Anyonic states in Chern-Simons theory

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We discuss the canonical quantization of Chern-Simons theory in $2+1$ dimensions, minimally coupled to a Dirac spinor field, first in the temporal gauge and then in the Coulomb gauge. In our temporal gauge formulation, Gauss's law and the gauge condition $A_0 = 0$ are implemented by embedding the formulation in an appropriate physical subspace. We construct a Fock space of charged particle states that satisfy Gauss's law, and show that they obey fermion, not fractional statistics. The gauge-invariant spinor field that creates these charged states from the vacuum obeys the anticommutation rules that generally apply to spinor fields. The Hamiltonian, when described in the representation in which the charged fermions are the propagating particle excitations that obey Gauss's law, contains an interaction between charge and transverse current densities. We observe that the implementation of Gauss's law and the gauge condition does not require us to use fields with graded commutator algebras or particle excitations with fractional statistics. In our Coulomb gauge formulation, we implement Gauss's law and the gauge condition $\partial_l A_l = 0$ by the Dirac-Bergmann procedure. In this formulation, the constrained gauge fields become functionals of the spinor fields, and are not independent degrees of freedom. The formulation in the Coulomb gauge confirms the results we obtained in the temporal gauge: The "Dirac-Bergmann" anticommutation rule for the charged spinor fields ψ and $\psi^\dagger$ that have both been constrained to obey Gauss's law is precisely identical to the canonical spinor anticommutation rule that generates standard fermion statistics. And we also show that the Hamiltonians for charged particle states in our temporal and Coulomb gauge formulations are identical, once Gauss's law has been implemented in both cases.

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I. INTRODUCTION

In recent work, attention has been focused on anyonic states in gauge theories with Chern-Simons (CS) interactions [1–5]. In this work, gauge-invariant fields have been constructed that create, from a vacuum state, charged particle states with arbitrary spin and fractional statistics. Considerable effort has been devoted to understanding the nature of these states, and the dynamical laws they obey. Some authors have argued that anyons are a consequence of the imposition of Gauss's law on charged states in $(2+1)$-dimensional gauge theories with CS terms in their Lagrangian [1–5]. In these authors' treatment of these models, local operator-valued fields that create charged particles from the vacuum obey either purely commutator or anticommutator algebras; the graded commutator algebras, and the consequent arbitrary spin and fractional statistics, arise with the construction of nonlocal gauge-invariant operators, which these authors consider essential for the imposition of Gauss's law. Other authors have questioned these conclusions [6–9]. The dynamical implications of the CS term in gauge theories coupled to charged matter have also been discussed [5,10–12]. Jackiw and Pi have shown that CS fields coupled to charged matter do not generally produce "pure gauge" interactions that have no effect on the equations of motion [10]. They point out that, in relativistic quantum field theories, the CS vector potential cannot be totally gauged away. In nonrelativistic quantum field theory, in which the CS interaction is pure gauge, Jackiw and Pi have exploited the pure gauge nature of the CS interaction to remove the entire gauge field from the Hamiltonian with a gauge transformation. The correspondingly transformed charged field operator $\Psi^0(\mathbf{r})$ does not commute with $\Psi^0(\mathbf{r}')$, but obeys a graded commutator algebra. N-particle orbitals, represented by appropriately selected matrix elements of products of these transformed Schrödinger field operators, are multivalued. The constraint imposed by the multivalued boundary condition carries the information contained in the gauge fields before they were eliminated by the gauge transformation, and produces charged N-particle orbitals that describe an interacting system of particles. Other investigators have used line integrals over gauge fields to construct gauge-invariant field operators that obey graded commutator algebras for relativistic quantum field theories interacting with a CS field [4]. These authors have identified the excitations of these gauge-invariant fields as anyonic states with arbitrary spin and fractional statistics.

In our work, we address this question from a somewhat different point of view. We investigate a $(2+1)$-dimensional gauge theory in which the gauge field is minimally coupled to a charged spinor field. The Lagrangian contains a CS term, but no Maxwell kinetic-energy term. The gauge field obeys canonical commutation rules, and the spinor field anticommutation rules. We construct a Fock space of N-particle charged states [the $(2+1)$-dimensional analogues of electrons and positrons] that satisfy Gauss's law. In the process we construct a gauge-invariant operator-valued spinor field that creates, from a vacuum state, the charged particle states that satisfy

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Gauss's law. We demonstrate that this gauge-invariant spinor field obeys anticommutation rules; and the excitations of the gauge-invariant spinor field, which satisfy Gauss's law, obey fermion rather than fractional statistics. Moreover, it is possible to define these states so that they change sign in a 2π rotation, regardless of the value of the CS coupling constant.

We do not argue that our results invalidate either the anyonic descriptions of particle states in CS theory, or the gauge invariance of the charged fields discussed in Ref. [4]. We do argue that an anyonic description is not the only possible one for this theory; and in particular, that it is not required for the implementation of Gauss's law. We demonstrate in this work that it is possible to formulate a consistent description of the charged particle excitations as "normal" fermions which obey Gauss's law and ordinary fermion statistics, and which interact through a nonlocal interaction mediated by the CS field. The availability, in this theory, of a Fock space of states with normal statistics is consistent with the work of Jackiw and Pi [10]; these authors have shown that nonrelativistic charged bosons coupled to a CS field can be described by either of two Schrödinger field operators, $\Psi(\mathbf{r})$ or $\Psi^0(\mathbf{r})$. $\Psi(\mathbf{r})$ obeys "ordinary" canonical commutation rules, while $\Psi^0(\mathbf{r})$ is subject to a graded commutator algebra, in which $\Psi^0(\mathbf{r})$ and $\Psi^0(\mathbf{r}')$ do not commute. In the representation in which $\Psi(\mathbf{r})$ is the appropriate Schrödinger field operator, explicit nonlocal charged particle interactions appear in the Hamiltonian. In the representation in which $\Psi^0(\mathbf{r})$ is the appropriate Schrödinger field operator, these explicit interactions have been replaced by equivalent boundary conditions. Both representations implement Gauss's law. We observe that in our work, an explicit interaction between charge and transverse current densities appears in the Hamiltonian, in the representation in which the electron and positron operators create (or annihilate) charged particles that obey Gauss's law. A similar interaction also is reported in Ref. [10] in the representation in which the orbitals of the boson field $\Psi(\mathbf{r})$ are used to describe the interacting particles. Our result, that CS theory coupled to relativistic charged fermions can be formulated in a Fock space of charged fermion states that satisfy Gauss's law as well as normal statistics, is consistent with the results of Jackiw and Pi for nonrelativistic charged bosons.

As has been noted, CS theories do not possess any observable propagating modes of the gauge field [13]. Only the charged fermion field gives rise to observable propagating particle excitations which interact with each other through their interaction with the gauge field. In our work in the temporal gauge, we treat this model much as we have previously treated the topologically massive Maxwell-Chern-Simons (MCS) theory [14]. We introduce a gauge-fixing field in such a way that A_0 has a conjugate momentum and obeys canonical commutation rules. Although, as in our treatment of MCS theory, Gauss's law and the gauge condition are not primary constraints, there are nevertheless other primary constraints in CS theory. Primary constraints relate the canonically conjugate momentum of A_1 to A_2, and vice versa, so that the constrained gauge field A_l will be subject to Dirac rather than Poisson commutation rules. Furthermore, all components of the CS gauge field, A_1 and A_2 as well as A_0, must be represented entirely in terms of ghost operators, which can mediate interactions between charges and currents but do not carry energy-momentum, and have no probability of being observed. Neither longitudinal nor transverse components of the CS fields have any propagating particle-like excitations.

In our Coulomb gauge formulation we implement all constraints, including Gauss's law and the gauge condition, $\partial_l A_l = 0$, by the Dirac-Bergmann (DB) procedure [15,16]. We include a gauge-fixing term $-G\partial_l A_l$ in the Lagrangian, in order to provide for the systematic development of all constraints, including the gauge condition, from the DB algorithm. In the Coulomb gauge, the gauge fields have no independent degrees of freedom whatsoever, but are reduced to functionals of the spinor fields. The constrained fields obey Dirac commutation (anticommutation) rules which must be evaluated, and which may, and often do, differ from the commutation (anticommutation) rules of the corresponding unconstrained fields. There is therefore an opportunity for discrepancies between the commutator (anticommutator) algebras for constrained and unconstrained fields to arise. The Dirac anticommutation rules among the constrained spinor fields are of particular significance, because a graded anticommutation algebra among the spinor fields may signal the development of "exotic" fractional statistics by their particle excitations.

II. FORMULATION OF THE THEORY IN THE TEMPORAL GAUGE

The Lagrangian for this model is given by

$$\mathcal{L} = \tfrac{1}{4}m\epsilon_{ln}(F_{ln}A_0 - 2F_{n0}A_l) - \partial_0 A_0 G + j_l A_l - j_0 A_0 + \bar{\psi}(i\gamma^\mu\partial_\mu - M)\psi \;, \tag{1}$$

where $F_{ln} = \partial_n A_l - \partial_l A_n$ and $F_{l0} = \partial_l A_0 + \partial_0 A_l$. We follow conventions identical to those in Ref. [14].

