c}\right)^{1/2} , \tag{61}$$

to agree with both the $\alpha = 0$ and $\alpha = \alpha_c$ limits, Eqs. (60) and (46). These equations determine our vertex for any W_i $(i = 1,2)$ that satisfy the constraints.

In [18] a plot is shown of the critical coupling, α_c, as a function of the covariant gauge parameter ξ, when the CP vertex is used. We refrain from showing the analogous graph for the presently constructed vertex, as α_c

would be boringly gauge independent. This has been achieved for any choice of the functions $W_i(x)$ $(i = 1, 2)$, that satisfy Eqs. (28), (33), (46), (59), and (60). A simple example of W_1 is $2\nu(1 - 2x)$. There are, of course, an infinity of such functions. In practice, we expect that W_1 should be expressible solely in terms of the ratio $F(k^2)/F(p^2)$, while W_2 should surely also involve $\mathcal{M}(k^2)/\mathcal{M}(p^2)$. However, we have not been able to find simple examples that achieve this. The exact form of the full vertex would, of course, determine these functions precisely. Thus solving the Schwinger-Dyson equation for the three-point function would specify the unknowns. However, that has not been our aim. Our aim is more limited. It is to construct a vertex that ensures the fermion propagator is gauge covariant, multiplicatively renormalizable, and has a gauge-independent chiral-symmetry-breaking phase transition. One does not need to know the exact form of the full vertex to achieve these properties, only the *effective* vertex for the fermion equation, Eq. (1). However, we believe that this effective vertex should nevertheless satisfy the appropriate Ward-Takahashi identity and agree with perturbation theory, at least in the leading logarithmic limit of the weak-coupling regime. This is the construction we have achieved for any functions $W_i(x)$ $(i = 1, 2)$. This effective vertex is thus given by Eqs. (5)–(7), (9), (10), (31), (51)–(58).

IV. CONCLUSIONS

The nonperturbative behavior of the fermion propagator is governed by its Schwinger-Dyson equation. In quenched QED, the self-consistent solution of this equation is determined by the fermion-boson interaction. This in turn satisfies a Schwinger-Dyson equation that relates it to the full four-point function and this four-point function satisfies its own Schwinger-Dyson equation relating it to the five-point function and so on. While the solution of this infinite set of equations represents the whole theory, the complete set is, of course, impossible to solve. Consequently, we need a systematic method of truncation that maintains the key features of the theory: its gauge invariance and multiplicative renormalizability. The only known truncation scheme consistently respecting these properties is perturbation theory. However, the bulk of strong interaction phenomena require a nonperturbative approach. Thus, for example, massless bare matter fields remain massless to all orders in perturbation theory. However, if the interactions are strong enough, a chiral-symmetry-breaking phase may become a possibility. Truncating the nested Schwinger-Dyson equations to just the fermion equation by the *rainbow* approximation, in which the fermion-boson vertex is simply treated as bare, this possibility is realized. However, this approximation is highly gauge dependent with the critical coupling for this phase transition varying by a factor of 2 from $\xi = 0$ to 3 [29]. The present paper defines a truncation of the fermion Schwinger-Dyson equation, which does respect the key properties of the theory. The vertex constructed satisfies the Ward-Takahashi identity, ensures the fermion propagator is multiplicatively renormalizable, agrees with one-loop perturbation theory for large momenta, and enforces a gauge-independent chiral-symmetry-breaking phase transition. This is a step on the way to a meaningful nonperturbative truncation scheme: meaningful in the sense that the fundamental aspects of the physics crucially determining the fermion propagator are thereby encoded in its Schwinger-Dyson equation.

Investigation of how, for a given coupling strength, the generated mass compares with that found using the rainbow approximation requires the solution of the coupled equations for $F(p^2)$ and $\mathcal{M}(p^2)$. Study of the chiral-symmetry-breaking phase transition, using bifurcation analysis, fortunately allows these equations to be uncoupled rigorously. The coupled solution is planned.

The fact that in a (more) realistic version of nonperturbative QED, mass generation is possible makes it more, rather than less, likely that such a phase transition has been observed in heavy-ion collisions [30]. Moreover, it motivates the need for a realistic calculation of $t\bar{t}$ condensates as the source of the electroweak symmetry breaking [31]. A realistic calculation, of course, requires the unquenching of the theory. This brings at once renormalizations of the transverse photon propagator and of the fermion-boson coupling. It is the renormalized coupling, which at the corresponding chiral-symmetry-breaking phase transition, is the physical quantity that must be gauge independent. The need to ensure the multiplicative renormalizability of the now coupled photon propagator, of the fermion-boson coupling, as well as of the fermion propagator, significantly complicates the problem. The fermion-boson vertex (in particular its transverse part) will intimately depend on the photon renormalization function in a nonperturbative way not yet understood.

Thus the complete multiplicative renormalizability of two- and three-point functions brings not merely greater algebraic but also methodological complexity. The results for quenched QED presented here provide the starting point for such an investigation of full QED. The solution to this problem will in turn be the starting point for a study of QCD, where boson self-interactions, so essential for both asymptotic freedom and confinement, will further complicate the analysis whether in covariant or axial gauges. All this is for the future.

ACKNOWLEDGMENTS

We are most grateful to our colleagues Jacques Bloch and Ayşe Kizilersü for lengthy discussions at the start of this work and David Atkinson for his useful comments on the draft version of this paper. A.B. wishes to thank the Government of Pakistan for financial support.

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