Wigner functional approach to quantum field dynamics

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(Received 13 May 1994)

We introduce the Wigner functional representing a quantum field in terms of the field amplitudes and their conjugate momenta. The equation of motion for the functional of a scalar field points out the relevance of solutions of the classical field equations to the time evolution of the quantum field. We discuss the field in thermodynamical equilibrium and find the explicit solution of the equations of motion for the so-called "rollover" phase transition. Finally, we briefly discuss the approximate methods for the evaluation of the Wigner functional that may be used to numerically simulate the initial value problem.

PACS number(s): 11.10.Ef, 03.65.Sq, 05.60.+w

I. INTRODUCTION

Dynamical processes in relativistic quantum fields are usually described in terms of the particle excitations of these fields. However, there are situations where more appropriate degrees of freedom are rather the field amplitudes themselves and their conjugate momenta. This happens when as a first approximation the dynamics is well described by the classical field equations. A well-known example here is the evolution of Higgs fields in the early Universe [1,2], which acquire a finite vacuum expectation value when the temperature falls below the critical temperature for the symmetry-breaking phase transition. The mechanism which drives the Higgs field from the symmetric phase, where the vacuum expectation value vanishes, to the asymmetric phase with a nonzero expectation value is essentially classical. Such phase transitions have attracted much attention in the context of inflationary cosmological models [3], models of galaxy formation [4], the creation of the cosmic baryon number asymmetry during the electroweak phase transition [5], and most recently due to the possibility of creating the so-called misaligned chiral condensates in ultrarelativistic heavy-ion collisions [6].

All these situations have in common that one must study the evolution of a quantum field far off thermal equilibrium, when the initial state of the quantum field is specified. Most theoretical approaches to this initial value problem for quantum fields have been based on dynamical equations for the field expectation values and the Gaussian fluctuations around those [4,6,7], assuming that fluctuations around the most probable path remain small. While this assumption may be correct in certain instances, it is a well-known fact that fluctuations usually do not remain small in dynamical phase transitions, where domain formation and clustering are regularly occurring phenomena.

In this paper we present a formalism which appears to be very convenient to study the evolution of quantum fields from an initial state and which goes beyond the classical description. The central object of our approach is the Wigner function, the *Wigner functional*, in fact, which is the quantum analogue of the classical distribution function in a phase space. Here the phase space, however, is spanned not by the particle coordinates and momenta, but by the field amplitudes and their conjugate momenta. The great advantage of the Wigner function formalism is that, while being fully quantum mechanical, it remains close in its spirit to the classical description, and that the classical limit can always be easily obtained. On the other hand, the simultaneous presence of many "quasiclassical" field configurations can be described with ease, and the fluctuations around these do not have to remain Gaussian.

An additional motivation for our work derives from the fact that there are many interesting investigations of nonlinear classical field equations in Minkowski space. One can mention here studies of the chaotic behavior of classical, in particular, Yang-Mills fields [8], searches for exact solutions of the equations of motion [9] (see, e.g., [10] for recent developments), or very specific predictions concerning multiparticle production in high-energy collisions [11]. Whereas the relevance of classical solutions in Euclidean space is safely grounded because they minimize the Euclidean action, as in the case of instantons [9], no such argument exists for solutions in Minkowski space. Attempts to understand the quantum nature of classical fields have been based on the coherent state representation of Fock space [12] or semiclassical methods [13]. For recent work in this direction, see [14].

In order to avoid unnecessary complications we restrict

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considerations in this paper to scalar quantum fields with quartic self-interactions. After discussing the definition of the Wigner functional we show that the equations of motion have the familiar form of a transport equation in phase space with quantum corrections. We then derive the Wigner functional for a free scalar field in thermal equilibrium. We discuss how the classical phase space distribution is obtained in the high temperature limit, and we obtain the two-point correlation function. We analyze the "rollover" of the scalar field in a second-order phase transition, where the power of the Wigner functional approach becomes apparent, allowing for the evolution of the quantum field along many simultaneous classical paths. We briefly discuss the mean field and semiclassical approximation, and finally we conclude with a suggestion how to obtain numerical solutions to the initial value problem for quantum fields in more complex situations.

II. WIGNER FUNCTIONAL

We consider a real scalar quantum field with the standard Lagrangian density

$$\hat{\mathcal{L}}(t,x)=\frac{1}{2}\partial^{\mu}\hat{\Phi}(t,x)\partial_{\mu}\hat{\Phi}(t,x)-\frac{1}{2}m^{2}\hat{\Phi}^{2}(t,x)+\hat{\mathcal{L}}_{I}(t,x)\,. \tag{1}$$

The interaction Lagrangian $\hat{\mathcal{L}}_I$ is assumed to be a polynomial in $\hat{\Phi}$ such as $\hat{\mathcal{L}}_I=-\frac{\lambda}{4!}\hat{\Phi}^4$. We use carets to distinguish operators from corresponding c numbers. For simplicity most formulas are written for $(1+1)$-dimensional field theory, but the generalization to higher dimensions is straightforward.

Before defining the Wigner functional of the field, let us write down the well-known definition of the single particle Wigner function as [15–18]

$$W(q,p;t)=\int du\, e^{-ipu}\langle q+\tfrac{1}{2}u|\,\hat{\rho}(t)\,|q-\tfrac{1}{2}u\rangle\,, \tag{2}$$

where p and q are the particle momentum and position, respectively, while $\hat{\rho}$ is the time-dependent density matrix operator in the Schrödinger picture. In analogy with (2) we define the *Wigner functional* as

$$W[\Phi(x),\Pi(x);t]=\int\mathcal{D}\varphi(x)\,\exp\left[-i\int dx\,\Pi(x)\varphi(x)\right]\langle\Phi(x)+\tfrac{1}{2}\varphi(x)|\,\hat{\rho}(t)\,|\Phi(x)-\tfrac{1}{2}\varphi(x)\rangle\,. \tag{3}$$

It sometimes appears easier to compute the Wigner functional of the fields in momentum space. However, we then face a slight complication. The field $\Phi(x)$ is real while $\Phi(p)$ is complex, but the real and imaginary components of $\Phi(p)$ are not independent from each other because of the constraint $\Phi(-p)=\Phi^*(p)$. Thus, we adopt the following procedure. The real and imaginary parts of $\Phi(p)$ are treated as independent variables, but $p\in(0,\infty)$ instead of $p\in(-\infty,\infty)$. The Wigner functional is then defined as

$$\widetilde{W}[\Phi(p),\Pi(p);t]=\int\mathcal{D}\varphi(p)\,\exp\left[-i\int_0^{\infty}dp\,[\Pi^*(p)\varphi(p)+\Pi(p)\varphi^*(p)]\right]\langle\Phi(p)+\tfrac{1}{2}\varphi(p)|\,\hat{\rho}(t)\,|\Phi(p)-\tfrac{1}{2}\varphi(p)\rangle\,, \tag{4}$$

with the functional integrations running over real and imaginary components of $\varphi(p)$. The transformation from (3) to (4) involves a Jacobian

$$\det\left[\frac{\delta\Phi(x)}{\delta\Phi(p)}\right]\,, \tag{5}$$

which equals unity on account of the unitarity of the Fourier transformation.