The Euler-Lagrange equations are

$$m\epsilon_{ln}F_{n0} - j_l = 0, \tag{2}$$

$$\tfrac{1}{2}m\epsilon_{ln}F_{ln} + \partial_0 G - j_0 = 0, \tag{3}$$

$$\partial_0 A_0 = 0, \tag{4}$$

and

$$(M - i\gamma^\mu D_\mu)\psi = 0, \tag{5}$$

where D_μ is the gauge-covariant derivative $D_\mu = \partial_\mu + ieA_\mu$. Current conservation leads to

$$\partial_0\partial_0 G = 0. \tag{6}$$

The momenta conjugate to the fields are given by

$$\Pi_0 = -G \tag{7}$$

and

$$\Pi_l = \tfrac{1}{2} m\epsilon_{ln} A_n. \tag{8}$$

The Hamiltonian density is given by

$$\mathcal{H} = -\tfrac{1}{2} m\epsilon_{ln} F_{ln} A_0 + j_0 A_0 - j_l A_l + \mathcal{H}_{e\bar{e}} \ , \tag{9}$$

where $\mathcal{H}_{e\bar{e}} = \psi^\dagger(\gamma^0 M - i\gamma^0\gamma^l\partial_l)\psi$ and the total derivative $\partial_n(\frac{1}{2}m\epsilon_{ln}A_lA_0)$ has been dropped.

The use of the gauge-fixing term $-\partial_0 A_0 G$ in the Lagrangian $\mathcal{L}$ leads to the equal-time commutation rule (ETCR)

$$[A_0(\mathbf{x}), G(\mathbf{y})] = -i\delta(\mathbf{x}-\mathbf{y}), \tag{10}$$

and elementary considerations lead to the equal-time anticommutation rule

$$\{\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{y})\} = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}), \tag{11}$$

for the fermion fields. But the naive use of Eq. (8) to set

$$\begin{aligned}[A_l(\mathbf{x}), \Pi_n(\mathbf{y})] &= [A_l(\mathbf{x}), \tfrac{1}{2} im\epsilon_{nk} A_k(\mathbf{y})] \\ &= i\delta_{ln}\delta(\mathbf{x}-\mathbf{y}) \ , \end{aligned} \tag{12}$$

and, after contraction over ϵ_{nk},

$$[A_l(\mathbf{x}), A_n(\mathbf{y})] = \frac{2i}{m}\epsilon_{ln}\delta(\mathbf{x}-\mathbf{y}), \tag{13}$$

is incorrect, because it ignores the fact that $\Pi_l - \frac{1}{2}m\epsilon_{ln}A_n = 0$ constitutes a primary constraint. There are various ways to arrive at the correct ETCR [15–17]. One way is to use the Dirac-Bergmann (DB) procedure [15,16], for which we need the Poisson commutator matrix

$$\mathcal{M}_{ln}(\mathbf{x},\mathbf{y}) = [\mathcal{C}_l(\mathbf{x}), \mathcal{C}_n(\mathbf{y})] = -im\epsilon_{ln}\delta(\mathbf{x}-\mathbf{y}) \tag{14}$$

for the primary constraints

$$\mathcal{C}_l = \Pi_l - \tfrac{1}{2} m\epsilon_{ln} A_n. \tag{15}$$

To implement the DB procedure we form the total Hamiltonian density

$$\mathcal{H}_{\mathrm{T}} = \mathcal{H} + \sum_{l=1}^{2} \mathcal{C}_l \mathcal{U}_l \ , \tag{16}$$

where the $\mathcal{U}_l$ are arbitrary c-number functions. The commutator $[H_{\mathrm{T}}, \mathcal{C}_i(\mathbf{x})]$ for $H_{\mathrm{T}} = \int d\mathbf{x}\ \mathcal{H}_{\mathrm{T}}(\mathbf{x})$, then is

$$[H_{\mathrm{T}}, \mathcal{C}_i(\mathbf{x})] = [H, \mathcal{C}_i(\mathbf{x})] + \sum_l \int d\mathbf{y}\ \mathcal{U}_l(\mathbf{y})[\mathcal{C}_l(\mathbf{y}), \mathcal{C}_i(\mathbf{x})] \ , \tag{17}$$

where the brackets represent canonical "Poisson" commutators. Equation (17) leads to

$$m\epsilon_{il}(\mathcal{U}_l + \partial_l A_0) - j_i = 0, \tag{18}$$

so that $\mathcal{U}_l$ is identified as $\mathcal{U}_l = \partial_0 A_l$, and no secondary constraints are generated. Having established that the two primary constraints given in Eq. (15) do not give rise to any secondary constraints, we recognize the two primary constraints $\mathcal{C}_l \approx 0$ as a system of second-class constraints, and use $\mathcal{Y}(\mathbf{x},\mathbf{y})$, the inverse of $\mathcal{M}(\mathbf{x},\mathbf{y})$, to obtain the Dirac commutator

$$\begin{aligned}[A_l(\mathbf{x}), An(\mathbf{y})]^{\mathrm{D}} = -\int d\mathbf{z}\, d\mathbf{z}'\ [A_l(\mathbf{x}), C_k(\mathbf{z})]\mathcal{Y}_{kk'}(\mathbf{z},\mathbf{z}') \\ \times[C_{k'}(\mathbf{z}'), A_n(\mathbf{y})]. \end{aligned} \tag{19}$$

The resulting correct expression for the commutator $[A_l(\mathbf{x}), A_n(\mathbf{y})]$ is the Dirac commutator

$$[A_l(\mathbf{x}), A_n(\mathbf{y})] = \frac{i}{m}\epsilon_{ln}\delta(\mathbf{x}-\mathbf{y}). \tag{20}$$

We now construct the following momentum space expansions of the gauge fields in such a way that the ETCR given in Eqs. (10) and (20) are satisfied:

$$\begin{aligned} A_l(\mathbf{x}) &= \sum_{\mathbf{k}} \frac{k_l}{2m^{3/2}}\left[a_R(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\star_R(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \\ &+ \sum_{\mathbf{k}} \frac{i\sqrt{m}\epsilon_{ln}k_n}{k^2}\left[a_Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\star_Q(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \\ &+ \sum_{\mathbf{k}} i\phi(\mathbf{k})k_l\left[a_Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\star_Q(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \end{aligned} \tag{21}$$

$$A_0(\mathbf{x}) = \sum_{\mathbf{k}} \frac{i}{\sqrt{m}}\left[a_Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\star_Q(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{22}$$

and

$$G(\mathbf{x}) = -\sum_{\mathbf{k}} \frac{\sqrt{m}}{2}\left[a_R(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\star_R(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \ , \tag{23}$$

where $\phi(\mathbf{k})$ is some arbitrary real and even function of $\mathbf{k}$. The magnetic field B and the electric field $\mathbf{E}$ are given by

$$B(\mathbf{x}) = -\sum_{\mathbf{k}} \sqrt{m}\left[a_Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\star_Q(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{24}$$

and by $E_l = -\partial_l A_0 - i[H, A_l]$ so that

$$E_l(\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{m}\epsilon_{ln} j_n(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \tag{25}$$

as shown in Eq. (2). The explicit form of $\phi(\mathbf{k})$ is immaterial to the commutation rules given in Eqs. (10) and (20); its form as well as its inclusion in Eq. (21) are therefore entirely optional. The operators $a_Q(\mathbf{k})$ and $a_R(\mathbf{k})$ and their Hermitian adjoints $a^\star_Q(\mathbf{k})$ and $a^\star_R(\mathbf{k})$ are the same ghost operators previously used for the MCS theory [14]; they obey the commutation rules

$$[a_Q(\mathbf{k}), a^\star_R(\mathbf{q})] = [a_R(\mathbf{k}), a^\star_Q(\mathbf{q})] = \delta_{\mathbf{kq}} \tag{26}$$

and

$$[a_Q(\mathbf{k}), a^\star_Q(\mathbf{q})] = [a_R(\mathbf{k}), a^\star_R(\mathbf{q})] = 0. \tag{27}$$

The use of ghosts is appropriate and necessary for components of gauge fields which have nonvanishing commutators with each other, but which do not exhibit any observable, propagating excitations. The representation of the gauge fields in terms of ghost excitations only, therefore, tests the principle that no observable excitation modes are required to represent the commutation rules given in Eqs. (10) and (20).

