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Strings propagating in the (2+1)-dimensional black hole anti–de Sitter spacetime

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We study the string propagation in the 2+1 black hole anti–de Sitter (2+1 BHAdS) background. We find the first- and second-order fluctuations around the string center of mass and obtain the expression for the string mass. The string motion is stable, all fluctuations oscillate with real frequencies and are bounded, even at $r=0$. We compare with the string motion in the ordinary black hole anti–de Sitter spacetime, and in the black string background, where string instabilities develop and the fluctuations blow up at $r=0$. We find the exact general solution for the circular string motion in all these backgrounds; it is given closely and completely in terms of elliptic functions. For the nonrotating black hole backgrounds the circular strings have a maximal bounded size r_m; they contract and collapse into $r=0$. No indefinitely growing strings nor multistring solutions are present in these backgrounds. In rotating spacetimes, both the 2+1 BHAdS and the ordinary Kerr-AdS backgrounds, the presence of angular momentum prevents the string from collapsing into r=0. The circular string motion is also completely solved in the black hole de Sitter spacetime and in the black string background (dual of the 2+1 BHAdS spacetime), in which expanding unbounded strings and multistring solutions appear.

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I. INTRODUCTION AND RESULTS

The study of string dynamics in curved spacetime and its associated physical phenomena that was started in Refs. [1,2] has received a systematic and increasing attention. Approximate [1–4] and exact [5–9] methods of solution have been developed. Classical and quantum string dynamics have been investigated in black hole backgrounds [10,11], cosmological spacetimes [1,12], cosmic string spacetime [13], gravitational wave backgrounds [14], supergravity backgrounds (which are necessary for fermionic strings) [15], and near spacetime singularities [16]. Physical phenomena such as the Hawking-Unruh effect in string theory [2,17], horizon string stretching [2,17], particle transmutation [10,18], string scattering [10,13], mass spectrum and critical dimension [1,10,13], string instability [1,5–8,12], and multistring solutions [6–8] have been found. It is also useful to consider simple tractable spacetimes of physical interest, and the restriction to lower dimensions. Although two-dimensional models have many attractive tractable aspects and can be used to test and get insight on particular features, $D=2$ is not a physically appealing dimension for string theory or gravity [19]. In contrast, $D=2+1$ theory possesses all the physical ingredients of string theory and gravity in higher dimensions [6–8,20–26].

In this paper we investigate the string dynamics in the 2+1 black hole anti–de Sitter (BHAdS) spacetime recently found by Bañados, Teitelboim, and Zanelli [20]. This spacetime background has stirred much interest recently [21–25]. It describes a two-parameter family (mass M and angular momentum J) of black holes in (2+1)-dimensional general relativity with the metric

$$ds^2 = \left(M - \frac{r^2}{l^2}\right) dt^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1} dr^2 - J dt d\phi + r^2 d\phi^2 \,. \tag{1.1}$$

It has two horizons

$$r_\pm = \left(\frac{Ml^2}{2} \pm \frac{l}{2}\sqrt{M^2l^2 - J^2}\right)^{1/2}$$

and a static limit $r_{\rm erg} = \sqrt{M}l$, defining an ergosphere, as for ordinary Kerr black holes. The spacetime is not asymptotically flat, however; it approaches anti–de Sitter spacetime asymptotically with a cosmological constant $\Lambda = -1/l^2$. The curvature is constant $R_{\mu\nu} = -(2/l^2)g_{\mu\nu}$ everywhere, except probably at $r=0$, where it has at most a δ-function singularity. Notice the weak nature of the singularity at $r=0$ in 2+1 dimensions as compared with the power law divergence of curvature scalars in $D>3$. (We will not discuss here the geometry near $r=0$. For a discussion, see Refs. [23,25].) The spacetime, Eq. (1.1), is also a solution of the low-energy effective action of string theory with a zero dilaton field $\Phi=0$, antisymmetric tensor field $H_{\mu\nu\rho} = (2/l^2)\epsilon_{\mu\nu\rho}$ (i.e., $B_{\phi t} = r^2/l^2$) and $k = l^2$ [21]. Moreover, it yields an exact solution of string theory in 2+1 dimensions, obtained by gauging the Wess-Zumino-Witten-Novikov (WZWN) σ model of the group SL(2,R) $\times$ R at level k [21,22] [for noncompact groups, k does not need to be an integer, so the central charge $c = 3k/(k-2)$=26 when k=52/23].

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This solution is the black string background [26]:

$$d\tilde{s}^2 = -\left(1-\frac{\mathcal{M}}{\tilde{r}}\right)d\tilde{t}^{\,2} + \left(1-\frac{\mathcal{Q}^2}{\mathcal{M}\tilde{r}}\right)d\tilde{x}^2 + \left(1-\frac{\mathcal{M}}{\tilde{r}}\right)^{-1}\left(1-\frac{\mathcal{Q}^2}{\mathcal{M}\tilde{r}}\right)^{-1}\frac{l^2 d\tilde{r}^2}{4\tilde{r}^2},$$

$$\tilde{B}_{\tilde{x}\tilde{t}} = \frac{\mathcal{Q}}{r}, \quad \tilde{\Phi} = -\tfrac{1}{2}\ln \tilde{r}l \; , \qquad (1.2)$$

which is related by duality [21,27] to the 2+1 BHAdS spacetime, Eq. (1.1). It has two horizons $\tilde{r}_\pm = r_\pm$, the same as the metric, Eq. (1.1), while the static limit is $\tilde{r}_{\rm erg} = J/(2\sqrt{M})$. Throughout the paper we use the sign conventions of Misner, Thorne, and Wheeler [28] and units where $G = 1$, $c = 1$, and the string tension $(2\pi\alpha')^{-1} = 1$.

We first investigate the string propagation in these backgrounds by considering the perturbation series around the exact center of mass of the string

$$x^\mu(\tau,\sigma) = q^\mu(\tau) + \eta^\mu(\tau,\sigma) + \xi^\mu(\tau,\sigma) + \cdots \; . \qquad (1.3)$$

The original method of Refs. [1,2] can be conveniently formulated in covariant form. Is also useful to introduce $D-1$ normal vectors n^μ_R $(R = 1,\ldots,D-1)$, (which can be chosen to be covariantly constant by gauge fixing), and consider comoving perturbations δx_R, i.e., those seen by an observer traveling with the center of mass; thus, $\eta^\mu = \delta x^R n^\mu_R$. After Fourier transforming $\delta x^R(\tau,\sigma) = \sum_n C^R_n(\tau)e^{-in\sigma}$, the first-order perturbations satisfy the matrix Schrödinger-type equation in τ:

$$\ddot{C}_{nR} + (n^2\delta_{RS} - R_{\mu\rho\sigma\nu}n^\mu_R n^\nu_S \dot{q}^\rho \dot{q}^\sigma)C^S_n = 0 \; . \qquad (1.4)$$

Second-order perturbations ξ^μ and constraints are similarly covariantly treated, ξ^μ also satisfying Schrödinger-type equations with source terms, see Eqs. (3.15)–(3.20).

For our purposes here it is enough to consider the nonrotating $(J = 0)$ 2+1 BHAdS background and a radially infalling string. We solve completely the c.m. motion $q^\mu(\tau)$ and the first- and second-order perturbations $\eta^\mu(\tau,\sigma)$ and $\xi^\mu(\tau,\sigma)$ in this background. Equation (1.4) becomes

$$\ddot{C}_{nR} + \left(n^2 + \frac{m^2}{l^2}\right)C_{nR} = 0, \quad R = \perp, \parallel \; . \qquad (1.5)$$

The first-order perturbations are independent of the black hole mass, only the AdS part emerges. All oscillation frequencies $\omega_n = \sqrt{n^2 + m^2/l^2}$ are real; there are no unstable modes in this case; the perturbations

$$\delta x_R(\tau,\sigma) = \sum_n [A_{nR}e^{-i(n\sigma+\omega_n\tau)} + \tilde{A}_{nR}e^{-i(n\sigma-\omega_n\tau)}] \qquad (1.6)$$

are completely finite and regular. This is also true for the second-order perturbations, which are bounded everywhere even for $r \to 0$ $(\tau \to 0)$. We also compute the conformal generators L_n, Eq. (3.61), and the string mass

$$m^2 = 2\sum_n \left(2n^2 + \frac{m^2}{l^2}\right)[A_{n\parallel}\tilde{A}_{-n\parallel} + A_{n\perp}\tilde{A}_{-n\perp}] \; . \qquad (1.7)$$

The mass formula is modified (by the term m^2/l^2) with respect to the usual flat spacetime expression. This is due to the asymptotic (here AdS) behavior of the spacetime. In ordinary $D > 3$ black hole spacetimes (without cosmological constant), which are asymptotically flat, the mass spectrum is the same as in flat space [10]. The quantum string dynamics and mass spectrum for the 2+1 BHAdS spacetime are to be discussed elsewhere.

We compare with the string perturbations in the ordinary $(D > 3)$ black hole anti–de Sitter spacetime. In this case Eqs. (1.4) become

$$\ddot{C}_{nS\perp} + \left(n^2 + m^2H^2 + \frac{Mm^2}{r^3}\right)C_{nS\perp} = 0, \quad S = 1,2 \qquad (1.8)$$

$$\ddot{C}_{n\parallel} + \left(n^2 + m^2H^2 - \frac{2Mm^2}{r^3}\right)C_{n\parallel} = 0 \; . \qquad (1.9)$$

The transverse $\perp$ perturbations are oscillating with real frequencies and are bounded even for $r \to 0$. For longitudinal $\parallel$ perturbations, however, imaginary frequencies arise and instabilities develop. The $(|n| = 1)$ instability sets in at

$$r_{\rm inst} = \left(\frac{2Mm^2}{1 + m^2H^2}\right)^{1/3} \; . \qquad (1.10)$$

Lower modes become unstable even outside the horizon, while higher modes develop instabilities at smaller r and eventually only for $r \approx 0$. For $r \to 0$ (which implies $\tau \to \tau_0$) we find $r(\tau) \approx (3m\sqrt{M/2})^{2/3}(\tau_0 - \tau)^{2/3}$ and

$$\ddot{C}_{nS\perp} + \frac{2}{9(\tau-\tau_0)^2}C_{nS\perp} = 0, \quad S = 1,2 \; , \qquad (1.11)$$

$$\ddot{C}_{n\parallel} - \frac{4}{9(\tau-\tau_0)^2}C_{n\parallel} = 0 \; . \qquad (1.12)$$

For $\tau \to \tau_0$ the $\parallel$ perturbations blow up while the string ends trapped into the $r = 0$ singularity. We see the important difference between the string evolution in the 2+1 BHAdS background and the ordinary 3+1 (or higher dimensional) black hole anti–de Sitter spacetime.

We also compare with the string propagation in the 2+1 black string background, Eq. (1.2) (with $J = 0$). In this case, Eqs. (1.4) become

$$\ddot{C}_{n\perp} + n^2 C_{n\perp} = 0 \; , \qquad (1.13)$$

$$\ddot{C}_{n\parallel} + \left(n^2 - \frac{2m^2M}{lr}\right)C_{n\parallel} = 0 \; . \qquad (1.14)$$

The $\perp$ modes are stable, while $C_{n\parallel}$ develop instabilities. For $r \to 0$ (which implies $\tau \to \tau_0$) we find $r(\tau) \approx \frac{m^2M}{l}(\tau_0 - \tau)^2$ and

$$\ddot{C}_{n\|} - \frac{2}{(\tau_0 - \tau)^2} C_{n\|} = 0 \;, \qquad (1.15)$$

with similar conclusions as for the ordinary 3+1 (or higher dimensional) black hole anti–de Sitter spacetime.