To clarify the physical meaning of the Wigner functional, we consider the expectation value of an operator $\mathcal{O}(\hat{\Phi},\hat{\Pi})$. The expectation value is defined as

$$\langle\mathcal{O}(\hat{\Phi},\hat{\Pi})\rangle=\frac{1}{Z}\,\mathrm{Tr}[\hat{\rho}(t)\,\mathcal{O}(\hat{\Phi},\hat{\Pi})]\,, \tag{6}$$

with

$$Z\equiv\mathrm{Tr}\hat{\rho}=\int\mathcal{D}\Phi(x)\frac{\mathcal{D}\Pi(x)}{2\pi}\,W[\Phi,\Pi;t]\,. \tag{7}$$

As is shown in Appendix A,

$$\langle\mathcal{O}(\hat{\Phi},\hat{\Pi})\rangle=\langle\mathcal{O}(\Phi,\Pi)\rangle\equiv\frac{1}{Z}\int\mathcal{D}\Phi(x)\frac{\mathcal{D}\Pi(x)}{2\pi}\,\mathcal{O}(\Phi,\Pi)\,W[\Phi,\Pi;t]\,, \tag{8}$$

provided that the noncommuting operators in $\mathcal{O}$ are properly symmetrized. Operators corresponding to an asymmetric ordering of the operators $\hat{\Phi}$ and $\hat{\Pi}$ must be explicitly expressed as sums of symmetrized terms.

Equation (8) shows that the Wigner functional has the same role in quantum field theory that the density distribution in the phase space spanned by Φ and Π has in the classical field theory. We emphasize, however, that

the Wigner functional is not always positive definite and thus cannot be interpreted as a probability density in phase space. The analytically tractable cases discussed below, when the Wigner functional is of Gaussian form in the variables Φ and Π, are somewhat exceptional in this respect. Of course, the limited classical interpretability of the Wigner functional does not diminish its usefulness as a representation of the quantum mechanical density matrix with a simple classical limit. This will become obvious in the next section, where we study the relevance of solutions of the classical field equations for the time evolution of the Wigner functional.

III. EQUATION OF MOTION

The density matrix operator satisfies the equation of motion

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)] \,. \tag{9}$$

In this section we refrain from setting $\hbar = 1$, because we want to discuss the classical limit of the evolution equation. The Hamiltonian $\hat{H}$ for the real scalar field is

$$\hat{H} = \frac{1}{2}\int dx\{\hat{\Pi}^2(x) + [\nabla\hat{\Phi}(x)]^2 + m^2\hat{\Phi}^2(x) - 2\hat{\mathcal{L}}_I(x)\} \,, \tag{10}$$

where $\hat{\Pi} \equiv \delta\hat{\mathcal{L}}/\delta\dot{\hat{\Phi}} = \dot{\hat{\Phi}}$ denotes the conjugate momentum operator, and the operators are in the Schrödinger picture.

As discussed in Appendix B, one derives from Eq. (9) the following equation of motion of the Wigner functional (3):

$$\left[\frac{\partial}{\partial t} + \int dx \left(\Pi(x)\frac{\delta}{\delta\Phi(x)} - [m^2\Phi(x) - \nabla^2\Phi(x)]\frac{\delta}{\delta\Pi(x)} + \mathcal{K}_I(x)\right)\right] W[\Phi,\Pi;t] = 0 \,, \tag{11}$$

where

$$\mathcal{K}_I(x) \equiv -\frac{i}{\hbar}\mathcal{L}_I\left(\Phi(x) + \frac{i\hbar}{2}\frac{\delta}{\delta\Pi(x)}\right) + \frac{i}{\hbar}\mathcal{L}_I\left(\Phi(x) - \frac{i\hbar}{2}\frac{\delta}{\delta\Pi(x)}\right) \,. \tag{12}$$

For $\mathcal{L}_I(\Phi) = -\frac{\lambda}{4!}\Phi^4$ we find

$$\mathcal{K}_I = \frac{\lambda}{6}\left(-\Phi^3(x)\frac{\delta}{\delta\Pi(x)} + \frac{\hbar^2}{4}\Phi(x)\frac{\delta^3}{\delta\Pi^3(x)}\right) \,. \tag{13}$$

In particular, the interaction term $\mathcal{K}_I$ always terminates for renormalizable quantum field theories in (3+1) space-time dimensions, since their Lagrangians contain at most quartic interaction terms.

One sees that Eq. (11) has the familiar structure of a transport equation. When the higher derivative term in the interaction (12) is neglected, Eq. (11) can be written in the form of the classical Liouville equation

$$\left[\frac{\partial}{\partial t} + \int dx \left(\frac{\delta H}{\delta\Pi(x)}\frac{\delta}{\delta\Phi(x)} - \frac{\delta H}{\delta\Phi(x)}\frac{\delta}{\delta\Pi(x)}\right)\right] \times W[\Phi,\Pi;t] = 0 \,, \tag{14}$$

where H is the classical Hamiltonian. The neglected term is proportional to $\hbar^2$, showing that corrections to the classical phase space evolution are of second order in $\hbar$. We will later (in Sec. VII) briefly discuss how the quantum corrections to the time evolution of the "classical" Wigner functional could be obtained.

IV. FREE FIELDS IN THERMAL EQUILIBRIUM

Let us consider the scalar field in thermodynamical equilibrium. The density operator is then

$$\hat{\rho} = \frac{1}{Z}e^{-\beta\hat{H}} \,, \tag{15}$$

where $\beta = T^{-1}$ is the inverse temperature and $Z = \mathrm{Tr}\, e^{-\beta\hat{H}}$ is the partition function. Since there is no conserved charge carried by the real scalar field, there is no chemical potential in Eq. (15).