The Hamiltonian $H = \int d\mathbf{x}\ \mathcal{H}(\mathbf{x}) = H_0 + H_{\rm I}$, where H_0 and $H_{\rm I}$ are given by

$$H_0 = -\int d\mathbf{x}\ \tfrac{1}{2}m\epsilon_{ln}F_{ln}A_0 + H_{e\bar{e}}$$
$$= \sum_{\mathbf{k}} im\left[a_Q(\mathbf{k})a_Q(-\mathbf{k}) - a_Q^\star(\mathbf{k})a_Q^\star(-\mathbf{k})\right] + H_{e\bar{e}} \tag{28}$$

with $H_{e\bar{e}} = \int d\mathbf{x}\ \mathcal{H}_{e\bar{e}}(\mathbf{x})$ and

$$H_{\rm I} = \sum_{\mathbf{k}} \frac{i}{\sqrt{m}}\left[a_Q(\mathbf{k})j_0(-\mathbf{k}) - a_Q^\star(\mathbf{k})j_0(\mathbf{k})\right]$$
$$-\sum_{\mathbf{k}} \frac{k_l}{2m^{3/2}}\left[a_R(\mathbf{k})j_l(-\mathbf{k}) + a_R^\star(\mathbf{k})j_l(\mathbf{k})\right]$$
$$-\sum_{\mathbf{k}} \frac{i\sqrt{m}\epsilon_{ln}k_n}{k^2}\left[a_Q(\mathbf{k})j_l(-\mathbf{k}) - a_Q^\star(\mathbf{k})j_l(\mathbf{k})\right]$$
$$-\sum_{\mathbf{k}} i\phi(\mathbf{k})k_l\left[a_Q(\mathbf{k})j_l(-\mathbf{k}) - a_Q^\star(\mathbf{k})j_l(\mathbf{k})\right]. \tag{29}$$

The total Hamiltonian $H_{\rm T}$ reduces to the canonical Hamiltonian H on the constraint surface on which all $\mathcal{C}_i$'s are zero, and it correctly implements time evolution when the Dirac commutation rule given in Eq. (20) is used. This can easily be demonstrated by observing that when the commutators $i[H, A_i]$, $i[H, A_0]$, and $i[H, G]$ are substituted for $\partial_0 A_i$, $\partial_0 A_0$, and $\partial_0 G$, respectively, the Euler-Lagrange equations are obtained. The other constraints, $\Pi_0 + G = 0$ and $\Pi_G = 0$, have no further effect on the commutation rules for the gauge fields.

We will implement the gauge constraint, $A_0 = 0$, and Gauss's law not by using the DB procedure but, as in earlier work [14,18], by confining the dynamical time evolution to an appropriately chosen subspace of the Hilbert space $\{|h\rangle\}$ in which the Hamiltonian H operates. The Hilbert space $\{|h\rangle\}$ very closely resembles the Hilbert space used in Ref. [14]; $\{|h\rangle\}$ is based on the perturbative vacuum $|0\rangle$ annihilated by all annihilation operators, $a_Q(\mathbf{k})$ and $a_R(\mathbf{k})$ as well as the electron and positron annihilation operators $e(\mathbf{k})$ and $\bar{e}(\mathbf{k})$, respectively. The Hilbert space $\{|h\rangle\}$ contains a subspace $\{|n\rangle\}$ that consists of all multiparticle electron-positron states of the form $|N\rangle = \bar{e}^\dagger(\mathbf{q}_1)\cdots\bar{e}^\dagger(\mathbf{q}_l)e^\dagger(\mathbf{p}_1)\cdots e^\dagger(\mathbf{p}_n)|0\rangle$, as well as all other states of the form $a_Q^\star(\mathbf{k}_1)\cdots a_Q^\star(\mathbf{k}_i)|N\rangle$. We note that the commutation rules for the ghost operators given in Eqs. (26) and (27) demonstrate that the states $a_Q^\star(\mathbf{k}_1)\cdots a_Q^\star(\mathbf{k}_i)|N\rangle$ have zero norm, since $\langle N|a_Q(\mathbf{k}_i)\cdots a_Q(\mathbf{k}_1)a_Q^\star(\mathbf{k}_1)\cdots a_Q^\star(\mathbf{k}_i)|N\rangle$ can be rewritten as $\langle N|a_Q^\star(\mathbf{k}_i)\cdots a_Q^\star(\mathbf{k}_1)a_Q(\mathbf{k}_1)\cdots a_Q(\mathbf{k}_i)|N\rangle$ and each of the $a_Q(\mathbf{k}_i)$ annihilates any state $|N\rangle$. The states in the subspace $\{|n\rangle\}$ therefore are either free of ghosts, or if they contain ghosts, they are zero norm states. H_0 time translates all states in $\{|n\rangle\}$ so that they remain contained within it; and the matrix elements of H_0 within $\{|n\rangle\}$, i.e., matrix elements of the form $\langle n_b|H_0|n_a\rangle$, always vanish when $|n_a\rangle$ or $|n_b\rangle$ contains any $a_Q^\star$ ghosts. States in which $a_R^\star(\mathbf{k})$ operators act on a state $|n\rangle$, such as $a_R^\star(\mathbf{q}_1)\cdots a_R^\star(\mathbf{q}_l)a_Q^\star(\mathbf{k}_1)\cdots a_Q^\star(\mathbf{k}_n)|N\rangle$, are included in $\{|h\rangle\}$, but excluded from $\{|n\rangle\}$. Such states are not probabilistically interpretable and their appearance in the course of time evolution signals an inconsistency in the theory. In the next section we will show how the implementation of Gauss's law and the gauge choice averts the development of this inconsistency. Lastly, it should be noted that the unit operator in the one-particle ghost (OPG) sector is given by

$$1_{\rm OPG} = \sum_{\mathbf{k}}\left[a_Q^\star(\mathbf{k})|0\rangle\langle 0|a_R(\mathbf{k}) + a_R^\star(\mathbf{k})|0\rangle\langle 0|a_Q(\mathbf{k})\right]. \tag{30}$$

For multiparticle ghost sectors, the obvious generalization of Eq. (30) applies.

III. THE ROLE OF GAUSS'S LAW

As in all other gauge theories, Gauss's law is not an equation of motion in CS theory. The operator $\mathcal{G}(\mathbf{x})$ used to implement Gauss's law is

$$\mathcal{G}(\mathbf{x}) = j_0(\mathbf{x}) - \tfrac{1}{2}m\epsilon_{ln}F_{ln}(\mathbf{x}), \tag{31}$$

and whereas $\partial_0 G = \mathcal{G}$, $\partial_0\partial_0 G = \partial_0\mathcal{G} = 0$ is the equation of motion that governs the behavior of this model. Further measures must be taken to implement $\mathcal{G} = 0$. We can conveniently express $\mathcal{G}$ in the form

$$\mathcal{G}(\mathbf{x}) = \sum_{\mathbf{k}} m^{3/2}\left[a_Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a_Q^\star(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} + \frac{j_0(\mathbf{k})}{m^{3/2}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\right], \tag{32}$$

where $j_0(\mathbf{k}) = \int d\mathbf{x}\ j_0(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$. We can define an operator $\Omega(\mathbf{k})$ as

$$\Omega(\mathbf{k}) = a_Q(\mathbf{k}) + \frac{1}{2m^{3/2}}j_0(\mathbf{k}), \tag{33}$$

so that

$$\mathcal{G}(\mathbf{x}) = \sum_{\mathbf{k}} m^{3/2}\left[\Omega(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \Omega^\star(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{34}$$

Similarly, we can write $A_0(\mathbf{x})$ as

$$A_0(\mathbf{x}) = \sum_{\mathbf{k}} \frac{i}{\sqrt{m}}\left[\Omega(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - \Omega^\star(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{35}$$

We can therefore implement Gauss's law and the gauge condition by embedding the theory in a subspace $\{|\nu\rangle\}$ of another Hilbert space. The subspace $\{|\nu\rangle\}$ consists of the states $|\nu\rangle$ which satisfy the condition

$$\Omega(\mathbf{k})|\nu\rangle = 0. \tag{36}$$

It can be easily seen from Eqs. (34) and (35) that, in the physical subspace $\{|\nu\rangle\}$, $\langle\nu'|\mathcal{G}|\nu\rangle = 0$ and $\langle\nu'|A_0|\nu\rangle = 0$, so that both Gauss's law and the gauge condition $A_0 = 0$ hold. Moreover, the condition $\Omega(\mathbf{k})|\nu\rangle = 0$, once established, continues to hold at all other times because

$$[H, \Omega(\mathbf{k})] = 0 \tag{37}$$

so that $\Omega(\mathbf{k})$ is an operator-valued constant. This demonstrates that a state initially in the physical subspace $\{|\nu\rangle\}$ will always remain entirely contained within it as it develops under time evolution.

Consider now the unitary transformation $U = e^D$, where

$$D = -i\int d\mathbf{x}\, d\mathbf{y}\, \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2}\, \partial_l A_l(\mathbf{x}) j_0(\mathbf{y}). \tag{38}$$

It is easy to show that

$$U^{-1}\Omega(\mathbf{k})U = a_Q(\mathbf{k}). \tag{39}$$

We can use U to establish a mapping that maps $\Omega(\mathbf{k}) \to a_Q(\mathbf{k})$ and $\{|\nu\rangle\} \to \{|n\rangle\}$, where $\{|n\rangle\}$ is the subspace described in the preceding section. In this mapping, operators $\mathcal{P}$ map into $\tilde{\mathcal{P}}$, i.e., $U^{-1}\mathcal{P}U = \tilde{\mathcal{P}}$. For example, $\tilde{\Omega}(\mathbf{k}) = a_Q(\mathbf{k})$, and $\tilde{H} = U^{-1}HU$ is given by

$$\tilde{H} = H_0 - \sum_{\mathbf{k}} \frac{i\epsilon_{ln}k_n}{mk^2}\, j_l(\mathbf{k}) j_0(-\mathbf{k}) - \sum_{\mathbf{k}} \frac{i\sqrt{m}\epsilon_{ln}k_n}{k^2}\left[a_Q(\mathbf{k})j_l(-\mathbf{k}) - a_Q^\star(\mathbf{k})j_l(\mathbf{k})\right]. \tag{40}$$