In order to extract more information about the string evolution in these backgrounds, in particular exact properties, we consider the circular string ansatz

$$t = t(\tau), \quad r = r(\tau), \quad \phi = \sigma + f(\tau) \;, \qquad (1.16)$$

in the equatorial plane ($\theta = \pi/2$) of the stationary axially symmetric backgrounds:

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + 2g_{t\phi}(r)dtd\phi + g_{\phi\phi}(r)d\phi^2 \;. \qquad (1.17)$$

This includes all the cases of interest here: the 2+1 BHAdS spacetime, the black string, as well as the equatorial plane of ordinary Einstein black holes. The string dynamics is then reduced to a system of second-order ordinary differential equations and constraints, also described as a Hamiltonian system:

$$\dot{r}^2 + V(r) = 0, \quad V(r) = g^{rr}(g_{\phi\phi} + E^2 g^{tt}) \;, \qquad (1.18)$$

$$\dot{t} = -Eg^{tt}, \quad \dot{f} = -Eg^{t\phi}, \quad E = -P_t = \text{const} \;, \qquad (1.19)$$

which in all backgrounds considered here are solved in terms of either elementary or elliptic functions. The dynamics of the circular strings takes place at the r axis in the $(r, V(r))$ diagram and from the properties of the potential $V(r)$ [minima, zeros, asymptotic behavior for large r and the value $V(0)$], general knowledge about the string motion can be obtained. On the other hand, the line element of the circular string turns out to be

$$ds^2 = g_{\phi\phi}(d\sigma^2 - d\tau^2), \quad \text{i.e., } S(\tau) = \sqrt{g_{\phi\phi}(r(\tau))} \;, \qquad (1.20)$$

$S(\tau)$ being the invariant string size. For all the static black hole AdS spacetimes (2+1 and higher dimensional) $S(\tau)=r(\tau)$, while for the black string background $S(\tau) = r(\tau)^{-1}$, reflecting the dual properties of the background on the circular test string.

For the rotating 2+1 BHAdS spacetime,

$$V(r) = r^2\left(\frac{r^2}{l^2} - M\right) + \frac{J^2}{4} - E^2 \qquad (1.21)$$

(see Fig. 1). $V(r)$ has a global minimum $V_{\min} < 0$ between the two horizons r_+, r_- (for $Ml^2 \geq J^2$, otherwise there are no horizons). The vanishing of $V(r)$ at $r = r_{01,2}$ [see Eq. (4.18)] determines three different types of solutions. (i) For $J^2 > 4E^2$, there are two positive zeros $r_{01} < r_{02}$, the string never comes outside the static limit, never falls into $r = 0$ neither (there is a barrier between $r = r_{01}$ and $r = 0$). The mathematical solution oscillates between r_{01} and r_{02} with $0 < r_{01} < r_- < r_+ < r_{02} < r_{\text{erg}}$. It may be interpreted as a string traveling between the different universes described by the maximal analytic extension of the manifold. (ii) For $J^2 < 4E^2$, there is only one positive zero r_0 outside the static limit and there is no barrier preventing the string from collapsing into $r = 0$. The string starts at $\tau = 0$ with maximal size $S^{(\text{ii})}_{\max}$ outside the static limit, it then contracts through the ergosphere and the two horizons and eventually collapses into a point $r = 0$. For $J \neq 0$, it may be still possible to continue this solution into another universe as in case (i). (iii) $J^2 = 4E^2$ is the limiting case where the maximal string size equals the static limit: $S^{(\text{iii})}_{\max} = l\sqrt{M}$. In this case $V(0) = 0$, thus the string contracts through the two horizons and eventually collapses into a point $r = 0$.

The exact general solution in the three cases (i)–(iii) is given by

$$r(\tau) = \left| r_m - \frac{1}{c_1 \wp(\tau - \tau_0) + c_2} \right| \;, \qquad (1.22)$$

where

$$r_m = S_{\max} = \sqrt{\frac{Ml^2}{2}} \left(1 + \sqrt{1 - \frac{4V(0)}{Ml^2}}\right)^{1/2} \;,$$
$$V(0) = \frac{J^2}{4} - E^2 \;. \qquad (1.23)$$

c_1, c_2 are constants in terms of (l, M, r_m), given by Eqs. (4.21), and $\wp$ is the Weierstrass elliptic $\wp$ function with invariants (g_2, g_3), discriminant Δ and roots (e_1, e_2, e_3), given by Eqs. (4.22)–(4.25). The three cases (i)–(iii) correspond to the cases $\Delta > 0$, $\Delta < 0$, and $\Delta = 0$, respectively. Notice that $S^{(\text{ii})}_{\max} > S^{(\text{iii})}_{\max} = l\sqrt{M} > S^{(\text{i})}_{\max}$. In the case (i), $r(\tau)$ can be written in terms of the Jacobian elliptic function $\text{sn}[\tau^*, k]$, $\tau^* = \sqrt{e_1 - e_3}\tau$, $k = \sqrt{(e_2 - e_3)/(e_1 - e_3)}$. It follows that the solution (i) oscillates between the two zeros r_{01} and r_{02} of $V(r)$, with period 2ω, where ω is the real semiperiod of the Weierstrass function, $\omega = K(k)/\sqrt{e_1 - e_3}$, in terms of the complete elliptic integral of the first kind $K(k)$. We have

$$r(0) = r_m, \quad r(\omega) = \sqrt{Ml^2 - r_m^2} \;,$$
$$r(2\omega) = r_m, \ldots \;. \qquad (1.24)$$

In case (ii) ($\Delta < 0$) two roots (e_1, e_3) become complex, the string collapses into a point $r = 0$, and we have

$$r(0) = r_m, \quad r\left(\frac{\omega_2}{2}\right) = 0, \quad r(\omega_2) = r_m, \ldots \;, \qquad (1.25)$$

where ω_2 is the real semiperiod of the Weierstrass function for this case. In case (iii) ($\Delta = 0$) the elliptic functions reduce to hyperbolic functions

$$r(\tau) = \frac{\sqrt{M}l}{\cosh(\sqrt{M}\tau)} \;, \qquad (1.26)$$

so that

$$r(-\infty) = 0, \quad r(0) = r_m = \sqrt{M}l, \quad r(+\infty) = 0 \;. \qquad (1.27)$$

Here, the string starts as a point, grows until $r = r_m$ (at τ=0), and then it contracts until it collapses again ($r = 0$) at $\tau = +\infty$. In this case the string makes only one oscillation between $r = 0$ and $r = r_m$.

Notice that for the static background ($J = 0$), the only allowed motion is (ii), i.e., $r_m > r_{\rm hor} = \sqrt{Ml}$ (there is no ergosphere and only one horizon in this case), with

$$r_m = \sqrt{\frac{Ml^2}{2}}\left(1+\sqrt{1+\frac{4E^2}{Ml^2}}\right)^{1/2} . \qquad (1.28)$$

For $J = 0$, the string collapses into $r = 0$ and stops there. The Penrose diagram of the 2+1 BHAdS spacetime for $J = 0$ is very similar to the Penrose diagram of the ordinary ($D > 3$) Schwarzschild spacetime, so the string motion outwards from $r = 0$ is unphysical because of the causal structure. The coordinate time $t(\tau)$ is expressed in terms of the incomplete elliptic integral of the third kind Π, Eq. (4.40). The string has its maximal size r_m at τ=0, passes the horizon at $\tau = \tau_{\rm hor}$ [expressed in terms of the incomplete elliptic integral of the first kind, Eq. (4.41)] and falls into $r = 0$ for $\tau = \omega_2/2$, ω_2 being the real semiperiod of the Weierstrass function, Eq. (4.36). That is, we have

$$r(0) = r_m, \quad r(\tau_{\rm hor}) = \sqrt{Ml}, \quad r(\omega_2/2) = 0 , \qquad (1.29)$$
$$t(0) = 0, \quad t(\tau_{\rm hor}) = \infty ,$$

and $t(\omega_2/2)$ is expressed in terms of the Jacobian ζ function Z, Eq. (4.42). We also study the circular strings in the ordinary $D > 3$ spacetimes. In the 3+1 Kerr–anti–de Sitter (or Kerr–de Sitter) spacetime, the potential $V(r)$ is given by Eq. (4.46), covering seven powers in r. The general circular string solution involves higher genus elliptic functions and it is not necessary to go into details here. We will compare with the nonrotating cases, only.

It is instructive to recall [29] the circular string in Minkowski (Min) spacetime, for which $V(r) = r^2 - E^2$ [Fig. 2(a)], the string oscillates between its maximal size $r_m = E$, and $r = 0$ with the solution $r(\tau) = r_m|\cos\tau|$.

In the Schwarzschild black hole (S) $V(r) = r^2 - 2Mr - E^2$ [Fig. 2(b)], the solution is remarkably simple: $r(\tau) = M + \sqrt{M^2+E^2}\cos\tau$. The mathematical solution oscillates between $r_m = M + \sqrt{M^2+E^2}$ and $M - \sqrt{M^2+E^2} < 0$, but because of the causal structure and the curvature singularity the motion cannot be continued after the string has collapsed into $r = 0$.

For anti–de Sitter spacetime (AdS), we find $V(r) = r^2(1 + H^2r^2) - E^2$ [Fig. 2(c)]. The string oscillates between $r_m = \frac{1}{\sqrt{2}H}(-1+\sqrt{1+4H^2E^2})^{1/2}$ and $r = 0$ with the solution

$$r(\tau) = r_m|\mathrm{cn}[(1+4H^2E^2)^{1/4}, k]| , \qquad (1.30)$$

which is periodic with period 2ω:

$$\omega = \frac{K(k)}{(1+4H^2E^2)^{1/4}}, \quad k = \left(\frac{\sqrt{1+4H^2E^2}-1}{2\sqrt{1+4H^2E^2}}\right)^{1/2} . \qquad (1.31)$$

For Schwarzschild–anti–de Sitter spacetime (SAdS), we find $V(r) = r^2(1 + H^2r^2) - 2Mr - E^2$ [Fig. 2(d)] and

$$r(\tau) = r_m - \frac{1}{d_1\wp(\tau)+d_2}, \quad r(0) = r_m . \qquad (1.32)$$

d_1, d_2 are constants given in terms of (M, H, r_m) by Eqs. (4.60) and (4.62) [r_m is the root of the equation $V(r)$=0, which has in this case exactly one positive solution]. The invariants, the discriminant and the roots are determined by Eqs. (4.63) and (4.64). The string starts with $r = r_m$ at $\tau = 0$, it then contracts and eventually collapses into the $r = 0$ singularity. The existence of elliptic function solutions for the string motion is characteristic of the presence of a cosmological constant. For $\Lambda = 0 = -3H^2$ the circular string motion reduces to simple trigonometric functions. From Fig. 2 and our analysis we see that the circular string motion is qualitatively very similar in all these backgrounds (Min, S, AdS, SAdS): the string has a maximal *bounded* size and then it contracts towards $r = 0$. There are however physical and quantitative differences: in Minkowski and pure anti–de Sitter spacetimes, the string truly oscillates between r_m and $r = 0$, while in the black hole cases (S, SAdS), there is only one half oscillation, the string motion stops at $r = 0$. This also holds for the 2+1 BHAdS spacetime with $J = 0$ [Fig. 1(b)]. Notice also that in all these cases, $V(0) = -E^2 < 0$ and $V(r) \sim r^\alpha$; (α=2,4) for $r \gg E$.

The similarity can be pushed one step further by considering small perturbations around the circular strings. We find

$$\ddot{C}_n + \left(n^2 + \frac{r}{2}\frac{da(r)}{dr} + \frac{r^2}{2}\frac{d^2a(r)}{dr^2} - \frac{2E^2}{r^2}\right)C_n = 0 , \qquad (1.33)$$

determining the Fourier components of the comoving perturbations. For the spacetimes of interest here, $a(r) = 1-2M/r+H^2r^2$ (Min, S, AdS, SAdS), or $a(r) = r^2/l^2-1$ (2+1 BHAdS), the comoving perturbations are regular except near $r = 0$, where we find (for all cases) $r(\tau) \approx -E(\tau - \tau_0)$ and

$$\ddot{C}_n - \frac{2}{(\tau-\tau_0)^2}C_n = 0 . \qquad (1.34)$$

It follows that not only the unperturbed circular strings, but also the comoving perturbations around them behave in a similar way in all these nonrotating backgrounds (2+1 and higher dimensional). This should be contrasted with the string perturbations around the center of mass, which behave differently in these backgrounds. It must be noticed that for rotating ($J \neq 0$) spacetimes, the circular string behavior is qualitatively different from the nonrotating ($J = 0$) spacetimes. For large J, both in the 2+1 BHAdS was well as in the 3+1 ordinary Kerr-AdS spacetimes, noncollapsing circular string solutions exist. The potential $V(r) \to +\infty$ for $r \to 0$ and no collapse into $r = 0$ is possible.