It appears easier first to compute the Wigner functional of the fields in the momentum space (4) and then transform it into the coordinate space representation (3). The Hamiltonian

$$\hat{H}_0 = \int_0^\infty \frac{dp}{2\pi}[\hat{\Pi}^\dagger(p)\hat{\Pi}(p) + (p^2 + m^2)\hat{\Phi}^\dagger(p)\hat{\Phi}(p)] \tag{16}$$

can be identified as the sum of independent harmonic oscillator Hamiltonians each representing the mode of momentum p. Using this analogy one finds, as shown in Appendix C, the equilibrium Wigner functional as

$$\widetilde{W}_\beta[\Phi,\Pi] = C\exp\left[-\frac{1}{2}\beta\int_{-\infty}^\infty \frac{dp}{2\pi}\,\tilde{\Delta}_\beta(p) \times\left(\Pi^*(p)\Pi(p) + E^2(p)\Phi^*(p)\Phi(p)\right)\right] \,, \tag{17}$$

with $E(p) \equiv \sqrt{p^2 + m^2}$, the thermal weight factor

$$\tilde{\Delta}_\beta(p) = \frac{2}{\beta E(p)}\tanh\frac{\beta E(p)}{2} \,, \tag{18}$$

and the normalization factor (V is the quantization volume)

$$C \equiv \exp\left[V\int_{-\infty}^\infty \frac{dp}{2\pi}\,\ln 2\tanh\frac{\beta E(p)}{2}\right] \,, \tag{19}$$

which represents the contributions from zero modes. Since the expressions under the momentum integrals in (17) and (19) are even functions of momentum, the in-

tegrals are extended from $-\infty$ to ∞ not from 0 to ∞, as in the definition (4). The Wigner functional (17) is normalized to unity.

In the high temperature limit, where $\tilde{\Delta}_\beta(p) \to 1$, we reproduce the classical result, i.e.,

$$\widetilde{W}_\beta^{\rm cl}[\Phi,\Pi] = \exp\left[-\frac{1}{2}\beta\int\frac{dp}{2\pi}\,[\Pi^*(p)\Pi(p) + E^2(p)\Phi^*(p)\Phi(p)]\right]. \tag{20}$$

In this case the Wigner functional depends only on the total energy of the system.

One immediately finds from (17) that

$$\langle\hat{\Phi}^\dagger(p)\hat{\Phi}(p)\rangle = \frac{V}{2\,E(p)\tanh\frac{\beta E(p)}{2}}\,,$$
$$\langle\hat{\Pi}^\dagger(p)\hat{\Pi}(p)\rangle = \frac{E(p)V}{2\tanh\frac{\beta E(p)}{2}}\,, \tag{21}$$

where the angular brackets are given by Eq. (8). Therefore,

$$\sqrt{\langle\hat{\Phi}^\dagger(p)\hat{\Phi}(p)\rangle\,\langle\hat{\Pi}^\dagger(p)\hat{\Pi}(p)\rangle} = \frac{V}{2\tanh\frac{\beta E(p)}{2}} \geq \frac{V}{2}\,, \tag{22}$$

showing that the uncertainty principle is built into the Wigner functional.

Knowing the Wigner functional (4) one easily finds the Wigner functional (3). Thus,

$$W_\beta[\Phi(x),\Pi(x)] = C\exp\left[-\beta\int dx\,dx'\,\mathcal{H}(x,x')\right], \tag{23}$$

with

$$\mathcal{H}(x,x') \equiv \frac{1}{2}\,\Delta_\beta(x-x')\,[\Pi(x)\Pi(x') + \nabla\Phi(x)\nabla\Phi(x') + m^2\Phi(x)\Phi(x')]\,, \tag{24}$$

where

$$\Delta_\beta(x) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\,e^{-ipx}\,\tilde{\Delta}_\beta(p)\,. \tag{25}$$

One sees that $\Delta_\beta(x) \approx \delta(x)$ for $m^{-1}, |x| \gg \beta$ which corresponds to the classical limit. For $m = 0$ the integral (25) can be computed analytically in one dimension [19] as

$$\Delta_\beta(x) = \frac{2}{\beta\pi}\,\ln\coth\frac{\pi|x|}{2\beta}\,, \tag{26}$$

and approximated as

$$\Delta_\beta(x) \approx \begin{cases} -\frac{2}{\beta\pi}\ln\frac{|x|}{\beta} & \text{for } |x| \ll \beta, \\ \frac{4}{\beta\pi}\exp(-\frac{\pi|x|}{\beta}) & \text{for } |x| \gg \beta\,. \end{cases} \tag{27}$$

One easily checks that the Wigner functional (23) represents a time-independent solution of the equation of motion (11).

V. EQUILIBRIUM CORRELATION FUNCTION

As usually, the correlation function is obtained from the generating functional, which for the free fields in equilibrium is defined as

$$\mathcal{Z}[j(x)] \equiv \int\mathcal{D}\Phi\frac{\mathcal{D}\Pi}{2\pi}\,W[\Phi,\Pi]\exp\left(\beta\int dx\;\Phi(x)j(x)\right)$$
$$= \int\mathcal{D}\Phi\frac{\mathcal{D}\Pi}{2\pi}\exp\left[-\beta\int dx\,dx'\,[\mathcal{H}(x,x') - \delta(x-x')\Phi(x)j(x')]\right], \tag{28}$$

where $\mathcal{H}(x,x')$ is given by Eq. (24) and $j(x)$ denotes the external classical current. Because of Eq. (8) the correlation function can be expressed as

$$\langle\hat{\Phi}(x)\hat{\Phi}(y)\rangle = \frac{1}{\beta^2}\,\frac{1}{\mathcal{Z}[j]}\,\frac{\delta^2\mathcal{Z}[j]}{\delta j(y)\delta j(x)}\bigg|_{j(x)=0}. \tag{29}$$