The similarly transformed fields are

$$\tilde{A}_l(\mathbf{x}) = A_l(\mathbf{x}) - \sum_{\mathbf{k}} \frac{ik_l}{m^{3/2}}\, \phi(\mathbf{k}) j_0(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - \sum_{\mathbf{k}} \frac{i\epsilon_{ln}k_n}{mk^2}\, j_0(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{41}$$

$$\tilde{A}_0(\mathbf{x}) = A_0(\mathbf{x}), \quad \tilde{G}(\mathbf{x}) = G(\mathbf{x}), \tag{42}$$

and

$$\tilde{\psi}(\mathbf{x}) = \exp\left[\mathcal{D}_{\mathrm{U}}(\mathbf{x})\right]\psi(\mathbf{x}), \tag{43}$$

where

$$\mathcal{D}_{\mathrm{U}}(\mathbf{x}) = -ie\int d\mathbf{y}\, \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2} \partial_l A_l(\mathbf{y}). \tag{44}$$

The transformed electric and magnetic fields are

$$\tilde{E}_l(\mathbf{x}) = E_l(\mathbf{x}) \tag{45}$$

and

$$\tilde{B}(\mathbf{x}) = B(\mathbf{x}) + \mathcal{B}(\mathbf{x}), \tag{46}$$

where $E_l(\mathbf{x})$ and $B(\mathbf{x})$ are given by Eqs. (25) and (24), respectively, and

$$\mathcal{B}(\mathbf{x}) = \frac{j_0(\mathbf{x})}{m}. \tag{47}$$

Equations (43) and (44) are of particular importance to one of the questions we are investigating, i.e., whether imposing Gauss's law on the charged particle states of this theory causes them to develop "exotic" fractional statistics. If the anticommutators for the spinor fields that implement Gauss's law, $\{\tilde{\psi}(\mathbf{x}), \tilde{\psi}^\dagger(\mathbf{y})\}$ and $\{\tilde{\psi}(\mathbf{x}), \tilde{\psi}(\mathbf{y})\}$, differ from the canonical spinor anticommutators $\{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})$ and $\{\psi(\mathbf{x}), \psi(\mathbf{y})\} = 0$ that account for the fact that the excitations of ψ and $\psi^\dagger$ are subject to Fermi statistics, then that difference may signal that the excitations of $\tilde{\psi}$ and $\tilde{\psi}^\dagger$ are subject to fractional statistics. We also note that $\tilde{\psi}(\mathbf{x})$ is gauge invariant; if we gauge transform $\tilde{\psi}(\mathbf{x})$ within the confines of the temporal gauge, then the effect of that gauge transformation on $\mathcal{D}_{\mathrm{U}}(\mathbf{x})$ and on $\psi(\mathbf{x})$ cancel, so that the spinor field $\tilde{\psi}(\mathbf{x})$ is gauge invariant. This gauge invariance is necessary for excitations of $\tilde{\psi}(\mathbf{x})$ to obey Gauss's law.

To show that the anticommutation rules for $\tilde{\psi}$ and $\tilde{\psi}^\dagger$ are identical to the anticommutation rules for the unconstrained ψ and $\psi^\dagger$, we observe that $\psi(\mathbf{x})$ and $A_l(\mathbf{y})$ [and therefore also $\psi(\mathbf{x})$ and $\mathcal{D}_{\mathrm{U}}(\mathbf{y})$] commute at equal times, so that $\{\tilde{\psi}(\mathbf{x}), \tilde{\psi}^\dagger(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})$ and $\{\tilde{\psi}(\mathbf{x}), \tilde{\psi}(\mathbf{y})\} = 0$. The constrained fields $\tilde{\psi}$ and $\tilde{\psi}^\dagger$ obey the same anticommutation rules as the unconstrained ψ and $\psi^\dagger$, and are not subject to any exotic graded anticommutator algebra. The electron and positron states that implement Gauss's law therefore obey standard Fermi, not fractional, statistics. This result can also be demonstrated from the fact the the transformed fields $\tilde{\psi}$, $\tilde{\psi}^\dagger$, $\tilde{A}_l(\mathbf{x})$, and $\tilde{\Pi}_l(\mathbf{x})$ are unitarily equivalent to ψ, $\psi^\dagger$, $A_l(\mathbf{x})$, and $\Pi_l(\mathbf{x})$ respectively, and that commutators and anticommutators that are equal to c numbers, are invariant to unitary transformations. It is, of course, important to keep in mind that the particular form of $\tilde{\psi}(\mathbf{x})$ given in Eq. (43) only applies to the temporal gauge and to this method of quantization. In other gauges, and with other methods of implementing constraints, the spinor fields that implement Gauss's law will have a different representation, and questions about the statistics of electron-positron states that obey Gauss's law arise in a different way. We will formulate this theory in the Coulomb gauge in later sections of this paper, and in that work confirm the result that the charged particle states obey standard Fermi statistics.

It is convenient to establish an entirely equivalent, alternative formalism, in which all operators and states are unitarily transformed by the unitary transformation U. Since all matrix elements and eigenvalues are invariant to such a similarity transformation, we can construct a map $\{|\nu\rangle\} \to \{|n\rangle\}$, $\Omega(\mathbf{k}) \to a_Q(\mathbf{k})$, and, in general, for all other operators $\mathcal{P}$, $\mathcal{P} \to \tilde{\mathcal{P}}$, where $\tilde{\mathcal{P}} = U^{-1}\mathcal{P}U$. We may then use the transformed representation as an equivalent formulation of the theory, in which Gauss's Law and the gauge constraint, $A_0 = 0$, have been implemented. In this equivalent alternative representation, $\{|n\rangle\}$ is the physical subspace in which Gauss's law and the gauge condition are implemented, and $\exp(-i\tilde{H}t)$ is the time-translation operator. A time-translation operator will time translate state vectors entirely within the physical subspace in the transformed representation, if it is entirely devoid of $a_R^\star(\mathbf{k})$ and $a_R(\mathbf{k})$ operators, or if it contains them at most in the combination $[a_R^\star(\mathbf{k})a_Q(\mathbf{k}) + a_Q^\star(\mathbf{k})a_R(\mathbf{k})]$. Inspection of Eq. (40) confirms that $\tilde{H}$ is, in fact, entirely devoid of $a_R^\star(\mathbf{k})$ and $a_R(\mathbf{k})$ operators, so that the time-translation operator, $\exp(-i\tilde{H}t)$, correctly satisfies this requirement. Observable multiparticle states in the alternative transformed representation are described by state vectors in $\{|n\rangle\}$ which we have previously designated by $|N\rangle$. In CS theory, the only such positive-norm observable states are charged excitations of the spinor field (we will refer to them as electrons and positrons for simplicity). The time-translation operator $e^{-i\tilde{H}t}$ translates state vectors $|N\rangle$ by transforming them into new state vectors, at a later time t; these time-translated state vectors consist of further positive-norm state vectors $|N'\rangle$, as well as additional ghost states. All of the latter are represented by products of $a_Q^\star(\mathbf{k})$ operators acting on positive-norm states $|N'\rangle$. At all times, the positive-norm states alone exactly saturate unitarity. We will refer to a quotient space, which is the set of all $|N\rangle$, and also is the residue of $\{|n\rangle\}$ after all zero-norm states have been excised from it.

We can define another Hamiltonian $\tilde{H}_{\rm quot}$, which consists of those parts of $\tilde{H}$ that remain after we have removed all the terms in which $a_Q^\star(\mathbf{k})$ or $a_Q(\mathbf{k})$ is a factor; $\tilde{H}_{\rm quot}$ is given by

$$\tilde{H}_{\rm quot} = H_{e\bar{e}} - \sum_{\mathbf{k}} \frac{i\epsilon_{ln}k_n}{mk^2}\, j_l(\mathbf{k})j_0(-\mathbf{k}). \qquad (48)$$

The Hamiltonian $\tilde{H}_{\rm quot}$ contains $H_{e\bar{e}}$, which describes the kinetic energy of noninteracting electrons and positrons; it also contains a part that describes a singular nonlocal interaction between the charge density and the transverse current density. The projections of $\exp[-i\tilde{H}t]|N\rangle$ and $\exp[-i\tilde{H}_{\rm quot}t]|N\rangle$ on other state vectors in the quotient space are identical. The parts of $\tilde{H}$ that contain $a_Q^\star(\mathbf{k})$ or $a_Q(\mathbf{k})$ as factors therefore do not play any role in the time evolution of state vectors within the quotient space of observable states, and cannot have any effect on the physical predictions of the theory. The time-evolution operator that time translates physical states in the quotient space of observable states can therefore be given as $\exp[-i\tilde{H}_{\rm quot}t]$.