The dynamics of circular strings in curved spacetimes is determined by the string tension, which tends to contract the string, and by the local gravity (which may be attractive or repulsive). In all the previous backgrounds, the local gravity is attractive (i.e., $da(r)/dr > 0$), and it acts together with the string tension in the sense of contraction. But in spacetimes with regions in which repulsion [i.e., $da(r)/dr < 0$] dominates, the strings can expand with unbounded radius (unstable strings [6,8]). It may also happen that the string tension and the local gravity be of the same order, i.e., the two opposite effects can balance, and the string is stationary. de Sitter spacetime provides an example in which all such type of solutions exist [6,8]. In de Sitter spacetime, $V(r)$ is unbounded from below for $r \to \infty$ [$V(r) \sim -r^4$] and unbounded expanding circular strings are present. In addition, an interesting new feature appears in the presence of a positive cosmological constant: the existence of multistring solutions [6–8]. The world-sheet time τ turns out to be a multivalued (finite or infinite) function of the physical time. That is, one single world sheet where $-\infty \leq \tau \leq +\infty$, can describe many (even infinitely many [8]) different and independent strings (in flat spacetime, one single world sheet describes only one string). In the S, AdS, and SAdS spacetimes, the multistring feature is absent.

We also study here the circular strings in Schwarzschild–de Sitter spacetime, where regions with $da(r)/dr > 0$ and $da(r)/dr < 0$ exist. The potential in this case is $V(r) = -H^2r^4 + r^2 - 2Mr - E^2$ with $V(0) = V(r_+) = V(r_-) = -E^2$, where the horizons $r_\pm$ are given by Eqs. (5.6)–(5.8), and $V(r) \sim -r^4$ for large r, see Fig. 3. It has a local minimum between $r = 0$ and r_-, and a local maximum at $r = r_0$ given by Eq. (5.12), $r_- < r_0 < r_+$. The motion is very complicated here, but again, it can be exactly determined in terms of elliptic functions, for which we analyze here only the degenerate case. We find two different types of solutions $r_\pm(\tau)$ given by Eqs. (5.15)–(5.19) with the properties

$$\begin{gathered} r_+(-\infty) = r_0, \quad r_+(0) = 0 , \\ r_-(-\infty) = r_0, \quad r_-(0) = \infty, \quad r_-(\infty) = r_0 . \end{gathered} \tag{1.35}$$

$r_+(\tau)$ describes one contracting string starting with maximal size r_0 at $\tau = -\infty$, passing the inner horizon r_- and falling into the $r = 0$ singularity at $\tau = 0$. The solution $r_-(\tau)$ describes two different and independent strings: String I starts with minimal size r_0 at $\tau = -\infty$ and grows until infinite size at τ=0 (unstable string). String II starts with infinite size at $\tau = 0$ and contracts until minimal size r_0 at $\tau = +\infty$. They never collapse into the $r = 0$ singularity. The $r_-(\tau)$ solution is very similar to the two-string solution discussed in Refs. [6,8] in the pure de Sitter case.

Finally, we study the circular string in the black string background. In this case (see Fig. 4),

$$\tilde{V}(r) = \frac{J^2}{4r^4} - \frac{M}{r^2} + \frac{1}{l^2} - E^2 , \tag{1.36}$$

with $V(r_\pm) = -E^2$, $V(\infty) = 1/l^2 - E^2$, and $V(0) = +\infty$ for $J \neq 0$, while $V(0) = -\infty$ for $J = 0$ (we only consider positive M). One effect of the dual transformation is to change the asymptotic behavior of $V(r)$. We see that if $E^2l^2 > 1$, then $V(\infty) < 0$, which gives rise to solutions of unbounded r. This is to be contrasted with the solutions in the BHAdS spacetimes in which the ring solutions are always bounded. Another effect of duality here is to change the invariant string size; we find $\tilde{S}(\tau) = 1/r(\tau)$. For $J \neq 0$, all solutions are bounded (finite $\tilde{S}$), while for $J = 0$, unbounded (infinite $\tilde{S}$) exists as well. For $J \neq 0$, the general solution can be expressed in terms of elliptic functions (elementary functions for $J = 0$), whose description and physical interpretation are to be described elsewhere.

This paper is organized as follows: In Sec. II, we review the 2+1 BHAdS and black string backgrounds. In Sec. III we describe and solve the string perturbations around the string center of mass in these backgrounds and in the ordinary black hole AdS spacetime. In Sec. IV we solve the exact circular string motion in all these backgrounds and compare between them, and in Sec. V we discuss the circular string motion in the black hole de Sitter case. A summary of our results and conclusions is presented in Tables I and II.

II. REVIEW OF THE (2+1)-DIMENSIONAL BLACK HOLE

In this section we give a short introduction to the black hole anti–de Sitter (BHAdS) solution of (2+1)-dimensional Einstein theory, recently found by Bañados, Teitelboim, and Zanelli [20]. There are now several ways to obtain this solution [20–22]. A simple way is to take as the starting point a line element in the form

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2d\phi^2 , \tag{2.1}$$

where $a(r)$ is an arbitrary function of r. The nonvanishing components of the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, take the form

$$G_{rr} = \frac{a_{,r}}{2ar}, \quad G_{tt} = -\frac{aa_{,r}}{2r}, \quad G_{\phi\phi} = \frac{r^2}{2}a_{,rr} . \tag{2.2}$$

The only vacuum solution to the Einstein equations is a=const, corresponding to flat spacetime, as is well known in three dimensions. However, if we introduce a cosmological constant

$$T^\mu{}_\nu = \text{diag}(1, 1, 1)\Lambda , \tag{2.3}$$

the solution to the Einstein equations becomes nontrivial:

$$a(r) = c - \Lambda r^2 , \tag{2.4}$$

where c is an arbitrary constant. Usually c is scaled to 1, and then a positive Λ represents de Sitter space, while a negative Λ represents anti–de Sitter space. On the other hand, since the constant c is completely arbitrary, we may as well take a negative c and then we also find solutions in the form

$$ds^2 = \left(1 - \frac{r^2}{l^2}\right) dt^2 + \left(\frac{r^2}{l^2} - 1\right)^{-1} dr^2 + r^2 d\phi^2 \ , \quad (2.5)$$

where l is a constant. This is in fact the simplest example of the 2+1 BHAdS solutions of Ref. [20]. This particular solution, where ϕ is identified with $\phi + 2\pi$, is a black hole spacetime with mass $M = 1$ and angular momentum $J = 0$. There is a horizon at $r = l$ and asymptotically it approaches anti–de Sitter space with $\Lambda = -1/l^2$. A two-parameter family (mass M and angular momentum J) of black holes is obtained by periodically identifying a linear combination of t and ϕ. This leads to the solution (the details can be found in Refs. [20,21])

$$ds^2 = \left(M - \frac{r^2}{l^2}\right) dt^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1} dr^2 - J dt d\phi + r^2 d\phi^2 \ , \quad (2.6)$$

with two horizons (provided $Ml^2 > J^2$)

$$r_\pm = \left(\frac{Ml^2}{2} \pm \frac{l}{2}\sqrt{M^2 l^2 - J^2}\right)^{1/2} \quad (2.7)$$

and a static limit

$$r_{\rm erg} = \sqrt{M} l \ , \quad (2.8)$$

defining an ergosphere, as for ordinary Kerr black holes. The Riemann tensor, corresponding to the line element (2.6), is given by

$$R_{\mu\rho\sigma\nu} = -\frac{1}{l^2}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\sigma\rho}) \ , \quad (2.9)$$

so that the curvature is constant

$$R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu} \ . \quad (2.10)$$

The geometry of the solution (2.6), near $r = 0$ in particular, is discussed in detail in Refs. [23,25].

We close this section with a few remarks on the relevance of this solution in string theory [21,22]. To lowest order in an expansion in α', the string action is [30]

$$S = \int d^3x \sqrt{-g} e^{-2\Phi} [4/k + R + 4(\nabla\Phi)^2 - \tfrac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}] \ , \quad (2.11)$$

where Φ is the dilation field and $H_{\mu\nu\rho}$ is the field strength of the Kalb-Ramond field $B_{\mu\nu}$ ($H_{\mu\nu\rho} = \partial_{[\rho} B_{\mu\nu]}$). It is easy to show that the metric (2.6) is a solution to the equations of motion corresponding to the action (2.11), when supplemented by [21]

$$B_{\phi t} = r^2/l^2, \quad \Phi = 0, \quad k = l^2 \ . \quad (2.12)$$

Following Horowitz and Welch [21] the connection to the WZWN σ-model approach is most easily established by dualizing the solution (2.6), (2.12) on the cyclic coordinate ϕ. According to the well-known procedure [27], the dual solution to (2.6), (2.12) is then given by [21]

$$d\tilde{s}^2 = \left(M - \frac{J^2}{4r^2}\right) dt^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1} dr^2 + \frac{2}{l} dt d\phi + \frac{d\phi^2}{r^2} \ ,$$

$$\tilde{B}_{\phi t} = -J/2r^2, \quad \tilde{\Phi} = -\ln r \ , \quad (2.13)$$

which after diagonalization of the metric becomes

$$d\tilde{s}^2 = -\left(1 - \frac{\mathcal{M}}{\tilde{r}}\right) d\tilde{t}^2 + \left(1 - \frac{\mathcal{Q}^2}{\mathcal{M}\tilde{r}}\right) d\tilde{x}^2 + \left(1 - \frac{\mathcal{M}}{\tilde{r}}\right)^{-1} \left(1 - \frac{\mathcal{Q}^2}{\mathcal{M}\tilde{r}}\right)^{-1} \frac{l^2 d\tilde{r}^2}{4\tilde{r}^2} \ ,$$

$$\tilde{B}_{\tilde{x}\tilde{t}} = \frac{\mathcal{Q}}{r}, \quad \tilde{\Phi} = -\tfrac{1}{2} \ln \tilde{r} l \ , \quad (2.14)$$

where

$$t = \frac{l(\tilde{x} - \tilde{t})}{\sqrt{r_+^2 - r_-^2}}, \quad \phi = \frac{r_+^2 \tilde{t} - r_-^2 \tilde{x}}{\sqrt{r_+^2 - r_-^2}} \ ,$$

$$\mathcal{M} = \tfrac{r_+^2}{l}, \quad \mathcal{Q} = \tfrac{J}{2}, \quad r^2 = \tilde{r} l \ . \quad (2.15)$$

The metric of (2.14) is exactly the black string solution in three dimensions of Horne and Horowitz [26], obtained by gauging the WZWN σ model of the group SL(2,R) $\times$ R.

Notice that the spacetime (2.13) is stationary and axially symmetric, and that it has the same nonvanishing components of the metric tensor as the original spacetime (2.6). This will be important when we consider circular strings in Sec. V. Another interesting observation is that the duality transformation does not change the two horizons (2.7), while the static limit is changed to $r_{\rm erg}$(black string)$=J/(2\sqrt{M})$.

III. PERTURBATIONS AROUND THE STRING CENTER OF MASS

One of the main purposes of the present paper is to consider the classical propagation of a bosonic test string in the 2+1 BHAdS spacetime (the spacetime is taken as a fixed background and no backreactions of the strings are included). The point particle geodesics were recently investigated in Ref. [24]. The string equations of motion are highly nonlinear coupled partial differential equations, so we will restrict ourselves by considering two different approaches. In this section we calculate first- and second-order string fluctuations around the string center of mass, following the approach originally developed by de Vega and Sánchez [1], and in Sec. IV we consider exact circular strings winding around the black hole. In both cases analysis and comparison with the ordinary black hole AdS and (dS) solutions in 3+1 dimensions is done.

A. General formalism

Let us solve the string equations of motion and constraints by considering perturbations around the exact string center of mass solution. In this subsection, we shortly review the method, and we demonstrate the simplifications arising at first order in the expansion, when considering only physical (perpendicular to the geodesic) perturbations.

In an arbitrary curved spacetime of dimension D, the string equations of motion and constraints, in the conformal gauge, take the form

$$\ddot{x}^\mu - x''^\mu + \Gamma^\mu_{\rho\sigma}(\dot{x}^\rho\dot{x}^\sigma - x'^\rho x'^\sigma) = 0 , \tag{3.1}$$

$$g_{\mu\nu}\dot{x}^\mu x'^\nu = g_{\mu\nu}(\dot{x}^\mu\dot{x}^\nu + x'^\mu x'^\nu) = 0 , \tag{3.2}$$

for $\mu = 0, 1, \ldots, (D-1)$ and a prime and overdot represent derivatives with respect to σ and τ, respectively. Consider first the equations of motion (3.1). A particular solution is provided by the string center of mass $q^\mu(\tau)$:

$$\ddot{q}^\mu + \Gamma^\mu_{\rho\sigma}\dot{q}^\rho\dot{q}^\sigma = 0 . \tag{3.3}$$

Then a perturbative series around this solution is developed:

$$x^\mu(\tau,\sigma) = q^\mu(\tau) + \eta^\mu(\tau,\sigma) + \xi^\mu(\tau,\sigma) + \cdots . \tag{3.4}$$

After insertion of Eq. (3.4) in Eq. (3.1) the equations of motion are to be solved order by order in the expansion.