As shown in Appendix D, the generating functional is

$$\mathcal{Z}[j(x)] = \mathcal{N}\exp\left[\frac{\beta}{2}\int dx\,dx'\,j(x)\mathcal{G}(x-x')j(x')\right], \tag{30}$$

where $\mathcal{N}$ is the normalization constant and

$$\mathcal{G}(x) = \int\frac{dp}{2\pi}\,\frac{e^{-ipx}}{\tilde{\Delta}_\beta(p)\,[p^2+m^2]}\,, \tag{31}$$

with $\tilde{\Delta}_\beta(p)$ given in Eq. (18). Thus, the correlation function (29) equals

$$\langle\hat{\Phi}(x)\hat{\Phi}(y)\rangle = \mathcal{G}(x-y)\,. \tag{32}$$

The integral (31) can be evaluated analytically [19] in three dimensions for $m = 0$:

$$\mathcal{G}(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}\,\frac{\sinh(2\pi|\mathbf{x}|/\beta)}{\cosh(2\pi|\mathbf{x}|/\beta)-1}\,, \tag{33}$$

$$\mathcal{G}(\mathbf{x}) \approx \begin{cases} \frac{1}{4\pi|\mathbf{x}|} & \text{for } \beta \ll |\mathbf{x}|, \\ \frac{\beta}{4\pi^2|\mathbf{x}|^2} & \text{for } \beta \gg |\mathbf{x}|. \end{cases} \tag{34}$$

For a finite mass the integral (31) can be calculated in an approximate way and for three dimensions one finds

$$\mathcal{G}(\mathbf{x}) \approx \frac{1}{4\pi|\mathbf{x}|}\,e^{-m|\mathbf{x}|} \tag{35}$$

for $m^{-1}, |\mathbf{x}| \gg \beta$, which corresponds to the classical limit, and

$$\mathcal{G}(\mathbf{x}) \approx \frac{\beta m}{4\pi^2|\mathbf{x}|} K_1(m|\mathbf{x}|) \approx \frac{\beta m^{1/2}}{2^{5/2}\pi^{3/2}|\mathbf{x}|^{3/2}} e^{-m|\mathbf{x}|}, \quad (36)$$

where the first approximate equality holds for $m^{-1} \ll \beta$, while the second one requires additionally $m^{-1} \ll |\mathbf{x}|$.

VI. "ROLLOVER" PHASE TRANSITION

Let us consider the scalar field in a variable heat bath, undergoing a phase transition with falling temperature. The mass squared of the field, effectively being temperature dependent, is initially assumed to be positive. We are interested in the evolution of the quantum field with spatially homogeneous initial conditions when, suddenly, due to a rapid decrease of the temperature, the mass squared acquires a negative value. This model is an idealization of instabilities arising in, e.g., inflationary cosmology and during the chiral phase transition in expanding dense hadronic matter. Thus, we study the time evolution of the field described by the Hamiltonian

$$\hat{H}(t) = \int_0^\infty \frac{dp}{2\pi} \left(\hat{\Pi}^\dagger(p)\hat{\Pi}(p) + [m^2(t) + p^2]\, \hat{\Phi}^\dagger(p)\hat{\Phi}(p) \right), \quad (37)$$

with

$$m^2(t) = \begin{cases} m^2 > 0 & \text{for } t < 0, \\ -\mu^2 < 0 & \text{for } t > 0. \end{cases} \quad (38)$$

At $t < 0$ the state of the system is described by the equilibrium Wigner functional (17). To find the system evolution starting with $t = 0$, we split the Hamiltonian (37) for $t > 0$ into stable and unstable modes as

$$\hat{H}(t>0) = \int_0^\mu \frac{dp}{2\pi} \left(\hat{\Pi}^\dagger(p)\hat{\Pi}(p) - \omega_-^2(p)\, \hat{\Phi}^\dagger(p)\hat{\Phi}(p) \right) + \int_\mu^\infty \frac{dp}{2\pi} \left(\hat{\Pi}^\dagger(p)\hat{\Pi}(p) + \omega_+^2(p)\, \hat{\Phi}^\dagger(p)\hat{\Phi}(p) \right), \quad (39)$$

where $\omega_\pm(p) \equiv \sqrt{\pm(p^2 - \mu^2)}$ is a real number.

One sees that the modes are independent from each other; thus, we discuss for a moment a single mode, stable or unstable. Since the Wigner functional equation of motion (14) with $m^2 = -\mu^2$ coincides with the classical Liouville equation, it can be solved in a way which is well known from classical mechanics. Specifically, the single mode equation of motion is solved by

$$\tilde{W}(\Phi_p, \Pi_p; t) = \tilde{W}(\Phi_p(-t), \Pi_p(-t); 0), \quad (40)$$

where $\Phi_p(t)$ is the solution of the classical field equation

$$\frac{d^2}{dt^2}\, \Phi_p(t) \pm \omega_\pm^2(p)\, \Phi_p(t) = 0, \quad (41)$$

with $\Pi_p(t) = \dot{\Phi}_p(t)$ and the initial conditions $\Phi_p(0) = \Phi_p$, $\Pi_p(0) = \Pi_p$.

Solving Eq. (41) one finds the single mode Wigner function (40), and then gets the whole functional as a product of the single mode functions. Thus,

$$\widetilde{W}[\Phi, \Pi; t] = C \exp\left[-\frac{1}{2}\beta \int_{-\infty}^\infty \frac{dp}{2\pi}\, \tilde{\Delta}_\beta(p) \left(\Pi_0^*(p)\Pi_0(p) + E^2(p)\Phi_0^*(p)\Phi_0(p) \right) \right], \quad (42)$$

where

$$\Phi_0(p) = \Phi(p)\cosh(\omega_-(p)t) - \frac{1}{\omega_-(p)}\, \Pi(p)\sinh(\omega_-(p)t) \quad (43)$$

$$\Pi_0(p) = \Pi(p)\cosh(\omega_-(p)t) - \omega_-(p)\, \Phi(p)\sinh(\omega_-(p)t) \quad (44)$$

for $0 < p < \mu$ and

$$\Phi_0(p) = \Phi(p)\cos(\omega_+(p)t) - \frac{1}{\omega_+(p)}\, \Pi(p)\sin(\omega_+(p)t), \quad (45)$$

$$\Pi_0(p) = \Pi(p)\cos(\omega_+(p)t) + \omega_+(p)\, \Phi(p)\sin(\omega_+(p)t), \quad (46)$$

for $p > \mu$. It should be stressed that the solution (42) is exact and fully quantum mechanical.