If we expand D in momentum space we get $D = D_1 + D_2$ where

$$D_1 = \sum_{\mathbf{k}} \frac{1}{2m^{3/2}}\,[a_R(\mathbf{k})j_0(-\mathbf{k}) - a_R^\star(\mathbf{k})j_0(\mathbf{k})] \qquad (49)$$

and

$$D_2 = \sum_{\mathbf{k}} i\phi(\mathbf{k})\,\left[a_Q(\mathbf{k})j_0(-\mathbf{k}) + a_Q^\star(\mathbf{k})j_0(\mathbf{k})\right]. \qquad (50)$$

Since D_2 commutes with $a_Q(\mathbf{k})$, it has no role in transforming $\Omega(\mathbf{k})$ into $a_Q(\mathbf{k})$, and the operator $V = e^{D_1}$ by itself achieves the same end as U, i.e.,

$$V^{-1}\Omega(\mathbf{k})V = a_Q(\mathbf{k}). \qquad (51)$$

We can use V to establish a second mapping of this theory, in which operators map according to $\mathcal{P} \to V^{-1}\mathcal{P}V = \hat{\mathcal{P}}$. $\hat{\Omega}(\mathbf{k}) = a_Q(\mathbf{k})$, so that $\hat{\Omega}$ and $\tilde{\Omega}$ are identical; under the mapping $P \to V^{-1}\mathcal{P}V = \hat{\mathcal{P}}$, the subspace $\{|\nu\rangle\}$ maps into the same subspace $\{|n\rangle\}$ as under the mapping $\mathcal{P} \to U^{-1}\mathcal{P}U = \tilde{\mathcal{P}}$. But, in the case of other operators, $\hat{P}$ differs from $\tilde{\mathcal{P}}$. For example, $\hat{H}$ is given by

$$\begin{aligned}\hat{H} = H_0 &- \sum_{\mathbf{k}} \frac{i\epsilon_{ln}k_n}{mk^2} j_l(\mathbf{k})j_0(-\mathbf{k}) \\ &- \sum_{\mathbf{k}} \frac{i}{m^{3/2}}\,\phi(\mathbf{k})k_l j_l(\mathbf{k})j_0(-\mathbf{k}) \\ &- \sum_{\mathbf{k}} \frac{i\sqrt{m}\epsilon_{ln}k_n}{k^2}\,[a_Q(\mathbf{k})j_l(-\mathbf{k}) - a_Q^\star(\mathbf{k})j_l(\mathbf{k})] \\ &- \sum_{\mathbf{k}} i\phi(\mathbf{k})k_l\,\left[a_Q(\mathbf{k})j_l(-\mathbf{k}) - a_Q^\star(\mathbf{k})j_l(\mathbf{k})\right]. \end{aligned} \qquad (52)$$

Similarly, $\tilde{\psi}$ and $\hat{\psi}$ differ from each other, although both are gauge invariant and project, from the correspondingly defined vacuum states, electron states that implement Gauss's law. $\hat{\psi}$ is given by $\hat{\psi}(\mathbf{x}) = \exp[\mathcal{D}_{\rm V}(\mathbf{x})]\,\psi(\mathbf{x})$ [19], where

$$\mathcal{D}_{\rm V}(\mathbf{x}) = -ie\int d\mathbf{y}\,\sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2}\Big[\partial_l A_l(\mathbf{y}) + \frac{k^2}{\sqrt{m}}\phi(\mathbf{k})\epsilon_{ln}\partial_l A_n(\mathbf{y})\Big]. \qquad (53)$$

Similarly, $\tilde{J}$ and $\hat{J}$ are the forms into which the Noether angular momentum operator J is mapped when it is unitarily transformed by U and V, respectively. Both these forms, $\tilde{J}$ and $\hat{J}$, are therefore significant for the rotation of states, and it is of particular importance to observe that $\tilde{J}$ and $\hat{J}$ differ from each other. J is given by

$$J = J_g + J_e, \qquad (54)$$

where J_g and J_e are the angular momenta of the gauge field and the spinors, respectively. J_g and J_e are given by

$$J_g = -\int d\mathbf{x}\,\epsilon_{ln}(\Pi_i x_l\partial_n A_i - G x_l \partial_n A_0 + \Pi_l A_n) \qquad (55)$$

and

$$J_e = -\int d\mathbf{x}\,\left(i\psi^\dagger x_l \epsilon_{ln}\partial_n\psi + \tfrac{1}{2}\psi^\dagger\gamma_0\psi\right). \qquad (56)$$

Under the transformation mediated by U, $J \to \tilde{J}$, and $\tilde{J} = J$, so that J remains untransformed. But, under the transformation mediated by V, $J \to \hat{J}$ where $\hat{J} = J + \mathcal{J}$ and

$$\mathcal{J} = -\sum_{\mathbf{k}} \epsilon_{ln} k_l \frac{\partial \phi(\mathbf{k})}{\partial k_n} \left[a_Q^\star(\mathbf{k}) j_0(\mathbf{k}) + a_Q(\mathbf{k}) j_0(-\mathbf{k})\right] + \sum_{\mathbf{k}} \frac{\epsilon_{ln} k_l}{2m^{3/2}} \frac{\partial \phi(\mathbf{k})}{\partial k_n} j_0(\mathbf{k}) j_0(-\mathbf{k}). \tag{57}$$

We can support the preceding demonstration that J transforms into itself under the unitary transformation mediated by U, whereas it transforms into $J + \mathcal{J}$ under the unitary transformation mediated by V, with the following observation: D is an integral over operators and functions which all transform as scalars under spatial rotations. Since J is the generator of spatial rotations, the commutator $[J, D]$ must vanish. D_1 is not necessarily such an integral over scalars, and there is therefore no similar requirement that $[J, D_1]$ vanishes.

Since U and V map $\Omega(\mathbf{k})$ into $a_Q(\mathbf{k})$ in identical ways, we can conclude that the implementation of Gauss's law is not responsible for the fact that J is transformed into $J + \mathcal{J}$ when V is used to effect the mapping. In fact, we can use the Baker-Hausdorff-Campbell relation to construct an operator $W = e^{D'}$, where

$$D' = \sum_{\mathbf{k}} \frac{i}{2m^{3/2}} \phi(\mathbf{k}) j_0(\mathbf{k}) j_0(-\mathbf{k}) - \sum_{\mathbf{k}} i\phi(\mathbf{k}) \left[a_Q(\mathbf{k}) j_0(-\mathbf{k}) + a_Q^\star(\mathbf{k}) j_0(\mathbf{k})\right], \tag{58}$$

so that $V = UW$. W has the same effect as V on J, i.e., we find that

$$W^{-1} J W = J + \mathcal{J}, \tag{59}$$

although W leaves $\Omega(\mathbf{k})$ and $\mathcal{G}(\mathbf{x})$ untransformed and does not play any role in implementing Gauss's law; $\phi(\mathbf{k})$ is arbitrary, and if we choose to set $\phi(\mathbf{k}) = 0$, U and V become identical. If we choose

$$\phi(\mathbf{k}) = \sqrt{m}\, \frac{\delta(k)}{k} \tan^{-1} \frac{k_2}{k_1}, \tag{60}$$

and if we assume that we can carry out the integration over $d\mathbf{k}$ while $j_0(\mathbf{k})$ is still operator valued, then $\mathcal{J}$ becomes $\mathcal{J} = Q^2/4\pi m$, and accounts for the well-known anyonic phase in the rotation of charged states through 2π.

In comparing $\tilde{H}$ with $\hat{H}$ we note that they differ by some terms that include $a_Q(\mathbf{k})$ or $a_Q^\star(\mathbf{k})$ as factors. Since both $\tilde{H}$ and $\hat{H}$ are entirely free of $a_R(\mathbf{k})$ and $a_R^\star(\mathbf{k})$ operators, $a_Q(\mathbf{k})$ and $a_Q^\star(\mathbf{k})$ commute with every other operator that appears in $\tilde{H}$ or $\hat{H}$. The terms which include $a_Q(\mathbf{k})$ or $a_Q^\star(\mathbf{k})$ as factors therefore do not affect the time evolution of state vectors in the previously defined quotient space of observable particles (i.e., electrons or positrons); they can neither produce projections on physical states, nor can they contribute internal loops to radiative corrections. They have no effect whatsoever on the physical predictions of the theory and if they are arbitrarily amputated from $\tilde{H}$ or $\hat{H}$, none of the physical predictions of the theory are affected. The only other difference between $\tilde{H}$ and $\hat{H}$ is

$$h = \sum_{\mathbf{k}} \frac{i}{m^{3/2}} \phi(\mathbf{k}) k_l j_l(\mathbf{k}) j_0(-\mathbf{k}); \tag{61}$$

h is a total time derivative in $\hat{H}$, which can be expressed alternatively as $h = i[H_0, \chi]$, as $h = i[H, \chi]$, as $h = i[\tilde{H}, \chi]$, or as $h = i[\hat{H}, \chi]$ where

$$\chi = -\sum_{\mathbf{k}} \frac{1}{2m^{3/2}} \phi(\mathbf{k}) j_0(\mathbf{k}) j_0(-\mathbf{k}). \tag{62}$$

We will discuss the significance of χ in the next section.