To zeroth order we just get Eq. (3.3). To first order we find

$$\ddot{\eta}^\mu + \Gamma^\mu_{\rho\sigma,\lambda}\dot{q}^\rho\dot{q}^\sigma\eta^\lambda + 2\Gamma^\mu_{\rho\sigma}\dot{q}^\rho\dot{\eta}^\sigma - \eta''^\mu = 0 . \tag{3.5}$$

The first three terms can be written in covariant form [3], c.f. the ordinary geodesic deviation equation:

$$\dot{q}^\lambda\nabla_\lambda(\dot{q}^\delta\nabla_\delta\eta^\mu) - R^\mu_{\epsilon\delta\lambda}\dot{q}^\epsilon\dot{q}^\delta\eta^\lambda\eta''^\mu = 0 . \tag{3.6}$$

However, we can go one step further. For a massive string, corresponding to the string center of mass satisfying

$$g_{\mu\nu}(q)\dot{q}^\mu\dot{q}^\nu = -m^2 , \tag{3.7}$$

there are $D-1$ physical polarizations of string perturbations around the geodesic $q^\mu(\tau)$. We therefore introduce $D-1$ normal vectors n^μ_R, $R = 1, 2, \ldots, (D-1)$,

$$g_{\mu\nu}n^\mu_R\dot{q}^\nu = 0, \quad g_{\mu\nu}n^\mu_R n^\nu_S = \delta_{RS} \tag{3.8}$$

and consider only first-order perturbations in the form

$$\eta^\mu = \delta x^R n^\mu_R , \tag{3.9}$$

where δx^R are the comoving perturbations, i.e., the perturbations as seen by an observer traveling with the center of mass of the string. The normal vectors are not uniquely defined by Eqs. (3.8). In fact, there is a gauge invariance originating from the freedom to make local rotations of the $(D-1)$-bein spanned by the normal vectors. For our purposes it is convenient to fix the gauge taking the normal vectors to be covariantly constant:

$$\dot{q}^\mu\nabla_\mu n^\nu_R = 0 . \tag{3.10}$$

This is achieved by choosing the basis (q^μ, n^μ_R) obeying conditions (3.8) at a given point, and defining it along the geodesic by means of parallel transport. Another useful formula is the completeness relation that takes the form

$$g^{\mu\nu} = -\frac{1}{m^2}\dot{q}^\mu\dot{q}^\nu + n^{\mu R}n^\nu_R . \tag{3.11}$$

Using Eqs. (3.7)–(3.10) in Eq. (3.6) we find after multiplication by $g_{\mu\nu}n^\nu_S$ the spacetime invariant formula

$$(\partial^2_\tau - \partial^2_\sigma)\delta x_R - R_{\mu\rho\sigma\nu}n^\mu_R n^\nu_S\dot{q}^\rho\dot{q}^\sigma\delta x^S = 0 . \tag{3.12}$$

Since the last term depends on σ only through δx^S it is convenient to make a Fourier expansion

$$\delta x_R(\tau,\sigma) = \sum_n C_{nR}(\tau)e^{-in\sigma} . \tag{3.13}$$

Then, Eq. (3.12) finally reduces to

$$\ddot{C}_{nR} + (n^2\delta_{RS} - R_{\mu\rho\sigma\nu}n^\mu_R n^\nu_S\dot{q}^\rho\dot{q}^\sigma)C^S_n = 0 , \tag{3.14}$$

which constitutes a matrix Schrödinger equation with τ playing the role of the spatial coordinate.

For the second-order perturbations the picture is a little more complicated. Since they couple to the first-order perturbations we consider the full set of perturbations ξ^μ [1,3]:

$$\dot{q}^\lambda\nabla_\lambda(\dot{q}^\delta\nabla_\delta\xi^\mu) - R^\mu_{\epsilon\delta\lambda}\dot{q}^\epsilon\dot{q}^\delta\xi^\lambda - \xi''^\mu = U^\mu , \tag{3.15}$$

where the source U^μ is bilinear in the first-order perturbations, and explicitly given by

$$\begin{aligned} U^\mu = & -\Gamma^\mu_{\rho\sigma}(\dot{\eta}^\rho\dot{\eta}^\sigma - \eta'^\rho\eta'^\sigma) - 2\Gamma^\mu_{\rho\sigma,\lambda}\dot{q}^\rho\eta^\lambda\dot{\eta}^\sigma \\ & - \tfrac{1}{2}\Gamma^\mu_{\rho\sigma,\lambda\delta}\dot{q}^\rho\dot{q}^\sigma\eta^\lambda\eta^\delta . \end{aligned} \tag{3.16}$$

After solving Eqs. (3.14) and (3.15) for the first- and second-order perturbations, the constraints (3.2) have to be imposed. In world-sheet light cone coordinates ($\sigma^\pm = \tau \pm \sigma$) the constraints take the form

$$T_{\pm\pm} = g_{\mu\nu}\partial_\pm x^\mu\partial_\pm x^\nu = 0 , \tag{3.17}$$

where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. The world-sheet energy-momentum tensor $T_{\pm\pm}$ is conserved, as can be easily verified using Eq. (3.1), and therefore can be written

$$\begin{aligned} T_{--} &= \frac{1}{2\pi}\sum_n \tilde{L}_n e^{-in(\sigma-\tau)} , \\ T_{++} &= \frac{1}{2\pi}\sum_n L_n e^{-in(\sigma+\tau)} . \end{aligned} \tag{3.18}$$

At the classical level under consideration in this paper, the constraints are then simply

$$L_n = \tilde{L}_n = 0 \ . \tag{3.19}$$

Up to second order in the expansion around the string center of mass we find

$$\begin{aligned} T_{\pm\pm} = & -\tfrac{1}{4}m^2 + g_{\mu\nu}\dot{q}^\mu\partial_\pm\eta^\nu + \tfrac{1}{4}g_{\mu\nu,\rho}\dot{q}^\mu\dot{q}^\nu\eta^\rho \\ & + g_{\mu\nu}\dot{q}^\mu\partial_\pm\xi^\nu + g_{\mu\nu}\partial_\pm\eta^\mu\partial_\pm\eta^\nu + g_{\mu\nu,\rho}\dot{q}^\mu\eta^\rho\partial_\pm\eta^\nu \\ & + \tfrac{1}{4}g_{\mu\nu,\rho}\dot{q}^\mu\dot{q}^\nu\xi^\rho + \tfrac{1}{8}g_{\mu\nu,\rho\sigma}\dot{q}^\mu\dot{q}^\nu\eta^\rho\eta^\sigma. \end{aligned} \tag{3.20}$$

In the following subsections we apply this formalism to the 2+1 BHAdS and to the black string solution of Sec. II, as well as to ordinary (3+1)-dimensional black hole AdS solutions.

B. Strings in the 2+1 BHAdS background

We now consider a string in the background of the (2+1)-dimensional BHAdS spacetime. For simplicity we take a non-rotating black hole ($J = 0$) and we consider a radially infalling string. This case is of sufficient complexity for our purposes; the more general case of a string with angular momentum in the rotating background is to be considered elsewhere.

Equations (3.3) and (3.7) determining the string center of mass lead to

$$\dot{t} = \frac{-E}{M - \frac{r^2}{l^2}} \ , \tag{3.21}$$

$$\dot{r}^2 + m^2\left(\frac{r^2}{l^2} - M\right) = E^2 \ , \tag{3.22}$$

where E is an integration constant. These two equations are solved by

$$t(\tau) = \frac{l}{2\sqrt{M}} \ln\left|\frac{1 - (E/m\sqrt{M})\tan m\tau/l}{1 + (E/m\sqrt{M})\tan m\tau/l}\right| \tag{3.23}$$

and

$$r(\tau) = -\frac{l}{m}\sqrt{Mm^2 + E^2}\sin\frac{m}{l}\tau \ . \tag{3.24}$$

Here the boundary conditions were chosen such that r takes its maximal value at $\tau = -\frac{l}{m}\frac{\pi}{2}$ and the string falls into $r = 0$ for $\tau \to 0_-$:

$$r_{\max} = r\left(-\frac{l}{m}\frac{\pi}{2}\right) = \frac{l}{m}\sqrt{Mm^2 + E^2}, \quad r(0) = 0 \ . \tag{3.25}$$

The string center of mass passes the horizon $r_{\text{hor}} = \sqrt{M}l$ at

$$\tau_{\text{hor}} = -\frac{l}{m}\arcsin\frac{m\sqrt{M}}{\sqrt{Mm^2 + E^2}} \tag{3.26}$$

and we find, from Eq. (3.23),

$$t\left(-\frac{l}{m}\frac{\pi}{2}\right) = 0, \quad t(\tau_{\text{hor}}) = \infty, \quad t(0) = 0 \ . \tag{3.27}$$

Let us now turn to the string perturbations around the solution (3.23) and (3.24). The two covariantly constant normal vectors satisfying Eqs. (3.8) and (3.10) are given by

$$\begin{aligned} n^\mu_\perp &= \left(0, 0, \frac{1}{r}\right) \ , \\ n^\mu_\parallel &= \left(\frac{\dot{r}}{m\left(M - \frac{r^2}{l^2}\right)}, -\frac{E}{m}, 0\right) \ , \end{aligned} \tag{3.28}$$

which define transverse and longitudinal comoving perturbations through Eq. (3.9), respectively. It is however remarkable that in this case we do not need the explicit expressions for the normal vectors and for the Riemann tensor to calculate the first-order perturbations (3.14). Using Eq. (2.9) and the normalization equations (3.7) and (3.8) we immediately get

$$\ddot{C}_{nR} + \left(n^2 + \frac{m^2}{l^2}\right)C_{nR} = 0 \ , \tag{3.29}$$

where R takes the values "$\perp$ and "$\parallel$". Equations (3.29) are easily solved and the comoving perturbations (3.13) are given by

$$\delta x_R(\tau, \sigma) = \sum_n [A_{nR}e^{-i(n\sigma + \omega_n\tau)} + \tilde{A}_{nR}e^{-i(n\sigma - \omega_n\tau)}] \ , \tag{3.30}$$

where

$$\omega_n = \sqrt{n^2 + m^2/l^2} \ , \tag{3.31}$$

$$A_{nR} = \tilde{A}^\dagger_{-nR} \ . \tag{3.32}$$

The string perturbations η^μ introduced in Eq. (3.4) are

$$\eta^t = \frac{\dot{r}}{m(M - r^2/l^2)}\delta x_\parallel, \quad \eta^r = -\frac{E}{m}\delta x_\parallel, \quad \eta^\phi = \frac{1}{r}\delta x_\perp \ , \tag{3.33}$$

and are plagued by coordinate singularities at $r = r_{\text{hor}}$ (for η^t) and at $r = 0$ (for η^ϕ). The comoving perturbations δx_R, (3.30), are however completely finite and regular trigonometric functions. Notice that in the "pure" de Sitter spacetime [1] the perturbations satisfy Eq. (3.30), but with frequency $\omega_n = \sqrt{n^2 - m^2/l^2}$, thus unstable modes (for $|n| < m/l$) appear and the perturbations blow up. The presence of such instabilities is a generic exact feature in the de Sitter spacetime [5,12]. In the present 2+1 BHAdS background, all frequencies ω_n are real and instabilities do not occur.

Notice also that the comoving perturbations (3.30) are independent of the black hole mass M (and of E). In fact, Eq. (3.29) is the "pure" anti–de Sitter result, where the perturbations are independent of the polarization as they of course should be in an isotropic spacetime. This suggests that we have to calculate at least the second-order perturbations to ensure that the effects of the black hole mass are included in the perturbations. The second-order perturbations are determined by Eqs. (3.15) and (3.16). It turns out that the ξ^ϕ equation decouples while the ξ^t and ξ^r equations constitute a set of two coupled partial second-order linear differential equations.