If one is interested only in the Φ dependence of the Wigner functional, Π is integrated over and one gets

$$\int \frac{\mathcal{D}\Pi}{2\pi} \widetilde{W}[\Phi, \Pi; t] = C' \exp\left[-\frac{\beta}{2} \int_0^\infty \frac{dp}{2\pi}\, \tilde{\Delta}_\beta(p)\, \chi(p,t)\, \Phi^*(p)\Phi(p) \right], \quad (47)$$

where

$$\chi(p,t) = \Theta(\mu - p)\,\frac{E^2(p)\omega_-^2(p)}{\omega_-^2(p)\cosh^2(\omega_-(p)t) + E^2(p)\sinh^2(\omega_-(p)t)} + \Theta(p-\mu)\,\frac{E^2(p)\omega_+^2(p)}{\omega_+^2(p)\cos^2(\omega_+(p)t) + E^2(p)\sin^2(\omega_+(p)t)}\,. \tag{48}$$

Using Eq. (47) one easily finds how the unstable modes grow:

$$\langle\hat\Phi^\dagger(p)\hat\Phi(p)\rangle_t = \langle\hat\Phi^\dagger(p)\hat\Phi(p)\rangle_0\left[\cosh^2(\omega_-(p)t) + \frac{E^2(p)}{\omega_-^2(p)}\sinh^2(\omega_-(p)t)\right], \tag{49}$$

with $\langle\hat\Phi^\dagger(p)\hat\Phi(p)\rangle_0$ given by Eq. (22). The unstable mode growth is usually obtained directly from the classical equation of motion (41). In such a case however, one finds only the first term on the right-hand side (RHS) of Eq. (49). The second term, which results from the proper incorporation of the initial condition and the interplay between Φ and Π in (42), is missing. Then it is argued that the zero-momentum mode grows the fastest. This statement, however, is not quite correct. Approximating the RHS of Eq. (49) as

$$\langle\hat\Phi^\dagger(p)\hat\Phi(p)\rangle_t = \langle\hat\Phi^\dagger(p)\hat\Phi(p)\rangle_0\,[1 + (\mu^2+m^2)t^2 + \mathcal{O}(\omega_-^4(p)t^4)]\,. \tag{50}$$

One sees that initially [for $t \ll \omega_-^{-1}(p)$] all unstable modes ($p^2 < \mu^2$) grow at the same rate, and the zero-momentum mode becomes dominant only at a later stage. This observation might be of physical significance since the experimental detection of the misaligned chiral condensates [6], if they are produced in heavy-ion collisions, would be difficult if the phenomenon occurred only for pions with approximately zero momentum. Equation (50) suggests that the pions with nonzero momentum have a chance to contribute to the condensate.

The two-point correlation function is also of interest here, because it contains information about domain growth after the phase transition. The generating functional $\mathcal{Z}[j]$ defined in (28) depends explicitly on time in the present case. A straightforward calculation shows that the generating functional again has the form (30), where the time-dependent correlation function now is given by

$$\mathcal{G}(x,t) = \int\frac{dp}{2\pi}\,\frac{e^{-ipx}}{\tilde\Delta_\beta(p)\,\chi(p,t)}\,. \tag{51}$$

The late-time properties of $\mathcal{G}(x,t)$ are determined by the exponentially growing modes with $p^2 < \mu^2$, for which $\chi(p,t) \propto \exp[-2\omega_-(p)t] \to 0$. Expanding $\omega_-(p)$ around $p = 0$ and carrying out the three-dimensional Gaussian integral over p one finds

$$\mathcal{G}(\mathbf{x},t) \to \frac{\beta m\mu}{64\tanh(\beta m/2)}\left(1+\frac{\mu^2}{m^2}\right)(\pi\mu t)^{-3/2} \times\exp\left(2\mu t - \frac{\mu\mathbf{x}^2}{4t}\right),\quad \text{as } t\to\infty\,. \tag{52}$$

From this result one can read off that the size of correlated scalar field domains grows as $\langle\mathbf{x}^2\rangle \sim 4t/\mu$.

VII. APPROXIMATE METHODS

There are only a few situations when the exact equation of motion (11) can be solved analytically. Thus, there is an obvious need to develop approximate methods to study problems more complicated than those discussed above. We briefly present in this section three approaches which seem to be promising.

A. Mean-field approximation

The mean-field approximation, which has been frequently used to study symmetry-breaking phase transitions, is implemented by replacing in the initial Lagrangian the terms which are cubic, quartic, etc., in fields by the products of the fields and their expectation values. In the case of $\hat{\mathcal{L}}_I(\Phi) = -\frac{\lambda}{4!}\hat\Phi^4$ the replacement is

$$\frac{\lambda}{4!}\hat\Phi^4(x) \to \frac{\lambda}{4}\langle\hat\Phi^2(x)\rangle\,\hat\Phi^2(x)\,, \tag{53}$$

or equivalently, we replace the mass m^2 in the free-field equation of motion by the effective mass $m_*^2 \equiv m^2 + \frac{\lambda}{2}\langle\hat\Phi^2(x)\rangle$. The combinatorial factor 4! changes into 4 in Eq. (53) because there are 6 ways to select 2 fields out of 4. Since $\langle\hat\Phi^2(x)\rangle$ is determined by the functional W via Eq. (8), we get a self-consistent nonlinear equation for W.

Let us briefly discuss the mean-field approximation for the case of thermodynamical equilibrium. The Wigner functional is then given by Eq. (17) derived for the free fields. However, instead of the mass there is the effective mass determined by the gap equation

$$m_*^2 = m^2 + \frac{\lambda}{2}\,\mathcal{G}(0)\,, \tag{54}$$

where $\mathcal{G}(0)$ is defined by Eq. (31) with m_* substituting m. Before Eq. (54) is solved one has to regulate $\mathcal{G}(0)$ subtracting from it a divergent zero temperature contribution. After this procedure we find the well-known gap equation

$$m_*^2 = m^2 + \frac{\lambda}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\frac{\sqrt{p^2+m_*^2}}{\exp\left(\beta\sqrt{p^2+m_*^2}\right)-1}\,\frac{1}{p^2+m_*^2}\,, \tag{55}$$

which is further analyzed in [2]. It is worth mentioning that the standard result is reproduced only after the correlation function is renormalized.