IV. ROTATIONAL ANOMALIES AND STATISTICS

Conclusions about the physical implications of this theory depend not only on the structure of the operators, but also on the properties of the Hilbert space in which these operators act. The choice of a Hilbert space can have significant consequences, even though these may not be reflected in the field equations or the commutation rules. We can, for example, decide to assign the previously defined Hilbert space $\{|n\rangle\}$ to the representation in which the Hamiltonian and the angular momentum take the forms $\tilde{H}$, given in Eq. (40), and $\tilde{J} = J$, given in Eq. (54), respectively. We will designate this representation of operators as the $\tilde{\mathcal{P}}$ representation, and the system consisting of operators in the $\tilde{\mathcal{P}}$ representation acting in the Hilbert space $\{|n\rangle\}$ as the $\mathcal{U}$ system. This system of operators and states constitutes a theory in which the charged fermions rotate "normally," acquiring a factor of $(-1)^N$ in a $2\pi N$ rotation. The sole interaction between electrons in the $\mathcal{U}$ system is given by

$$\mathsf{H} = -\sum_{\mathbf{k}} \frac{i\epsilon_{ln} k_n}{mk^2} j_l(\mathbf{k}) j_0(-\mathbf{k}). \tag{63}$$

We can use W to unitarily transform the operators in the $\tilde{\mathcal{P}}$ representation so that all operators $\tilde{\mathcal{P}}$ are transformed to the $\hat{\mathcal{P}}$ representation, as shown by

$$W^{-1} \tilde{\mathcal{P}} W = \hat{\mathcal{P}}. \tag{64}$$

The Hamiltonian $\hat{H}$, and the angular momentum $\hat{J}$, will then have the forms given in Eqs. (52) and (59), respectively. By itself, this change in the form of the operators has no significance. For example, if we combine these transformed operators with the correspondingly transformed Hilbert space $\{|n'\rangle\}$, where each $|n_i'\rangle = W^{-1}|n_i\rangle$, then we have merely regenerated the $\mathcal{U}$ system of operators and states in another representation. There will be no rotational anomaly, although $\mathcal{J}$ appears as part of the angular momentum $\hat{J}$. The combined transformation of operators and states guarantees that the rotated state $e^{i(J+\mathcal{J})\theta}|n_i'\rangle$ returns to $(-1)^N|n_i'\rangle$ in a $2\pi N$ rotation, in spite of the arbitrary parameter in $\mathcal{J}$, which appears to imply arbitrary phases in 2π rotations. To demonstrate this in detail, we examine

$$\hat{R}(\theta)|n_i'\rangle = e^{i(J+\mathcal{J})\theta}|n_i'\rangle = e^{iJ\theta} e^{i\mathcal{J}\theta} W^{-1}|n_i\rangle. \tag{65}$$

$\mathcal{J}$ commutes with W^{-1}; and, for $\phi(\mathbf{k})$ given by Eq. (60), J commutes with $\mathcal{J}$, but not with W^{-1}; we can show that

$$e^{iJ\theta}W^{-1} = W^{-1}e^{iJ\theta}e^{-i\mathcal{J}\theta} \tag{66}$$

so that

$$e^{i(J+\mathcal{J})\theta}|n_i'\rangle = W^{-1}e^{iJ\theta}|n_i\rangle \tag{67}$$

and the state $|n_i'\rangle$ rotates "normally," to acquire a factor of $(-1)^N$ in a $2\pi N$ rotation, as is required by the similarity transformation.

Alternatively, we could just as well assign the Hilbert space $\{|n\rangle\}$ to the $\hat{\mathcal{P}}$ representation of operators. We will designate the system consisting of operators in the $\hat{\mathcal{P}}$ representation and the Hilbert space $\{|n\rangle\}$ as the $\mathcal{W}$ system. The systems $\mathcal{W}$ and $\mathcal{U}$ both implement the equations of motion, as well as Gauss's law and the gauge choice $A_0 = 0$. There is no reason to prefer one system over the other on the basis of dynamics or constraints. In the $\mathcal{W}$ system of operator and states, however, the $\mathcal{J}$ in $\hat{J}$ would be responsible for an anomalous rotational phase that we associate with anyonic behavior. Subsequent similarity transformations that transform to the operator representation $\tilde{\mathcal{P}}$ and the states $W|n_i\rangle$, would preserve that rotational phase anomaly, although then the arbitrary parameter would not reside in J. We can conclude from these observations that rotational phase anomalies are possible; but they are not an inevitable feature of this gauge theory. However, contrary to what has been suggested by some authors [1,2,12], it is not the implementation of Gauss's law that is responsible for anyonic rotational phases. Gauss's law can be implemented with or without producing arbitrary rotational phases. Moreover, in corroboration of a result obtained by other means [6], we find that regardless of whether the arbitrary rotational phase develops, the anticommutation rule that governs the electron field operator remains unchanged by the unitary transformations (U or V) that are instrumental in implementing Gauss's law in the $\{|n\rangle\}$ space. And that observation applies equally to the free Dirac field and to the gauge-invariant electron field that projects electrons that obey Gauss's law. The "normal" and the "anyonic" operators are unitarily equivalent and both obey Fermi-Dirac statistics. Graded commutator algebras and "exotic" fractional statistics do not arise in the process of implementing Gauss's law and establishing the $\mathcal{U}$ and $\mathcal{W}$ systems of operators and states.

We next turn our attention to the extra term h that appears in the Hamiltonian $\hat{H}$ in the $\mathcal{W}$ system; h is the only part of $(\hat{H} - \tilde{H})$ that describes interactions between charges and currents, and which therefore might possibly account for physically observable discrepancies between $\tilde{H}$ and $\hat{H}$. Since the $\mathcal{U}$ and $\mathcal{W}$ systems of operators and states both implement the same dynamical equations and constraints, the question whether both these systems make identical physical predictions is of considerable interest. The fact that h given in Eq. (61) is a total time derivative gives us *a priori* confidence that it will not affect the S matrix produced by this theory. A formal argument that confirms this result is based on a theorem about the relationship between two representations, in which the Hamiltonian in one is a unitary transform of the Hamiltonian in the other, but the states in both representations are left untransformed, and are identical [18,20]. When this theorem is applied to the question we have raised here, we find that the on-shell transition amplitudes determined by the two Hamiltonians are related by

$$\tilde{T}_{f,i} = \hat{T}_{f,i} + i\epsilon R \,, \tag{68}$$

where $\tilde{T}_{f,i}$ and $\hat{T}_{f,i}$ are the scattering amplitudes

$$\tilde{T}_{f,i} = \left\langle f \left| \left(\tilde{H}_{\rm I} + \tilde{H}_{\rm I}\frac{1}{E - \tilde{H} + i\epsilon}\tilde{H}_{\rm I}\right) \right| i \right\rangle \tag{69}$$

and

$$\hat{T}_{f,i} = \left\langle f \left| \left(\hat{H}_{\rm I} + \hat{H}_{\rm I}\frac{1}{E - \hat{H} + i\epsilon}\hat{H}_{\rm I}\right) \right| i \right\rangle; \tag{70}$$

$\tilde{H}_{\rm I}$ and $\hat{H}_{\rm I}$ are given by

$$\tilde{H}_{\rm I} = \tilde{H} - H_0 \tag{71}$$

and

$$\hat{H}_{\rm I} = \hat{H} - H_0 \,, \tag{72}$$

and R is given by

$$R = \left\langle f \left| \left(\tilde{H}_{\rm I}\frac{1}{E - \tilde{H} + i\epsilon}(1 - e^{-i\chi}) - (1 - e^{-i\chi})\frac{1}{E - \hat{H} + i\epsilon}\hat{H}_{\rm I}\right) \right| i \right\rangle ; \tag{73}$$

$|i\rangle$ and $|f\rangle$ are two multiparticle electron-positron states in $\{|n\rangle\}$ with identical energy E, and represent initial and final states in scattering events, respectively. We note that if H_0 and χ commute, then h vanishes. If H_0 and χ fail to commute, then R will consist of terms proportional to

$$\int dE_n \, \langle f|\tilde{H}_{\rm I} + \tilde{H}_{\rm I}(E - \tilde{H} + i\epsilon)^{-1}\tilde{H}_{\rm I}|n\rangle \times \langle n|(-i\chi)^k|i\rangle(E - E_n + i\epsilon)^{-1}$$

and

$$\int dE_n \, \langle f|(-i\chi)^k|n\rangle \times \langle n|\hat{H}_{\rm I} + \hat{H}_{\rm I}(E - \hat{H} + i\epsilon)^{-1}\hat{H}_{\rm I}|i\rangle(E - E_n + i\epsilon)^{-1} \,,$$

where k is an integer. When χ and H_0 do not commute, R will not give rise to any $1/i\epsilon$ singularities to cancel the $i\epsilon$ on the right-hand side of Eq. (68), except perhaps, for contributions that represent self-energy insertions in external lines. These contributions only affect renormalization constants and do not affect physical quantities.

We conclude, therefore, that the physical predictions of this model, i.e., the S-matrix elements that determine scattering amplitudes and energy level shifts for electron-positron systems, are insensitive to whether the charged states develop anomalous phases.