We first consider the ξ^ϕ equation. Explicitly it is given by

$$\ddot{\xi}^\phi - \xi''^\phi + \frac{2}{r}\dot{r}\dot{\xi}^\phi = U^\phi \ , \tag{3.34}$$

where

$$U^\phi = -\frac{2}{r}(\dot{\eta}^r\dot{\eta}^\phi - \eta'^r\eta'^\phi) + 2\frac{\dot{r}}{r^2}\eta^r\dot{\eta}^\phi \tag{3.35}$$

The source U^ϕ is here written in terms of η^r and η^ϕ and its explicit expression as a function of τ and σ can be obtained using Eqs. (3.30), (3.33), and (3.24). It is convenient to make the redefinitions

$$\Sigma^\phi \equiv r\xi^\phi, \quad \tilde{U}^\phi \equiv rU^\phi \tag{3.36}$$

and the Fourier expansions

$$\Sigma^\phi(\tau,\sigma) = \sum_n \Sigma_n^\phi(\tau)e^{-in\sigma} \ , \qquad \tilde{U}^\phi(\tau,\sigma) = \sum_n \tilde{U}_n^\phi(\tau)e^{-in\sigma} \ . \tag{3.37}$$

Equation (3.34) then reduces to

$$\ddot{\Sigma}_n^\phi + \left(n^2 + \frac{m^2}{l^2}\right)\Sigma_n^\phi = \tilde{U}_n^\phi \ , \tag{3.38}$$

that is solved by

$$\Sigma_n^\phi(\tau) = B_n e^{-i\omega_n\tau} + \tilde{B}_n e^{i\omega_n\tau} + \frac{e^{i\omega_n\tau}}{2i\omega_n} \times \int^\tau \tilde{U}_n^\phi(\tau')e^{-i\omega_n\tau'}d\tau' - \frac{e^{-i\omega_n\tau}}{2i\omega_n}\int^\tau \tilde{U}_n^\phi(\tau')e^{i\omega_n\tau'}d\tau' \ , \tag{3.39}$$

where $B_n = \tilde{B}_{-n}^\dagger$ and ω_n is defined in Eq. (3.31).

The perturbations ξ^t and ξ^r are somewhat more complicated to derive. By redefining ξ^r and U^r,

$$\xi^r \equiv \left(\frac{r^2}{l^2} - M\right)\xi^* \ , \tag{3.40}$$

$$U^r \equiv \left(\frac{r^2}{l^2} - M\right)U^* \ , \tag{3.41}$$

we find, from Eq. (3.15),

$$\begin{pmatrix}\ddot{\xi}^t \\ \ddot{\xi}^*\end{pmatrix} - \begin{pmatrix}\xi''^t \\ \xi''^*\end{pmatrix} + 2\mathcal{A}\begin{pmatrix}\dot{\xi}^t \\ \dot{\xi}^*\end{pmatrix} + \mathcal{B}\begin{pmatrix}\xi^t \\ \xi^*\end{pmatrix} = \begin{pmatrix}U^t \\ U^*\end{pmatrix} \ , \tag{3.42}$$

where the matrices $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A} = \begin{pmatrix} a & b \\ b & a\end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & c \\ 0 & d\end{pmatrix};$$

$$a = \frac{r\dot{r}}{l^2\left(\frac{r^2}{l^2} - M\right)}, \quad b = \frac{Er}{l^2\left(\frac{r^2}{l^2} - M\right)} \ , \tag{3.43}$$

$$c = \frac{2E\dot{r}}{l^2\left(\frac{r^2}{l^2} - M\right)}, \quad d = \frac{2E^2}{l^2\left(\frac{r^2}{l^2} - M\right)} - \frac{m^2}{l^2} \ .$$

The first-order τ derivatives in Eq. (3.42) are eliminated by the transformation

$$\begin{pmatrix}\xi^t \\ \xi^*\end{pmatrix} \equiv \mathcal{G}\begin{pmatrix}\Sigma^t \\ \Sigma^*\end{pmatrix}, \quad \mathcal{G} = \exp\left(-\int^\tau \mathcal{A}(\tau')d\tau'\right) \ , \tag{3.44}$$

i.e.,

$$\mathcal{G} = \frac{-1}{m(r^2/l^2 - M)}\begin{pmatrix}\dot{r} & E \\ E & \dot{r}\end{pmatrix} \ . \tag{3.45}$$

We now Fourier expand the second-order perturbations and the sources,

$$\Sigma^t(\tau,\sigma) = \sum_n \Sigma_n^t(\tau)e^{-in\sigma} \ , \qquad \Sigma^*(\tau,\sigma) = \sum_n \Sigma_n^*(\tau)e^{-in\sigma} \ , \tag{3.46}$$

$$U^t(\tau,\sigma) = \sum_n U_n^t(\tau)e^{-in\sigma} \ , \qquad U^*(\tau,\sigma) = \sum_n U_n^*(\tau)e^{-in\sigma} \ , \tag{3.47}$$

and the matrix equation (3.42) reduces to

$$\begin{pmatrix}\ddot{\Sigma}_n^t \\ \ddot{\Sigma}_n^*\end{pmatrix} + \mathcal{V}\begin{pmatrix}\Sigma_n^t \\ \Sigma_n^*\end{pmatrix} = \mathcal{G}^{-1}\begin{pmatrix}U_n^t \\ U_n^*\end{pmatrix} \equiv \begin{pmatrix}\tilde{U}_n^t \\ \tilde{U}_n^*\end{pmatrix} \ , \tag{3.48}$$

where

$$\mathcal{V} = \mathcal{G}^{-1}(n^2 I + \mathcal{B} - \mathcal{A}^2 - \dot{\mathcal{A}})\mathcal{G} = \begin{pmatrix} n^2 + \frac{m^2}{l^2} & 0 \\ 0 & n^2\end{pmatrix} \ , \tag{3.49}$$

i.e., two decoupled inhomogeneous second-order linear differential equations with *constant* coefficients. It follows that the complete solution is known, and given explicitly by

$$\Sigma_n^t(\tau) = C_n e^{-i\omega_n\tau} + \tilde{C}_n e^{i\omega_n\tau} + \frac{e^{i\omega_n\tau}}{2i\omega_n}\int^\tau \tilde{U}_n^t(\tau')e^{-i\omega_n\tau'}d\tau' - \frac{e^{-i\omega n\tau}}{2i\omega_n}\int^\tau \tilde{U}_n^t(\tau')e^{i\omega_n\tau'}d\tau' \tag{3.50}$$

and

$$\Sigma^{*}_{n}(\tau) = D_n e^{-in\tau} + \tilde{D}_n e^{in\tau} + \frac{e^{in\tau}}{2in}\int^{\tau} \tilde{U}^{*}_{n}(\tau')e^{-in\tau'}d\tau' - \frac{e^{-in\tau}}{2in}\int^{\tau} \tilde{U}^{*}_{n}(\tau')e^{in\tau'}d\tau' , \quad (3.51)$$

where

$$C_n = \tilde{C}^{\dagger}_{-n}, \;\; D_n = D^{\dagger}_{-n}, \;\; \tilde{D}_n = \tilde{D}^{\dagger}_{-n} . \quad (3.52)$$

The string perturbations ξ^{μ} introduced in Eq. (3.4) are finally

$$\xi^{t} = \frac{1}{m(M - r^2/l^2)}\left[\dot{r}\sum_n \Sigma^{t}_{n}(\tau)e^{-in\sigma} + E\sum_n \Sigma^{*}_{n}(\tau)e^{-in\sigma}\right] ,$$
$$\xi^{\phi} = \frac{1}{r}\sum_n \Sigma^{\phi}_{n}(\tau)e^{-in\sigma} , \quad (3.53)$$
$$\xi^{r} = -\frac{E}{m}\sum_n \Sigma^{t}_{n}(\tau)e^{-in\sigma} - \frac{\dot{r}}{m}\sum_n \Sigma^{*}_{n}(\tau)e^{-in\sigma} .$$

These expressions are quite complicated because of the integrals of the sources in Eqs. (3.39), (3.50), and (3.51). To allow comparison with the ordinary (3+1)-dimensional black hole cases (see next subsection) it is enough to consider the region $r \to 0$ (equivalent to the region $\tau \to 0_-$). The sources are calculated from Eq. (3.16) using the first-order perturbations (3.30) and (3.33). To leading order in τ for $\tau \to 0_-$ we find

$$U^{\phi}(\tau,\sigma) = \frac{2E}{m(Mm^2+E^2)\tau^4}\sum_n\sum_p (A_{p\parallel} + \tilde{A}_{p\parallel})(A_{n-p\perp} + \tilde{A}_{n-p\perp})e^{-in\sigma} + O\left(\frac{1}{\tau^3}\right) , \quad (3.54)$$

$$U^{r}(\tau,\sigma) = \frac{M}{\sqrt{Mm^2+E^2}\,\tau^3}\sum_n\sum_p (A_{p\perp} + \tilde{A}_{p\perp})(A_{n-p\perp} + \tilde{A}_{n-p\perp})e^{-in\sigma} + O\left(\frac{1}{\tau^2}\right) , \quad (3.55)$$

while U^t is finite for $\tau \to 0$. It follows that

$$\tilde{U}^{t}_{n}(\tau) = \frac{E}{m\sqrt{Mm^2+E^2}\,\tau^3}\sum_p (A_{p\perp} + \tilde{A}_{p\perp})(A_{n-p\perp} + \tilde{A}_{n-p\perp}) + O\left(\frac{1}{\tau^2}\right) ,$$

$$\tilde{U}^{*}_{n}(\tau) = \frac{1}{m\tau^3}\sum_p (A_{p\perp} + \tilde{A}_{p\perp})(A_{n-p\perp} + \tilde{A}_{n-p\perp}) + O\left(\frac{1}{\tau^2}\right) , \quad (3.56)$$

$$\tilde{U}^{\phi}_{n}(\tau) = \frac{-2E}{m\sqrt{Mm^2+E^2}\,\tau^3}\sum_p (A_{p\parallel} + \tilde{A}_{p\parallel})(A_{n-p\perp} + \tilde{A}_{n-p\perp}) + O\left(\frac{1}{\tau^2}\right) .$$

From Eqs. (3.39), (3.50), and (3.51) we find the asymptotic behavior of the Σ_n's:

$$\Sigma^{t}_{n}(\tau) = \frac{E}{2m\sqrt{Mm^2+E^2}\,\tau}\sum_p (A_{p\perp} + \tilde{A}_{p\perp})(A_{n-p\perp} + \tilde{A}_{n-p\perp}) + O(1) ,$$

$$\Sigma^{*}_{n}(\tau) = \frac{1}{2m\tau}\sum_p (A_{p\perp} + \tilde{A}_{p\perp})(A_{n-p\perp} + \tilde{A}_{n-p\perp}) + O(1) , \quad (3.57)$$

$$\Sigma^{\phi}_{n}(\tau) = \frac{-E}{m\sqrt{Mm^2+E^2}\,\tau}\sum_p (A_{p\parallel} + \tilde{A}_{p\parallel})(A_{n-p\perp} + \tilde{A}_{n-p\perp}) + O(1) ,$$

and from Eqs. (3.36), (3.44), and (3.45) we have

$$\xi^\phi(\tau,\sigma) = \frac{E}{mr^2}\sum_n\sum_p(A_{p\|}+\tilde{A}_{p\|})(A_{n-p\perp}+\tilde{A}_{n-p\perp})e^{-in\sigma}+O\left(\frac{1}{r}\right) , \tag{3.58}$$

$$\xi^r(\tau,\sigma) = \frac{-M}{2r}\sum_n\sum_p(A_{p\perp}+\tilde{A}_{p\perp})(A_{n-p\perp}+\tilde{A}_{n-p\perp})e^{-in\sigma}+O(1) , \tag{3.59}$$

while ξ^t is finite for $r \to 0$ ($\tau \to 0_-$). The singularities of ξ^ϕ and ξ^r for $r = 0$ are coordinate artifacts like the singularity of η^ϕ in Eq. (3.33). Such singularities appear even in flat Minkowski space when parametrized in terms of polar coordinates. In the present case the coordinate singularities are removed by introducing pseudo-Cartesian coordinates near $r = 0$:

$$\begin{aligned} T &= \sqrt{M}x^t , \\ X &= \frac{x^r}{\sqrt{M}}\sinh(\sqrt{M}x^\phi) , \\ Y &= \frac{x^r}{\sqrt{M}}\cosh(\sqrt{M}x^\phi) . \end{aligned} \tag{3.60}$$