B. Semiclassical approximation

This approximation might be particularly useful when the system of interest is, obviously, "semiclassical" and the "classical" solution, analytically or numerically, is known. Then, the quantum correction can be found in the following way. The equation of motion (11) is written down in symbolic notation as

$$\left(\hat{K}_{\rm c} + \hbar^2 \hat{K}_{\rm q}\right) W = 0\,, \tag{56}$$

where $\hat{K}_{\rm c}$ and $\hat{K}_{\rm q}$ represent the "classical" and "quantum" parts of the operator acting on the Wigner functional, which is expanded as $W = W_{\rm c} + \hbar^2 W_{\rm q}$. Since $\hat{K}_{\rm c} W_{\rm c} = 0$, we find $W_{\rm q}$ as a solution of the equation

$$\hat{K}_{\rm c} W_{\rm q} = -\hat{K}_{\rm q} W_{\rm c} + O(\hbar^2)\,. \tag{57}$$

For a single particle Wigner function one finds

$$W_{\rm q}(q,p;t) = \int dt'dq'dp'\; G(q,q',p,p';t,t') \times \hat{K}_{\rm q} W_{\rm c}(q',p';t')\,, \tag{58}$$

where the Green function $G(q,q',p,p';t,t')$ solves the equation

$$\hat{K}_{\rm c} G(q,q',p,p';t,t') = \frac{\partial G}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial G}{\partial p} = \delta(t-t')\,\delta(q-q')\,\delta(p-p')\,. \tag{59}$$

Since solutions of the classical equations of motion are assumed to be known, the Green function can be explicitly written as

$$G(q,q',p,p';t,t') = \Theta(t-t')\,\delta(q-\tilde{q}(t)-q'+\tilde{q}(t')) \times \delta(p-\tilde{p}(t)-p'+\tilde{p}(t'))\,, \tag{60}$$

with $(\tilde{q}(t),\tilde{p}(t))$ being the classical solution. The generalization of Eqs. (58) and (60) to the case of a quantum field is trivial, only the notation is more complicated.

C. Numerical simulation of the initial value problem

The main goal of this study has been to find a formalism to analyze temporal evolution of the quantum fields beyond the classical limit. Here we argue that the equation of motion (11) is indeed numerically tractable. The first step is to define the field $\Phi(x)$ on a lattice. Then, the equation of motion (11) resembles a many particle transport equation with every site of the lattice representing a single particle that interacts with an external polynomial potential (due to the mass and interaction terms of the Lagrangian) and with its nearest neighbors (via the gradient term). We can then adopt, e.g., the method developed by John and Remler [20] to solve the quantum transport or quantum Liouville equation of a many particle system. The method is sketched below for a single particle; a generalization to many particles is straightforward.

A single particle Wigner function is represented by a finite sum over discrete points in phase space. Since the Wigner function is not positive definite, the points may have positive or negative signature. Then, each point representing a test particle is evolved in such a way that every step of the classical evolution is followed by a quantum step, which converts a single momentum point into several ones. Specifically, the δ function in momentum space is represented by a Gaussian distribution centered around the classical position. The higher momentum gradients, which are responsible for the quantum evolution, act on this distribution. The resulting function, which has positive and negative components, is again represented by discrete points, i.e., by new test particles with positive or negative signature. To avoid unlimited proliferation of the test particles, those with closely neighboring trajectories can be allowed to merge. It was demonstrated in [20] that the method successfully works provided the initial Wigner function is sufficiently smooth in phase space. We intend to apply this method in future work to study the quantum field evolution.

VIII. SUMMARY AND OUTLOOK

We have introduced the formalism which describes the quantum field dynamics in terms of the field amplitudes Φ and their conjugate momenta Π. The Wigner functional playing the central role in our approach provides a density in a phase space spanned by Φ and Π. We have derived the equation of motion of the functional, which is of the form of the quantum transport or Liouville equation. It has been shown in the last section how the knowledge of the classical solution of this equation can be used to find quantum corrections. The free fields in thermodynamical equilibrium have been discussed in detail, and we found an explicit analytical solution of the field analogue of the "upside-down" harmonic oscillator. The solution incorporates in a very natural way the "thermal" initial conditions; its physical meaning is intuitive and transparent. In the last section we briefly discussed the approximate methods which will be needed to solve more complicated problems.

We intend to apply our approach to study the role of quantum fluctuations in the temporal evolution of the symmetry-breaking phase transitions beyond the Gaussian approximation [7]. As mentioned in the Introduction our formalism also appears useful to investigate the quantum content of solutions of the classical field equations of motion.

ACKNOWLEDGMENTS

The authors are grateful to C. Gong and S. G. Matinyan for fruitful discussions, and to D. B. Fairlie for useful comments about the history of the Wigner function approach to quantum mechanics. St.M. would like to thank to the Physics Department of Duke University for its warm hospitality. This work was partially supported by the U.S. Department of Energy (Grant No. DE-FG05-90ER40592) and by the Polish Committee of Scientific Research (Grant No. 20960-91-01).

APPENDIX A

In this appendix we prove Eqs. (8) and (7), which show that the Wigner functional can be treated as a density in the phase space spanned by Φ and Π. Formula (7) is obvious if one observes that

$$\int \frac{\mathcal{D}\Pi}{2\pi} \exp\left[-i\int dx\, \Pi(x)\varphi(x)\right] = \delta(\varphi(x)) \,. \tag{A1}$$

When the operator $\hat{O}$ depends only on $\hat{\Phi}$ but not $\hat{\Pi}$, one easily gets Eq. (8) since the states $|\Phi\rangle$ are by definition eigenstates of $\hat{\Phi}$ and consequently $\mathcal{O}(\hat{\Phi})|\Phi\rangle = \mathcal{O}(\Phi)|\Phi\rangle$. It is assumed that $\mathcal{O}(\hat{\Phi})$ can be expanded in the power series of $\hat{\Phi}$.