V. FORMULATION OF THE THEORY IN THE COULOMB GAUGE

To confirm the results we obtained in the preceding discussion of CS theory in the temporal gauge, we now also formulate the same model in the Coulomb gauge. The Coulomb gauge formulation makes use of a quantization procedure that is different from the one we used for the temporal gauge, and therefore can provide independent corroboration of our earlier conclusions. In the Coulomb gauge, the gauge field A_0 is not involved in the gauge condition, so that a gauge-fixing term cannot be used to generate a canonical momentum conjugate to A_0. The most convenient ways to quantize CS theory in the Coulomb gauge are the Dirac-Bergmann (DB) procedure [15,16] and the symplectic method of Faddeev and Jackiw [17]. We will here use the DB procedure to impose the constraints. In this method, the canonical "Poisson" commutators (anticommutators) are replaced by their respective Dirac commutators (anticommutators), which apply to the fields that obey all the constraints of the theory. Since the Dirac and the canonical commutators (anticommutators) can, and often do, differ from each other, this method enables us to investigate whether the Dirac anticommutator for the spinor field ψ and its adjoint $\psi^\dagger$ differ from the corresponding canonical anticommutator. A discrepancy between the Dirac and canonical anticommutators for the spinor fields could signal the development of "exotic" fractional statistics due to the imposition of Gauss's law. On the other hand, identity of the Dirac and the canonical anticommutators for the spinor fields demonstrate that the excitations of the charged spinor field that obey Gauss's law (as well as all other constraints) also obey standard Fermi statistics. The question of whether the imposition of Gauss's law produces charged particle excitations that are subject to exotic statistics, therefore, arises in a new way in the Coulomb gauge. In this section we will carry out this quantization procedure and demonstrate explicitly that the implementation of Gauss's law for the charged spinor field does not change the anticommutation rule for ψ and $\psi^\dagger$, and does not cause the excitations of these fields to develop exotic fractional statistics.

The Lagrangian for CS theory in the Coulomb gauge is given by

$$\mathcal{L} = \tfrac{1}{4}m\epsilon_{ln}(F_{ln}A_0 - 2F_{n0}A_l) - G\partial_l A_l + j_l A_l - j_0 A_0 + \bar{\psi}(i\gamma^\mu\partial_\mu - M)\psi. \quad (74)$$

This Lagrangian differs from Eq. (1) only in that the gauge-fixing term $-G\partial_0 A_0$ is replaced by $-G\partial_l A_l$. We have included a gauge-fixing term for the Coulomb gauge in Eq. (74) to enable us to develop all the constraints systematically from the Lagrangian.

The Euler-Lagrange equations generated by the Lagrangian are

$$m\epsilon_{ln}F_{n0} - j_l - \partial_l G = 0, \quad (75)$$

$$\tfrac{1}{2}m\epsilon_{ln}F_{ln} - j_0 = 0, \quad (76)$$

$$\partial_l A_l = 0, \quad (77)$$

and

$$(M - i\gamma^\mu D_\mu)\psi = 0. \quad (78)$$

The momenta conjugate to the gauge fields are $\Pi_l = \frac{1}{2}m\epsilon_{ln}A_n$, for $l = 1,2$; $\Pi_0 = 0$, where Π_0 is the momentum conjugate to A_0; and $\Pi_G = 0$, where Π_G is the momentum conjugate to the gauge-fixing field G. For the spinor fields, we obtain the momenta $\Pi_\psi = i\psi^\dagger$ and $\Pi_{\psi^\dagger} = 0$ which are conjugate to ψ and $\psi^\dagger$, respectively. The corresponding primary constraints can be expressed as $\mathcal{C}_i \approx 0$ for $i = 1,\ldots,4$, as well as $\mathcal{C}_\psi \approx 0$ and $\mathcal{C}_{\psi^\dagger} \approx 0$, where

$$\mathcal{C}_1 = \Pi_1 - \tfrac{1}{2}mA_2, \quad (79)$$

$$\mathcal{C}_2 = \Pi_2 + \tfrac{1}{2}mA_1, \quad (80)$$

$$\mathcal{C}_3 = \Pi_0, \quad (81)$$

$$\mathcal{C}_4 = \Pi_G, \quad (82)$$

$$\mathcal{C}_\psi = \Pi_\psi - i\psi^\dagger, \quad (83)$$

and

$$\mathcal{C}_{\psi^\dagger} = \Pi_{\psi^\dagger}. \quad (84)$$

The total Coulomb gauge Hamiltonian H_T^C is given by

$$H_T^C = \int d\mathbf{x}\ \psi^\dagger(\gamma_0 M - i\gamma_0\gamma_l\partial_l)\psi + \int d\mathbf{x}\ [-\tfrac{1}{2}m\epsilon_{ln}F_{ln}A_0 + G\partial_l A_l + j_0A_0 - j_lA_l - \mathcal{U}_1\mathcal{C}_1 - \mathcal{U}_2\mathcal{C}_2 - \mathcal{U}_3\mathcal{C}_3 - \mathcal{U}_4\mathcal{C}_4 - \mathcal{U}_\psi\mathcal{C}_\psi - \mathcal{U}_{\psi^\dagger}\mathcal{C}_{\psi^\dagger}]\ , \quad (85)$$

where $\mathcal{U}_1,\ldots,\mathcal{U}_4$ designate arbitrary functions that commute with all operators; $\mathcal{U}_\psi$ and $\mathcal{U}_{\psi^\dagger}$ designate arbitrary functions that are Grassmann numbers, which anticommute with all fermion fields and with Grassmann numbers, but commute with bosonic operators and with $\mathcal{U}_1,\ldots,\mathcal{U}_4$. In the imposition of constraints we will use the Poisson brackets $[\![A,B]\!]$ of two operators A and B, defined as $[\![A,B]\!] = AB-(-1)^{n(A)n(B)}BA$, where $n(P)$ is an index for the operators P;[1] $n(P) = 0$ if P is a bosonic

[1] We generally follow the conventions in Sundermeyer [21]. The definition of Poisson brackets used here, however, differs from Sundermeyer's definition by a factor i.

operator, such as a gauge field or a bilinear combination of fermion fields; and $n(P) = 1$ if P is a Grassmann number, or a fermionic operator such as ψ or $\psi^\dagger$. The Poisson bracket $[\![A, B]\!]$ is the commutator $[A, B]$ when A and B are both bosonic operators, or if one is bosonic and the other fermionic. But $[\![A, B]\!]$ is the anticommutator $\{A, B\}$ when A and B are both fermionic operators.

We use the total Hamiltonian to generate the further constraints needed to maintain the stability of the primary constraints under time evolution. For this purpose we evaluate time derivatives of the primary constraints by using the equation $\partial_0 C_i = [\![H_T^C, C_i]\!]$, and set $\partial_0 C_i \approx 0$. In this way we find that $\partial_0 \mathcal{C}_1 \approx 0$ and $\partial_0 \mathcal{C}_2 \approx 0$ lead to equations for $\mathcal{U}_1$ and $\mathcal{U}_2$, but do not generate any secondary constraints. The equation $\partial_0 \mathcal{C}_3 \approx 0$ leads to the secondary constraint $\mathcal{C}_5 \approx 0$ where

$$\mathcal{C}_5 = m\epsilon_{ln}\partial_l A_n + j_0, \tag{86}$$

which implements Gauss's law. $\partial_0 \mathcal{C}_5 \approx 0$ leads to a further, tertiary constraint, $\mathcal{C}_6 \approx 0$ where

$$\mathcal{C}_6 = \partial_l \partial_l G. \tag{87}$$

This constraint is necessary for consistency between Eq. (75) and Gauss's law. To demonstrate this we observe that taking the two-dimensional divergence of Eq. (75), and applying current conservation, lead to $\partial_0 \left[\frac{1}{2} m\epsilon_{ln}F_{ln} - j_0\right] + \partial_l\partial_l G = 0$, which is inconsistent with Gauss's law unless $\partial_l\partial_l G = 0$. The tertiary constraint $\mathcal{C}_6 = \partial_l\partial_l G \approx 0$ ends this particular chain of unfolding constraints; imposing the condition that $\partial_0(\partial_l\partial_l G) \approx 0$, leads to an equation for $\mathcal{U}_4$, but generates no further constraints. To ensure the stability of the constraint $\Pi_G \approx 0$, we set $\partial_0 \Pi_G \approx 0$ and obtain $\mathcal{C}_7 \approx 0$ where

$$\mathcal{C}_7 = \partial_l A_l, \tag{88}$$

which implements the gauge condition for the Coulomb gauge. The equation $\partial_0 \mathcal{C}_7 \approx 0$ leads to the tertiary constraint $\mathcal{C}_8 \approx 0$ where

$$\mathcal{C}_8 = \partial_l\partial_l A_0 + \frac{1}{m}\epsilon_{ln}\partial_l j_n. \tag{89}$$

The constraint $\mathcal{C}_8 \approx 0$ is an expression of the fact that Eq. (75), which is an equation of motion for the gauge field A_l in most gauges, reduces to a constraint in the Coulomb gauge. Equation (75), which has a longitudinal as well as a transverse component, can be expressed as $m\epsilon_{ln}(\partial_n A_0 + \partial_0 A_n) - j_l - \partial_l G = 0$. The longitudinal component has just been shown to lead to the constraint $\partial_l\partial_l G \approx 0$. The transverse component can be extracted by contracting over $\epsilon_{il}\partial_i$, and noting that $\epsilon_{il}\epsilon_{ln} = -\delta_{in}$. In the resulting equation, the sole remaining time derivative, $\partial_0\partial_l A_l$, vanishes because of the Coulomb gauge condition, leaving $\mathcal{C}_8 \approx 0$ as a constraint. The equation $\partial_0 \mathcal{C}_8 \approx 0$ leads to no further constraint, so that $\mathcal{C}_8 \approx 0$ terminates the chain of constraints that develops from $\Pi_G \approx 0$. In the case of the spinor fields, the equations $\partial_0 C_\psi \approx 0$ and $\partial_0 C_{\psi^\dagger} \approx 0$ lead to equations for the Grassmann functions $\mathcal{U}_\psi$ and $\mathcal{U}_{\psi^\dagger}$; but they do not lead to any further constraints.