It is easy to show that T, X, and Y are finite for $r \to 0$ when x^t, x^r, and x^ϕ are expanded up to second order (3.4) using Eqs. (3.33), (3.58), and (3.59). Having calculated the first- and second-order perturbations, we can now also calculate the world-sheet energy-momentum tensor $T_{\pm\pm}$ (3.18)–(3.20). This calculation is simplified using the fact that $T_{\pm\pm}$ are functions of $n(\sigma \pm \tau)$ while the first-order perturbations η^μ are functions of $(n\sigma \pm \omega_n\tau)$. The first-order perturbations can therefore only give constant contributions to $T_{\pm\pm}$. A straightforward but tedious calculation gives

$$L_n = -2\pi imnD_n, \quad \tilde{L}_n = 2\pi imn\tilde{D}_n; \quad n \neq 0 ,$$

$$L_0 = \pi\sum_n(\omega_n+n)^2[A_{n\|}\tilde{A}_{-n\|}+A_{n\perp}\tilde{A}_{-n\perp}] - \frac{\pi}{2}m^2 , \tag{3.61}$$

$$\tilde{L}_0 = \pi\sum_n(\omega_n-n)^2[A_{n\|}\tilde{A}_{-n\|}+A_{n\perp}\tilde{A}_{-n\perp}] - \frac{\pi}{2}m^2 ,$$

and from Eq. (3.19) we obtain the constraints

$$D_n = \tilde{D}_n = 0, \quad \sum_n n\omega_n[A_{n\|}\tilde{A}_{-n\|}+A_{n\perp}\tilde{A}_{-n\perp}] = 0 ,$$

as well as

$$m^2 = 2\sum_n\left(2n^2+\frac{m^2}{l^2}\right)[A_{n\|}\tilde{A}_{-n\|}+A_{n\perp}\tilde{A}_{-n\perp}] , \tag{3.62}$$

determining the mass of the string. Notice that the mass formula of the string is modified with respect to the usual flat space expression ($m^2 = 4\sum_n n^2[A_{n\|}\tilde{A}_{-n\|} + A_{n\perp}\tilde{A}_{-n\perp}]$). The reason for this modification is the asymptotic character of the spacetime; in particular, the presence of a cosmological constant. In the ordinary black hole spacetime without a Λ term, which is asymptotically flat (and in which bounded orbits do not appear), the mass formula is the same as in flat spacetime [10]. In the "pure" de Sitter spacetime, however, the mass formula is modified exactly in the same way as Eq. (3.62), but for a positive Λ, i.e., with a term $(2n^2-m^2/l^2)$ in the sum. The quantization of the string in the 2+1 BHAdS background and the consequences of Eq. (3.62) for the quantum mass spectrum, are to be discussed elsewhere.

This concludes our analysis of the first- and second-order perturbations around the string center of mass, for a string embedded in the 2+1 BHAdS spacetime of Sec. II.

C. Strings in the ordinary black hole anti–de Sitter spacetime

String perturbations around the center of mass of a string embedded in ordinary higher dimensional black hole and de Sitter backgrounds were already considered in Refs. [1,10,11]. Here we take the equatorial plane of (3+1)-dimensional Schwarzschild anti–de Sitter space, to better allow comparison with the results of subsection III B. We furthermore use the formalism of subsection III A, where only physical first-order perturbations are considered. This simplifies the analysis considerably as compared to Refs. [1,10,11].

The line element is taken in the form

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 . \tag{3.63}$$

In the first place we take $a(r)$ to be an arbitrary function of r, but we will eventually be interested in the case

$$a(r) = 1 - \frac{2M}{r} + H^2r^2 , \tag{3.64}$$

corresponding to Schwarzschild–anti–de Sitter space. For a radially infalling string in the equatorial plane ($\theta = \pi/2$, ϕ=const) the geodesic equations (3.3) and (3.7) for the string center of mass are solved by

$$\dot{t} = E/a(r) , \tag{3.65}$$

$$\dot{r}^2 + m^2a(r) = E^2 , \tag{3.66}$$

generalizing Eqs. (3.21) and (3.22). The $\dot{r}$ equation

(3.66) is solved by

$$\tau-\tau_0=\int_{r_0}^{r}\frac{dx}{\sqrt{E^2-m^2a(x)}}\;, \tag{3.67}$$

and after inversion, giving r as an explicit function of τ, the coordinate time t is obtained by the integration of Eq. (3.65). In the case of a Schwarzschild black hole this gives the well-known results in terms of elementary functions (see, for instance Ref. [28], section 25.5), while in the case under consideration here, where the function $a(r)$ is given by Eq. (3.64), $r(\tau)$ and $t(\tau)$ will be expressed in terms of elliptic functions.

There are now three covariantly constant normal vectors satisfying Eqs. (3.8) and (3.10):

$$\begin{aligned} n^{\mu}_{1\perp}&=\left(0,0,\frac{1}{r},0\right)\;,\\ n^{\mu}_{2\perp}&=\left(0,0,0,\frac{1}{r}\right)\;,\\ n^{\mu}_{\parallel}&=\left(\frac{\dot r}{ma(r)},\frac{E}{m},0,0\right)\;. \end{aligned} \tag{3.68}$$

The nonvanishing components of the Riemann tensor, corresponding to the line element (3.63), are

$$\begin{aligned} &R_{rtrt}=\tfrac{1}{2}a_{,rr},\quad R_{r\theta r\theta}=\frac{-r}{2a}a_{,r},\quad R_{r\phi r\phi}=\frac{-r}{2a}a_{,r}\sin^2\theta\;,\\ &R_{t\theta t\theta}=\frac{r}{2}aa_{,r},\quad R_{t\phi t\phi}=\frac{r}{2}aa_{,r}\sin^2\theta\;,\\ &R_{\theta\phi\theta\phi}=r^2(1-a)\sin^2\theta\;. \end{aligned} \tag{3.69}$$

Equations (3.14), determining the Fourier components of the comoving first-order perturbations, separate and reduce to

$$\ddot C_{nS\perp}+\left(n^2+\frac{m^2a_{,r}}{2r}\right)C_{nS\perp}=0,\quad S=1,2\;, \tag{3.70}$$

$$\ddot C_{n\parallel}+\left(n^2+\frac{m^2a_{,rr}}{2}\right)C_{n\parallel}=0\;. \tag{3.71}$$

For the two transverse polarizations we find, from Eq. (3.64),

$$\ddot C_{nS\perp}+\left(n^2+m^2H^2+\frac{Mm^2}{r^3}\right)C_{nS\perp}=0,\quad S=1,2\;, \tag{3.72}$$

while, for the longitudinal polarization,

$$\ddot C_{n\parallel}+\left(n^2+m^2H^2-\frac{2Mm^2}{r^3}\right)C_{n\parallel}=0\;. \tag{3.73}$$

From these equations it is obvious that we need only look for singularities in the region $r\to 0$. However, for the transverse perturbations (3.72) the bracket is always positive. We therefore expect that the solution is oscillating and bounded even for $r\to 0$. For the longitudinal perturbations (3.73), on the other hand, the bracket can be negative. In that case imaginary frequencies arise and instabilities develop. The ($|n|$=1) instability sets in at

$$r_{\rm inst}=\left(\frac{2Mm^2}{1+m^2H^2}\right)^{1/3}\;, \tag{3.74}$$

which may be inside or outside the horizon, depending on the relation between the various parameters (M,H,m) involved. The higher modes develop instabilities for smaller r.

These results are easily confirmed by the exact time evolution of the perturbations. For $r\to 0$ we find, from Eq. (3.66),

$$r(\tau)\approx(3m\sqrt{M/2})^{2/3}(\tau_0-\tau)^{2/3}\;, \tag{3.75}$$

where the integration constant is chosen such that $r\to 0$ corresponds to $\tau\to\tau_0$. Then for $\tau\to\tau_0$ Eqs. (3.72) and (3.73) are

$$\ddot C_{nS\perp}+\frac{2}{9(\tau_0-\tau)^2}C_{nS\perp}=0,\quad S=1,2\;, \tag{3.76}$$

$$\ddot C_{n\parallel}-\frac{4}{9(\tau_0-\tau)^2}C_{n\parallel}=0\;, \tag{3.77}$$

with complete solutions

$$C_{nS\perp}(\tau)=\alpha_{nS\perp}(\tau_0-\tau)^{1/3}+\beta_{nS\perp}(\tau_0-\tau)^{2/3}\;,\qquad S=1,2\;, \tag{3.78}$$

$$C_{n\parallel}(\tau)=\gamma_{n\parallel}(\tau_0-\tau)^{4/3}+\delta_{n\parallel}(\tau_0-\tau)^{-1/3}\;, \tag{3.79}$$

where $(\alpha_{nS\perp},\beta_{nS\perp},\gamma_{n\parallel},\delta_{n\parallel})$ are constants. It follows that $C_{nS\perp}(\tau)$ is finite for $r\to 0$ $(\tau\to\tau_0)$ while $C_{n\parallel}(\tau)$ blows up because of the $\delta_{n\parallel}$ term. This result demonstrates the important difference between the 2+1 BHAdS solution of Sec. II and the equatorial plane of an ordinary higher-dimensional black hole. In the first case we found finite bounded perturbations, while in the latter case instabilities develop already in the first-order comoving perturbations near $r=0$. For the ordinary black hole anti–de Sitter spacetime it is then meaningless to analyze higher-order perturbations for $r\to 0$, in contrast, as we have seen, to the 2+1 BHAdS background. In the ordinary black hole anti–de Sitter spacetime the string falls to the center ($r=0$) and is trapped by the singularity. For $\tau\to\tau_0$ the potential for the radial perturbations, Eq. (3.77), is of the type $-\gamma/(\tau-\tau_0)^2$ with γ=4/9, which is a singular attractive potential.

D. Strings in the black string background

We close this section with the analysis of the string propagation in the black string background. It is convenient to take the metric in the form (2.14) and for simplicity we consider the uncharged ($\mathcal{Q}=0$) black string:

$$ds^2 = -\left(1-\frac{\mathcal{M}}{r}\right)dt^2 + \left(1-\frac{\mathcal{M}}{r}\right)^{-1}\frac{l^2dr^2}{4r^2} + dx^2 \,, \tag{3.80}$$

where the tildes have been deleted [cf. Eq. (2.14)]. This spacetime is just the direct product of Witten's two-dimensional black hole [31] and the real line space. It has a horizon at $r = \mathcal{M} = Ml$ and, contrary to its dual, the 2+1 BHAdS solution, it has a strong curvature singularity at $r = 0$. To compare with the results of subsections III B and III C we consider a radially infalling string, corresponding to x=const. Equations (3.3) and (3.7) for the string center of mass become

$$\dot{t} = \frac{E}{1-\mathcal{M}/r} \,, \tag{3.81}$$

$$\dot{r}^2 = \frac{4r^2}{l^2}\left[E^2 - m^2 + \frac{m^2\mathcal{M}}{r}\right] \,, \tag{3.82}$$

which can be solved in terms of elementary functions (taking for simplicity $m^2 > E^2$)

$$r(\tau) = \frac{m^2\mathcal{M}}{2(m^2-E^2)}\left[1 - \sin\frac{2\sqrt{m^2-E^2}}{l}\tau\right] \tag{3.83}$$

and

$$t(\tau) = E\tau - \frac{l}{2}\ln\left|\frac{1+[(m^2-2E^2)/(m^2-2E\sqrt{m^2-E^2})]\tan(\sqrt{m^2-E^2}/l)\tau}{1+[(m^2-2E^2)/(m^2+2E\sqrt{m^2-E^2})]\tan(\sqrt{m^2-E^2}/l)\tau}\right| \,, \tag{3.84}$$

where the integration constants were chosen such that $t(0)$=0 and $r(\tau_0)$=0 for

$$\tau_0 = \frac{\pi l}{4\sqrt{m^2-E^2}} \,. \tag{3.85}$$

Notice that the horizon is passed for $\tau = \tau_{\rm hor}$:

$$\tau_{\rm hor} = \frac{l}{2\sqrt{m^2-E^2}}\arcsin\left(\frac{2E^2}{m^2} - 1\right) \,, \tag{3.86}$$

and that $t(\tau_{\rm hor}) = \infty$.