Let us now consider $\hat{O}$ which depends only on $\hat{\Pi}$. Then, we introduce the complete set of the momentum eigenstates and the RHS of Eq. (8) is

$$\frac{1}{Z}\int \mathcal{D}\Phi\mathcal{D}\varphi \frac{\mathcal{D}\Pi}{2\pi}\frac{\mathcal{D}\Pi_1}{2\pi}\frac{\mathcal{D}\Pi_2}{2\pi}\mathcal{O}(\Pi_2)\exp\left[-i\int dx\,\Pi(x)\varphi(x)\right]\langle\Phi+\tfrac{1}{2}\varphi|\Pi_1\rangle\langle\Pi_1|\,\hat{\rho}(t)\,|\Pi_2\rangle\langle\Pi_2|\Phi-\tfrac{1}{2}\varphi\rangle \,. \tag{A2}$$

Further we use the momentum eigenfunction

$$\langle\Phi|\Pi\rangle = \exp\left[i\int dx\,\Pi(x)\Phi(x)\right] \tag{A3}$$

and observe that

$$\mathcal{O}(\Pi_2(x))\exp\left[i\int dx\,\left(\left[\Pi_1-\Pi_2\right]\Phi+\frac{1}{2}\left[\Pi_1+\Pi_2\right]\varphi\right)\right] = \mathcal{O}\left(\frac{1}{i}\frac{\delta}{\delta\varphi(x)}-\frac{1}{2i}\frac{\delta}{\delta\Phi(x)}\right) \times\exp\left[i\int dx\,\left(\left[\Pi_1-\Pi_2\right]\Phi+\frac{1}{2}\left[\Pi_1+\Pi_2\right]\varphi\right)\right] \,. \tag{A4}$$

Performing repeatedly the partial integration we effectively convert $\mathcal{O}\left(\frac{1}{i}\frac{\delta}{\delta\varphi(x)}-\frac{1}{2i}\frac{\delta}{\delta\Phi(x)}\right)$ into $\mathcal{O}(\Pi(x))$ and finally prove Eq. (8) for $\mathcal{O}(\hat{\Pi}(x))$.

When the operator $\hat{O}$ depends simultaneously on $\hat{\Phi}$ and $\hat{\Pi}$, the situation gets somewhat complicated because the operators $\hat{\Phi}$ and $\hat{\Pi}$ do not commute with each other. The direct computation, in particular, shows that

$$\langle\hat{\Pi}(x)\hat{\Phi}(y)\rangle = \langle\Pi(x)\Phi(y)\rangle - \frac{i}{2}\,\delta(x-y) \,, \tag{A5}$$

while

$$\langle\hat{\Phi}(y)\hat{\Pi}(x)\rangle = \langle\Pi(x)\Phi(y)\rangle + \frac{i}{2}\,\delta(x-y) \,. \tag{A6}$$

Therefore,

$$\langle\{\hat{\Pi}(x),\hat{\Phi}(y)\}\rangle = \langle\Pi(x)\Phi(y)\rangle \,, \tag{A7}$$

where $\{\hat{A},\hat{B}\} \equiv \frac{1}{2}\left(\hat{A}\hat{B}+\hat{B}\hat{A}\right)$.

Generalizing the result (A7), one proves the equality $\langle\mathcal{O}(\hat{\Phi},\hat{\Pi})\rangle = \langle\mathcal{O}(\Phi,\Pi)\rangle$ assuming that the pairs of noncommuting operators are symmetrized.

APPENDIX B

We derive here the equation of motion (3.3). For this purpose the Hamiltonian (10) is split as

$$\hat{H} = \hat{H}_\Pi + \hat{H}_\nabla + \hat{H}_m + \hat{H}_I \,, \tag{B1}$$

where $\hat{H}_\Pi$, $\hat{H}_\nabla$, $\hat{H}_m$, and $\hat{H}_I$ correspond to the first, second, third, and fourth terms, respectively, on the RHS of Eq. (10). Now one has to calculate four expressions:

$$G_i \equiv \int \mathcal{D}\varphi \exp\left[-\frac{i}{\hbar}\int dx\,\Pi(x)\varphi(x)\right] \times\langle\Phi+\tfrac{1}{2}\varphi|\,[\hat{H}_\alpha,\hat{\rho}]\,|\Phi-\tfrac{1}{2}\varphi\rangle \,, \tag{B2}$$

with the index $\alpha = \Pi$, ∇, m, and I.

The evaluation of G_m is straightforward since $|\Phi\rangle$ is, by definition, the eigenstate of $\hat{\Phi}$. Thus,

$$G_m = \frac{m^2}{2}\int dx \int \mathcal{D}\varphi \exp\left[-\frac{i}{\hbar}\int dx\, \Pi(x)\varphi(x)\right] \{[\Phi(x)+\tfrac{1}{2}\varphi(x)]^2 - [\Phi(x)-\tfrac{1}{2}\varphi(x)]^2\}\langle\Phi+\tfrac{1}{2}\varphi|\,\hat{\rho}\,|\Phi-\tfrac{1}{2}\varphi\rangle\ . \quad \text{(B3)}$$

Since

$$\varphi(x)\exp\left[-\frac{i}{\hbar}\int dx\,\Pi(x)\varphi(x)\right] = i\hbar\frac{\delta}{\delta\Pi(x)}\exp\left[-\frac{i}{\hbar}\int dx\,\Pi(x)\varphi(x)\right], \quad \text{(B4)}$$

one finds

$$G_m = i\hbar m^2\int dx\,\Phi(x)\,\frac{\delta}{\delta\Pi(x)}\,W[\Phi,\Pi;t]\ . \quad \text{(B5)}$$

Keeping in mind that $\hat{\mathcal{L}}_I$ is a polynomial in $\hat{\Phi}$, one also easily computes G_I as

$$G_I = -\int dx\left[\mathcal{L}_I\Big(\Phi(x)+\frac{i\hbar}{2}\frac{\delta}{\delta\Pi(x)}\Big) - \mathcal{L}_I\Big(\Phi(x)-\frac{i\hbar}{2}\frac{\delta}{\delta\Pi(x)}\Big)\right]W[\Phi,\Pi;t]\ . \quad \text{(B6)}$$

To find G_Π we introduce the complete set of the momentum eigenstates $|\Pi\rangle$ and then

$$G_\Pi = \frac{1}{2}\int dx\int \mathcal{D}\varphi\frac{\mathcal{D}\Pi_1}{2\pi}\frac{\mathcal{D}\Pi_2}{2\pi}\exp\left[-\frac{i}{\hbar}\int dx\,\Pi(x)\varphi(x)\right](\Pi_1^2(x)-\Pi_2^2(x))$$

$$\times\langle\Phi+\tfrac{1}{2}\varphi|\Pi_1\rangle\langle\Pi_1|\,\hat{\rho}\,|\Pi_2\rangle\langle\Pi_2|\Phi-\tfrac{1}{2}\varphi\rangle\ . \quad \text{(B7)}$$