The preceding analysis leads to ten second-class constraints for this gauge theory. Imposition of the constraints requires that we form the matrix $\mathcal{M}(\mathbf{x}, \mathbf{y})$, whose elements are $\mathcal{M}_{ij}(\mathbf{x}, \mathbf{y}) = [\![\mathcal{C}_i(\mathbf{x}), \mathcal{C}_j(\mathbf{y})]\!]$. We assign the values $\mathcal{C}_1, \ldots, \mathcal{C}_{10}$ to the descending horizontal rows of the matrix, as well as to the sequence of vertical columns, where $\mathcal{C}_1, \ldots, \mathcal{C}_8$ refer to the previously defined constraints; for simplicity we will designate $\mathcal{C}_\psi$ and $\mathcal{C}_{\psi^\dagger}$ as $\mathcal{C}_9$ and $\mathcal{C}_{10}$, respectively. The matrix $\mathcal{M}(\mathbf{x}, \mathbf{y})$ is evaluated and inverted; its inverse, $\mathcal{Y}(\mathbf{x}, \mathbf{y})$, which obeys

$$\int d\mathbf{z}\; \mathcal{M}_{ik}(\mathbf{x}, \mathbf{z})\mathcal{Y}_{kj}(\mathbf{z}, \mathbf{y}) = \int d\mathbf{z}\; \mathcal{Y}_{ik}(\mathbf{x}, \mathbf{z})\mathcal{M}_{kj}(\mathbf{z}, \mathbf{y}) = \delta_{ij}\delta(\mathbf{x} - \mathbf{y}) \tag{90}$$

is used to calculate the Dirac commutators (anticommutators) by applying the equation

$$[\![\xi(\mathbf{x}), \zeta(\mathbf{y})]\!]^D = [\![\xi(\mathbf{x}), \zeta(\mathbf{y})]\!] - \sum_{i,j=1}^{10} \int d\mathbf{z}\, d\mathbf{z}'\; [\![\xi(\mathbf{x}), \mathcal{C}_i(\mathbf{z})]\!] \times \mathcal{Y}_{ij}(\mathbf{z}, \mathbf{z}')[\![\mathcal{C}_j(\mathbf{z}'), \zeta(\mathbf{y})]\!]. \tag{91}$$

We observe that the resulting Dirac commutators (anticommutators) are given by

$$[\![\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})]\!]^D = \{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \tag{92}$$

$$[\![\psi(\mathbf{x}), \psi(\mathbf{y})]\!]^D = \{\psi(\mathbf{x}), \psi(\mathbf{y})\} = 0, \tag{93}$$

$$[\![A_0(\mathbf{x}), \psi(\mathbf{y})]\!]^D = [A_0(\mathbf{x}), \psi(\mathbf{y})] = -\frac{ie}{m}\epsilon_{ln}\frac{(x-y)_l}{2\pi|\mathbf{x} - \mathbf{y}|^2}\gamma_0\gamma_n\psi(\mathbf{y}), \tag{94}$$

$$[\![A_l(\mathbf{x}), \psi(\mathbf{y})]\!]^D = [A_l(\mathbf{x}), \psi(\mathbf{y})] = -\frac{ie}{m}\epsilon_{ln}\frac{(x-y)_n}{2\pi|\mathbf{x} - \mathbf{y}|^2}\psi(\mathbf{y}), \tag{95}$$

$$[\![A_l(\mathbf{x}), A_0(\mathbf{y})]\!]^D = [A_l(\mathbf{x}), A_0(\mathbf{y})] = 0\;, \tag{96}$$

and

$$[\![A_l(\mathbf{x}), A_n(\mathbf{y})]\!]^D = [A_l(\mathbf{x}), A_n(\mathbf{y})] = 0. \tag{97}$$

Equations (92) and (93) demonstrate that the constrained spinor field obeys standard anticommutation rules, and not a graded anticommutator algebra; and that the charged excitations of that spinor field are subject to standard Fermi statistics, and not the exotic fractional statistics that would result from a graded anticommutator algebra. In contrast with the spinor field, the Dirac commutators of the gauge fields differ substantially both from the unconstrained canonical commutators, and also from their corresponding values in the temporal gauge. The observation that the spinor anticommutation rule is unaffected by constraints, and identical in the Coulomb and temporal gauges, therefore is not trivial.

The Dirac commutators (anticommutators) imply relationships among the constrained operators that reduce the independent degrees of freedom of this theory. There are various procedures for making these relationships explicit, but in this model the simplest way is to use the constraint equations to express the gauge fields as functionals of the charge and current densities. Since the gauge fields have no observable, propagating degrees of freedom, this procedure can completely eliminate all gauge fields from the Hamiltonian. We find, for example, that $\mathcal{C}_5 = 0$ can be solved to yield

$$A_l(\mathbf{x}) = -\frac{1}{m}\epsilon_{ln}\frac{1}{\nabla^2}\frac{\partial}{\partial x_n}j_0(\mathbf{x}), \tag{98}$$

and $\mathcal{C}_8 = 0$ can be solved for

$$A_0 = -\frac{1}{m}\epsilon_{ln}\frac{1}{\nabla^2}\partial_l j_n. \tag{99}$$

When we use Eqs. (98) and (99) and the fermion anticommutation relations for the constrained ψ and $\psi^\dagger$, we exactly reproduce the Dirac commutators given in Eqs. (94)–(97). The value G^D of the gauge-fixing field G on the constraint surface can be shown to be zero using the relation

$$G^{\rm D}(\mathbf{x}) = G(\mathbf{x}) - \sum_{i,j=1}^{10}\int d\mathbf{y}\,d\mathbf{z}\;[\![G(\mathbf{x}),\mathcal{C}_i(\mathbf{y})]\!]\mathcal{Y}_{ij}(\mathbf{y},\mathbf{z})\mathcal{C}_j(\mathbf{z}). \tag{100}$$

Since $G^D = 0$, then Eq. (75) reduces to $m\epsilon_{ln}F_{0n} - j_l = 0$ under the influence of the constraints. When these constrained representations of the gauge fields are substituted into the Hamiltonian, and all the other constraint functions, $\mathcal{C}_i$, for $i = 1, \ldots, 10$, are set to zero, we obtain the result that

$$H_C = \int d\mathbf{x}\;\psi^\dagger(\gamma_0 M - i\gamma_0\gamma_l\partial_l)\psi - \sum_{\mathbf{k}}\frac{i\epsilon_{ln}k_n}{mk^2}j_l(\mathbf{k})j_0(-\mathbf{k}). \tag{101}$$

We observe that this form of the Hamiltonian is exactly identical to $\tilde{H}_{\rm quot}$ in the temporal gauge. This exact identity of H_C and $\tilde{H}_{\rm quot}$ provides a very compelling demonstration that the charged states that obey Gauss's law are subject to the identical dynamics in both our temporal gauge and Coulomb gauge formulations of this model. Since the gauge independence of the physical predictions of this theory is a firm requirement, this identity serves as a significant corroboration of the consistency and correctness of both of our formulations of this model. Similarly, our separate demonstrations, in the temporal gauge and the Coulomb gauge formulations, of the anticommutation rules for the spinor fields that create and annihilate charged particles that obey Gauss's law, show that these charged particle states obey standard Fermi rather than fractional statistics. The formalism employed for each of these demonstrations is specific to each gauge—we observe, for example, that $\mathcal{D}_{\rm U}(\mathbf{x})$, given in Eq. (44), whose properties play an essential role in the argument that pertains to the temporal gauge, would vanish in the Coulomb gauge. But the fact that the same result is reached in both gauges, confirms that our conclusion about the statistics of charged particle states is gauge independent, and makes the argument particularly persuasive.

VI. DISCUSSION

In the preceding work, we have demonstrated that in Chern-Simons theory coupled to a charged-fermion field, the imposition of Gauss's law does not cause the spinor fields to develop an "exotic" graded commutator algebra. In the process, we have constructed a Fock space of charged excitations of these fields that are subject to standard Fermi, and not fractional statistics, even when they obey Gauss's law. Our work applies to both the temporal and the Coulomb gauge formulations of the theory. We have also obtained a time evolution operator for the single and multiparticle electron-positron states that obey Gauss's law, and have shown that this time evolution operator is identical in the two calculations we have carried out, one in the temporal and the other in the Coulomb gauge. This result provides further confirmation that the quantization procedures and the conclusions based on them are correct. We have also shown that the charged states may or may not acquire arbitrary phases in 2π rotations, depending upon the way we choose to represent them. However, that choice of representation can be made independently of whether the charged states of the theory obey Gauss's law; it is not a consequence of the imposition of the constraints. Moreover, the choice of representation that determines whether arbitrary rotational phases result from 2π rotations does not have any implications for the physical predictions of the theory.

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