The two covariantly constant normal vectors, satisfying Eqs. (3.8) and (3.10), are given by

$$n_\perp^\mu = (0,0,1) \,,$$

$$n_\parallel^\mu = \left(\frac{l\dot{r}}{2m(r-\mathcal{M})}, \frac{2Er}{ml}, 0\right) \,. \tag{3.87}$$

The only nonvanishing component of the Riemann tensor, corresponding to the line element (3.80), is

$$R_{trtr} = \frac{-\mathcal{M}}{2r^3} \,, \tag{3.88}$$

and then Eqs. (3.14), determining the string perturbations, take the form

$$\ddot{C}_{n\perp} + n^2 C_{n\perp} = 0 \,, \tag{3.89}$$

$$\ddot{C}_{n\parallel} + \left(n^2 - \frac{2m^2\mathcal{M}}{l^2 r}\right)C_{n\parallel} = 0 \,. \tag{3.90}$$

Not surprisingly, the perturbations in the x direction are completely finite and regular. For the longitudinal perturbations we see that the term in the parentheses in Eq. (3.90) becomes negative and approaches $-\infty$ for $r \to 0$, suggesting an instability. This is confirmed by the exact time evolution near r =0. Using Eq. (3.82) we find, for $r \to 0$,

$$r(\tau) \approx \frac{m^2\mathcal{M}}{l^2}(\tau_0 - \tau)^2 \,, \tag{3.91}$$

where the integration constant is chosen such that $r \to 0$ corresponds to $\tau \to \tau_0$. Equation (3.90) is now, for $\tau \to \tau_0$,

$$\ddot{C}_{n\parallel} - \frac{2}{(\tau_0-\tau)^2}C_{n\parallel} = 0 \,, \tag{3.92}$$

with the solution

$$C_{n\parallel}(\tau) = \alpha_{n\parallel}(\tau_0-\tau)^2 + \frac{\beta_{n\parallel}}{\tau_0-\tau} \,. \tag{3.93}$$

The solution indeed blows up for $r \to 0$ ($\tau \to \tau_0$) with conclusions similar to the ordinary black hole case, subsection III C.

IV. CIRCULAR STRINGS IN STATIONARY AXIALLY SYMMETRIC BACKGROUNDS

In this section we consider circular strings embedded in stationary axially symmetric backgrounds. Circular strings in curved spacetimes have attracted important interest recently [6,8,32–38]. The analysis is carried out in 2+1 dimensions, but the results will hold for the equatorial plane of higher-dimensional backgrounds as well. To be more specific we consider the line element

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + 2g_{t\phi}(r)dtd\phi + g_{\phi\phi}(r)d\phi^2 \,, \tag{4.1}$$

that will be general enough for our purposes here. It obviously includes as special cases the BHAdS solution of Sec. II (as well as the black string) and the equatorial plane of the black hole solutions of Einstein theory in 3+1 dimensions.

The circular string ansatz, consistent with the symmetries of the background, is taken to be

$$t = t(\tau), \quad r = r(\tau), \quad \phi = \sigma + f(\tau) , \tag{4.2}$$

where the three functions $t(\tau)$, $r(\tau)$, and $f(\tau)$ are to be determined by the equations of motion and constraints (3.1)–(3.2). The equations of motion (3.1) for the ansatz (4.2) and the background (4.1) lead to

$$\begin{gathered} \ddot{t} + 2\Gamma^t_{tr}\dot{t}\dot{r} + 2\Gamma^t_{\phi r}\dot{r}\dot{f} = 0 , \\ \ddot{r} + \Gamma^r_{rr}\dot{r}^2 + \Gamma^r_{tt}\dot{t}^2 + \Gamma^r_{\phi\phi}(\dot{f}^2 - 1) + 2\Gamma^r_{t\phi}\dot{t}\dot{f} = 0 , \\ \ddot{f} + 2\Gamma^\phi_{tr}\dot{t}\dot{r} + 2\Gamma^\phi_{\phi r}\dot{f}\dot{r} = 0 , \end{gathered} \tag{4.3}$$

while the constraints become

$$\begin{gathered} g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\phi\phi}(\dot{f}^2 + 1) + 2g_{t\phi}\dot{t}\dot{f} = 0 , \\ g_{t\phi}\dot{t} + g_{\phi\phi}\dot{f} = 0 . \end{gathered} \tag{4.4}$$

This system of second-order ordinary differential equations and constraints is most easily described as a Hamiltonian system:

$$\mathcal{H} = \tfrac{1}{2}g^{tt}P_t^2 + \tfrac{1}{2}g^{rr}P_r^2 + \tfrac{1}{2}g^{\phi\phi}P_\phi^2 + g^{t\phi}P_tP_\phi + \tfrac{1}{2}g_{\phi\phi} , \tag{4.5}$$

supplemented by the constraints

$$\mathcal{H} = 0, \quad P_\phi = 0 . \tag{4.6}$$

The function $f(\tau)$ introduced in Eq. (4.2) does not represent any physical degrees of freedom. It describes the "longitudinal" rotation of the circular string and is therefore a pure gauge artifact. This interpretation is consistent with Eq. (4.6) saying that there is no angular momentum P_ϕ.

The Hamilton equations of the two cyclic coordinates t and f are

$$\dot{f} = g^{\phi\phi}P_\phi + g^{t\phi}P_t, \quad \dot{t} = g^{tt}P_t + g^{t\phi}P_\phi , \tag{4.7}$$

as well as

$$P_\phi = \text{const} = 0, \quad P_t = \text{const} \equiv -E , \tag{4.8}$$

where E is an integration constant and we used Eq. (4.6). The two functions $t(\tau)$ and $f(\tau)$ are then determined by

$$\dot{f} = -Eg^{t\phi} , \tag{4.9}$$

$$\dot{t} = -Eg^{tt} , \tag{4.10}$$

that can be integrated provided $r(\tau)$ is known. Using Eqs. (4.6), (4.9), and (4.10) the Hamilton equation of r becomes, after one integration,

$$\dot{r}^2 = -g^{rr}(E^2g^{tt} + g_{\phi\phi}) , \tag{4.11}$$

so that $r(\tau)$ can be obtained by inversion of

$$\tau - \tau_0 = \int_{r_0}^{r} \frac{dx}{\sqrt{-g^{rr}(x)[E^2g^{tt}(x) + g_{\phi\phi}(x)]}} . \tag{4.12}$$

For the cases that we will consider in the following, Eq. (4.12) will be solved in terms of either elementary or elliptic functions.

We close this subsection by the following interesting observation: Insertion of the ansatz (4.2), using the results (4.9)–(4.11), in the line element (4.1) leads to

$$ds^2 = g_{\phi\phi}(d\sigma^2 - d\tau^2) . \tag{4.13}$$

We can then identify the invariant string size as

$$S(\tau) = \sqrt{g_{\phi\phi}(r(\tau))} . \tag{4.14}$$

For the (2+1)-dimensional BHAdS spacetime the invariant string size is then simply r, as well as for the ordinary (3+1)-dimensional Schwarzschild and Reissner-Nordström black hole backgrounds. For the black string background (2.13) the invariant string size is actually r^{-1}.

A. Circular strings in the 2+1 BHAdS spacetime

In the (2+1)-dimensional BHAdS spacetime (2.6), Equation (4.11) determining the invariant string size $r(\tau)$, takes the explicit form

$$\dot{r}^2 + V(r) = 0; \quad V(r) = r^2\left(\frac{r^2}{l^2} - M\right) + \frac{J^2}{4} - E^2 . \tag{4.15}$$

Here we have defined the potential $V(r)$ such that the dynamics takes place at the r axis in a $(r, V(r))$ diagram, see Fig. 1. The potential (4.15) has a global minimum between the two horizons:

$$V_{\min} = V\left(\sqrt{\frac{Ml^2}{2}}\right) = -\tfrac{1}{4}(M^2l^2 - J^2 + 4E^2) < 0 , \tag{4.16}$$

which is always negative, since we only consider the case when $Ml^2 \geq J^2$ [otherwise there are no horizons, see Eq. (2.7)]. For large values of r the potential goes as r^4 and at $r = 0$ we have

$$V(0) = \frac{J^2}{4} - E^2 , \tag{4.17}$$

that can be either positive, negative, or zero. Notice also that the potential vanishes provided

$$V(r_0) = 0 \Leftrightarrow r_{01,2} = \left(\frac{Ml^2}{2} \pm \frac{l}{2}\sqrt{M^2l^2 - J^2 + 4E^2}\right)^{1/2} . \tag{4.18}$$

Equation (4.18) leads to three fundamentally different types of solutions.

(i) For $J^2 > 4E^2$ there are two positive-r zeros of the potential [Fig. 1(a)]. The smallest zero is located between the inner horizon and $r = 0$, while the other

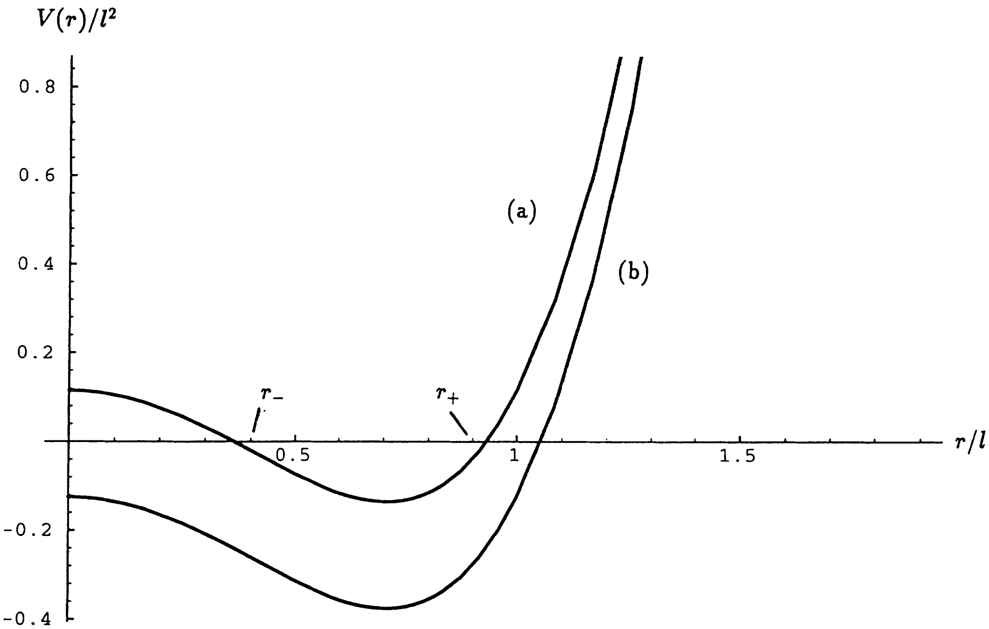

FIG. 1. The potential $V(r)$, Eq. (4.15), for a circular string in the (2+1)-dimensional black hole anti–de Sitter (BHAdS) spacetime. In (a) we have $J^2 > 4E^2$ and a barrier between the inner horizon and $r = 0$, while (b) represents a case where $J^2 < 4E^2$ and a string will always fall into $r = 0$. In the cases shown, the values of the various parameters are $M = 1$, $J = l/\sqrt{2}$ as well as E=0.1 [case (a)] and E=0.5 [case (b)]. The static limit is $r_{\rm erg} = l$.

zero is between the outer horizon and the static limit. Therefore, this string solution never comes outside the static limit. On the other hand, it never falls into $r = 0$. The mathematical solution oscillating between these two positive zeros of the potential may be interpreted as a string traveling between the different universes described by the maximal analytic extension of the spacetime (2.6) [the Penrose diagram of the (2+1)-dimensional BHAdS spacetime is discussed in Refs. [23,25]]. Such type of circular string solutions also exist in other stringy black hole backgrounds [36].

(ii) For $J^2 < 4E^2$ there is only one positive-r zero of the potential, which is always located outside the static limit [Fig. 1(b)]. The potential is negative for $r = 0$, so there is no barrier preventing the string from collapsing into $r = 0$. By suitably fixing the initial conditions the string starts with its maximal size outside the static limit at τ=0. It then contracts through the ergosphere and the two horizons and eventually falls into $r = 0$. If $J \neq 0$ it may, however, still be possible to continue this solution into another universe as in case (i).

(iii) $J^2 = 4E^2$ is the limiting case where the maximal string radius equals the static limit. The potential is exactly zero for r=0 so also in this case the string contracts through the two horizons and eventually falls into $r = 0$.