Next we make use of the explicit form of the momentum eigenfunctions (A3) and observe that

$$\frac{2\hbar^2}{i^2}\frac{\delta}{\delta\Phi(x)}\,\frac{\delta}{\delta\varphi(x)}\exp\left\{\frac{i}{\hbar}\int dx\,\left[(\Pi_1-\Pi_2)\Phi+\frac{1}{2}(\Pi_1+\Pi_2)\varphi\right]\right\}$$

$$= (\Pi_1^2-\Pi_2^2)\exp\left\{\frac{i}{\hbar}\int dx\,\left[(\Pi_1-\Pi_2)\Phi+\frac{1}{2}(\Pi_1+\Pi_2)\varphi\right]\right\}\ . \quad \text{(B8)}$$

Since the derivative over φ appears under the integral over φ, we perform the partial integration and as a result we obtain

$$G_\Pi = -i\hbar\int dx\,\Pi(x)\,\frac{\delta}{\delta\Phi(x)}\,W[\Phi,\Pi;t]\ . \quad \text{(B9)}$$

We best find G_∇ defining the field $\Phi(x)$ on a lattice. The Hamiltonian is then

$$\hat{H}_\nabla = \frac{a}{2}\sum_i\left(\frac{\hat{\Phi}_{i+1}-\hat{\Phi}_i}{a}\right)^2, \quad \text{(B10)}$$

where a is the lattice spacing and i numerates the lattice sites. With such a Hamiltonian the computation of G_∇ is very similar to that of G_m and one finds

$$G_\nabla = i\hbar a\sum_i\frac{1}{a^2}(\Phi_{i+1}-\Phi_i)\left(\frac{\partial}{\partial\Pi_{i+1}}-\frac{\partial}{\partial\Pi_i}\right)W(\{\Phi_j\},\{\Pi_j\};t)\ . \quad \text{(B11)}$$

In the continuum limit the above expression reads

$$G_\nabla = i\hbar\int dx\,\nabla\Phi(x)\,\nabla\frac{\delta}{\delta\Pi(x)}\,W[\Phi,\Pi;t]\ . \quad \text{(B12)}$$

After the partial integration we finally get

$$G_\nabla = -i\hbar \int dx\, \nabla^2\Phi(x)\, \frac{\delta}{\delta\Pi(x)}\, W[\Phi,\Pi;t]\,. \tag{B13}$$

Collecting G_Π, G_∇, G_m, and G_I we find the desired equation of motion (11).

APPENDIX C

We derive here the equilibrium Wigner functional of the fields in the momentum space. We introduce the complete set of the energy eigenstates of the two-dimensional (isotropic) oscillator $|n_1,n_2\rangle$ and rewrite Eq. (4) as

$$\widetilde{W}[\Phi,\Pi] = \frac{1}{\mathcal{Z}} \sum_{n_1,n_2} \int \mathcal{D}\varphi \exp\left\{ -\int_0^\infty dp[i(\Pi^*(p)\varphi(p) + \Pi(p)\varphi^*(p)) + \beta E_{n_1,n_2}(p)] \right\}$$

$$\times \langle \Phi + \tfrac{1}{2}\varphi | n_1,n_2\rangle\langle n_2,n_1|\Phi - \tfrac{1}{2}\varphi\rangle\,, \tag{C1}$$

with $E_{n_1,n_2}(p) \equiv \sqrt{p^2+m^2}\,(n_1+n_2+1)$. The eigenfunctions for the two-dimensional (isotropic) oscillator are [21]

$$\langle \Phi_R,\Phi_I|n_1,n_2\rangle = \left(\frac{\alpha^2}{\pi\, 2^{n_1}\, n_1!\, 2^{n_2}\, n_2!}\right)^{1/2} H_{n_1}(\alpha\Phi_R)\, H_{n_2}(\alpha\Phi_I) \exp\left[-\frac{\alpha^2}{2}\left(\Phi_R^2+\Phi_I^2\right)\right]\,, \tag{C2}$$

where $\Phi_{R,I}$ is the real or imaginary part of the field. H_n denotes the Hermite polynomial, while

$$\alpha \equiv \left(\frac{p^2+m^2}{4}\right)^{1/4}\,. \tag{C3}$$

Substituting (C2) into (C1) and using the identity [19]

$$\sum_n^\infty \frac{a^n}{n!}\, H_n(x)H_n(y) = \frac{1}{\sqrt{1-4a^2}} \exp\left[\frac{4axy - 4a^2(x^2+y^2)}{1-4a^2}\right] \tag{C4}$$

which holds for $a < 1/2$, one finds after elementary integration and proper normalization the final formula of the equilibrium Wigner functional (17). The same result can be obtained by analytic continuation and generalization to infinitely many degrees of freedom of the formula given in the Appendix of Ref. [17] for the generating function $G(s)$ for the Wigner functions associated with the energy eigenstates of the harmonic oscillator.

APPENDIX D

The purpose of this appendix is to derive the generating functional (30). We apply the method described in the literature [22] modifying it slightly. After integration over Π and partial integrations with respect to x' and then x, we rewrite Eq. (28) as

$$\mathcal{Z}[j] = \mathcal{C} \int \mathcal{D}\Phi \exp\left\{ -\beta \int dx\, dx' \left[\frac{1}{2}\Delta(x-x')\Phi(x)\left(m^2-\nabla^2\right)\Phi(x') - \delta(x-x')\Phi(x)j(x')\right]\right\}\,. \tag{D1}$$

It is important for these manipulations that $\Delta(x) = \Delta(-x)$. Now we change the variable $\Phi(x)$ into $\Phi(x) + \Phi_0(x)$ demanding that the field $\Phi_0(x)$ satisfies the equation

$$\int dx'\Delta(x-x')\left(-\nabla^2+m^2\right)\Phi_0(x') = j(x)\,. \tag{D2}$$

After integration over Φ and the partial integrations with respect to x and x', we get the result

$$\mathcal{Z}[j] = \mathcal{N} \exp\left[\frac{\beta}{2}\int dx\, \Phi_0(x)j(x)\right]\,. \tag{D3}$$

Substituting into Eq. (D3) the solution of Eq. (D2) in the form

$$\Phi_0(x) = \int dx'\, \mathcal{G}(x-x')j(x')\,, \tag{D4}$$

with $\mathcal{G}$ being the Green function given by Eq. (31), one finally finds the generating functional (30).

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