Let us now look at the exact mathematical solution of Eq. (4.15). In the general case (arbitrary J) the non-negative solution can be represented as

$$r(\tau) = \left| r_m - \frac{1}{c_1 \wp(\tau - \tau_0) + c_2} \right| , \tag{4.19}$$

where r_m is the maximal string radius,

$$r_m = \left(\frac{Ml^2}{2} + \frac{l}{2}\sqrt{M^2l^2 - J^2 + 4E^2} \right)^{1/2} , \tag{4.20}$$

c_1 and c_2 are two constants given by

$$c_1 = \left(\frac{r_m^3}{l^2} - \frac{Mr_m}{2} \right)^{-1} ,$$
$$c_2 = \frac{1}{12}\left(\frac{6r_m^2}{l^2} - M \right)\left(\frac{r_m^3}{l^2} - \frac{Mr_m}{2} \right)^{-1} , \tag{4.21}$$

and $\wp$ is the Weierstrass elliptic $\wp$ function [39],

$$\dot{\wp}^2 = 4\wp^3 - g_2\wp - g_3 , \tag{4.22}$$

with invariants

$$g_2 = \frac{M^2}{12} + \frac{Mr_m^2}{l^2} - \frac{r_m^4}{l^4},$$
$$g_3 = -\frac{M^3}{216} + \frac{M^2r_m^2}{6l^2} - \frac{Mr_m^4}{6l^4} , \tag{4.23}$$

discriminant

$$\Delta \equiv g_2^3 - 27g_3^2 = \frac{r_m^2}{l^2}\left(M - \frac{r_m^2}{l^2} \right)\left(\frac{M}{2} - \frac{r_m^2}{l^2} \right)^4 , \tag{4.24}$$

and roots

$$\left\{ -\frac{M}{6}, \frac{1}{2}\left(\frac{M}{6} \pm \frac{r_m}{l^2}\sqrt{Ml^2 - r_m^2} \right) \right\} . \tag{4.25}$$

The qualitatively different solutions (i), (ii), and (iii) discussed before, are now distinguished by the sign of the discriminant Δ:

$$J^2-4E^2>(<)0 \Leftrightarrow r_m^2<(>)Ml^2 \Leftrightarrow \Delta>(<)0\ , \quad (4.26)$$

where we used $Ml^2<2r_m^2$ (otherwise there are no horizons). We then analyze the three types of solutions in terms of the mathematical formalism above.

(i), $\Delta>0$: In this case the roots (4.25) are given by

$$\begin{aligned} e_1 &\equiv \frac{M}{12}+\frac{r_m}{2l^2}\sqrt{Ml^2-r_m^2}>0\geq e_2 \\ &\equiv \frac{M}{12}-\frac{r_m}{2l^2}\sqrt{Ml^2-r_m^2}>e_3\equiv -\frac{M}{6}\ , \end{aligned} \quad (4.27)$$

and the solution (4.19) can be written in terms of Jacobian elliptic functions:

$$r(\tau)=r_m\frac{\delta-\mathrm{sn}^2[\tau^*,k]}{\delta+\mathrm{sn}^2[\tau^*,k]}\ , \quad (4.28)$$

where

$$\tau^*=\sqrt{e_1-e_3}\,\tau,\quad \delta=\frac{4l^2(e_1-e_3)}{2r_m^2-Ml^2},$$
$$k=\left(\frac{e_2-e_3}{e_1-e_3}\right)^{1/2}\ . \quad (4.29)$$

It follows that

$$r(0)=r_m,\quad r(\omega)=\sqrt{Ml^2-r_m^2},\quad r(2\omega)=r_m,\ldots\ . \quad (4.30)$$

where ω is the real semiperiod of the Weierstrass function,

$$\omega=\frac{K(k)}{\sqrt{e_1-e_3}}\ , \quad (4.31)$$

and $K(k)$ is the complete elliptic integral of first kind. From Eqs. (4.28) and (4.30) it is seen that the solution oscillates between the two positive zeros (4.18) of the potential, with the period 2ω.

(ii), $\Delta<0$: Now two of the roots (4.25) become nonreal:

$$\begin{aligned} e_1 &\equiv \frac{M}{12}+i\frac{r_m}{2l^2}\sqrt{r_m^2-Ml^2}, \\ e_3 &\equiv \frac{M}{12}-i\frac{r_m}{2l^2}\sqrt{r_m^2-Ml^2}\ , \\ e_2 &\equiv -\frac{M}{6}\ , \end{aligned} \quad (4.32)$$

and Eq. (4.19) leads to

$$r(\tau)=r_m|\mathrm{cn}[2\sqrt{H_2}\tau,k]|\ , \quad (4.33)$$

where

$$H_2=\sqrt{e_2^2-e_1e_3},\quad k=\sqrt{\tfrac{1}{2}-3e_2/4H_2}\ . \quad (4.34)$$

It follows that

$$r(0)=r_m,\quad r(\omega_2/2)=0,\quad r(\omega_2)=r_m,\ldots, \quad (4.35)$$

where the real semiperiod of the Weierstrass function is now

$$\omega_2=\frac{K(k)}{\sqrt{H_2}}\ . \quad (4.36)$$

(iii), Δ=0: In this limiting case the elliptic functions reduce to hyperbolic functions. Explicitly we find

$$r(\tau)=\frac{\sqrt{M}l}{\cosh(\sqrt{M}\tau)}\ , \quad (4.37)$$

so that

$$r(-\infty)=0,\quad r(0)=r_m=\sqrt{M}l,\quad r(\infty)=0\ . \quad (4.38)$$

In this limiting case Δ=0, the mathematical solution only makes one oscillation between $r=0$ and the maximal radius.

Let us close this section with a few more words on the nonrotating black hole case. For J=0 we are always in case (ii); i.e., if the black hole has no angular momentum the circular string has its maximal invariant size larger than the horizon (there is no ergosphere and only one horizon in this case) and it always falls into $r=0$. The Penrose diagram for $J=0$ [23] is similar to the Penrose diagram of ordinary Schwarzschild spacetime, so the string motion outwards from $r=0$ is unphysical because of the causal structure. The string motion stops when the string falls into $r=0$.

The physical string size is given by Eq. (4.33) for $J=0$, and the coordinate time $t(\tau)$ is then obtained from Eq. (4.10),

$$t(\tau)=-El^2\int^{\tau}\frac{dx}{Ml^2-r_m^2\mathrm{cn}^2[2\sqrt{H_2}x,k]}\ , \quad (4.39)$$

which leads to

$$t(\tau)=\frac{El^2}{2(r_m^2-Ml^2)\sqrt{H_2}}\Pi\left(\frac{r_m^2}{r_m^2-Ml^2},\sqrt{H_2}\tau,k\right)\ . \quad (4.40)$$

Here Π is the incomplete elliptic integral of third kind and r_m, k, H_2 and ω_2 are given by Eqs. (4.20), (4.34), and (4.36), respectively, with $J=0$. The string has its maximal size for τ=0, passes the horizon $r_{\mathrm{hor}}=\sqrt{M}l$ at

$$\tau_{\mathrm{hor}}=\frac{1}{2\sqrt{H_2}}F\left(\arcsin\sqrt{1-\frac{Ml^2}{r_m^2}},k\right)\ , \quad (4.41)$$

where F is the incomplete elliptic integral of first kind, and falls into $r=0$ for $\tau=\omega_2/2$ [Eq. (4.35)]. The corresponding values of the coordinate time are given by

$$t(0)=0,\quad t(\tau_{\mathrm{hor}})=\infty\ ,$$
$$t\left(\frac{\omega_2}{2}\right)=\frac{Elr_m}{2\sqrt{MH_2}\sqrt{r_m^2-Ml^2}}\frac{Z[\epsilon,k]}{\sqrt{r_m^2(1-k^2)+k^2Ml^2}}\ , \quad (4.42)$$

where $Z[\epsilon,k]$ is the Jacobian ζ function [39] and $\mathrm{sn}[\epsilon,k]=(r_m^2-Ml^2)/r_m^2$.

B. Circular strings in ordinary spacetimes

We will now compare the circular strings in the (2+1)-dimensional BHAdS spacetime and in the equatorial plane of ordinary (3+1)-dimensional black holes. In the most general case it is natural to compare the spacetime metric (2.6) to the ordinary (3+1)-dimensional Kerr–anti–de Sitter spacetime with metric components

$$g_{tt} = \frac{a^2\Delta_\theta \sin^2\theta - \Delta_r}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\Delta_r},$$

$$g_{t\phi} = [\Delta_r - (r^2+a^2)\Delta_\theta]\frac{a\sin^2\theta}{\Delta_0\rho^2},$$

$$g_{\phi\phi} = [\Delta_\theta(r^2+a^2)^2 - a^2\Delta_r\sin^2\theta]\frac{\sin^2\theta}{\Delta_0^2\rho^2}, \quad g_{\theta\theta} = \frac{\rho^2}{\Delta_\theta}, \tag{4.43}$$

where we have introduced the notation

$$\Delta_r = (1-\tfrac{1}{3}\Lambda r^2)(r^2+a^2) - 2Mr, \quad \Delta_\theta = 1 + \tfrac{1}{3}\Lambda a^2\cos^2\theta,$$

$$\Delta_0 = 1 + \tfrac{1}{3}\Lambda a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta. \tag{4.44}$$

Here the mass is represented by M while a is the specific angular momentum, and a positive Λ corresponds to de Sitter while a negative Λ corresponds to anti–de Sitter spacetime. In the equatorial plane ($\theta = \pi/2$) the metric (4.43) is in the general form (4.1) and it is easy to see that the analysis of Sec. IV goes through, so that we can take over the general results of Eqs. (4.9)–(4.11) for the circular strings. In the most general case the t equation (4.10) and the r equation (4.11) take the form

$$\dot{t} = \frac{E}{\Delta_r r^2}[(r^2+a^2)^2 - a^2\Delta_r], \quad \dot{r}^2 + V(r) = 0, \tag{4.45}$$

where the potential is given explicitly by

$$V(r) = -\frac{\Lambda}{3\Delta_0}r^4 + \frac{1-2\Lambda a^2/3}{\Delta_0}r^2 - \frac{2M(1+2\Lambda a^2/3)}{\Delta_0^2}r + \frac{2a^2 - \Lambda a^2/3 - E^2\Delta_0^2}{\Delta_0} - \frac{4M\Lambda a^4}{3\Delta_0^2}\frac{1}{r} + \frac{a^2[\Delta_0(a^2 - E^2\Delta_0^2) - 4M^2]}{\Delta_0^2}\frac{1}{r^2} + \frac{2Ma^2(a^2 - E^2\Delta_0^2)}{\Delta_0^2}\frac{1}{r^3}; \tag{4.46}$$

i.e., the potential covers seven powers in r. The general solution will therefore involve higher genus elliptic functions and we shall not go into any detail here. It is furthermore very complicated to deduce the physical properties of the circular strings from the shape of the potential (the zeros, etc.) since the invariant string size (4.14) is nontrivially connected to r:

$$S(\tau) = \left(\frac{r^2+a^2}{1+\Lambda a^2/3} + \frac{2M}{r}\frac{a^2}{(1+\Lambda a^2/3)^2}\right)^{1/2}. \tag{4.47}$$

We will leave the general Kerr–anti–de Sitter case for analysis elsewhere, and concentrate here on the nonrotating case $a = 0$, that should be compared with the $J = 0$ case of the 2+1 BHAdS solution discussed at the end of subsection IV A. Our strategy will be to start with the simplest case of a circular string in flat Minkowski spacetime and then introduce a mass (Schwarzschild) and a negative cosmological constant (anti–de Sitter).

1. Minkowski space

This case was originally discussed by Vilenkin [29], and is obtained from Eqs. (4.45) and (4.46) taking $\Lambda = 0 = a = M$,

$$\dot{t} = E, \quad V(r) = r^2 - E^2, \tag{4.48}$$

i.e., [see Fig. 2(a)]:

$$V(0) = -E^2, \quad r_m = E, \quad V(r) \propto r^2 \quad \text{for } r \gg E. \tag{4.49}$$

The string oscillates between its maximal size $r = E$ and $r = 0$, with the solution of Eqs. (4.45) given explicitly by

$$r(\tau) = r_m|\cos\tau|, \quad t = E\tau. \tag{4.50}$$

2. Schwarzschild black hole

This case was already considered in Ref. [36] for $E = 0$ and in Ref. [38] for arbitrary E. The potential and coordinate time are obtained from Eqs. (4.45) and (4.46) with $a = 0 = \Lambda$:

$$\dot{t} = \frac{E}{1-2M/r}, \quad V(r) = r^2 - 2Mr - E^2, \tag{4.51}$$

i.e. [see Fig. 2(b)],

$$V(0) = -E^2, \quad r_m = M + \sqrt{M^2+E^2} > E, \quad V(r) \propto r^2 \quad \text{for } r \gg (M,E). \tag{4.52}$$

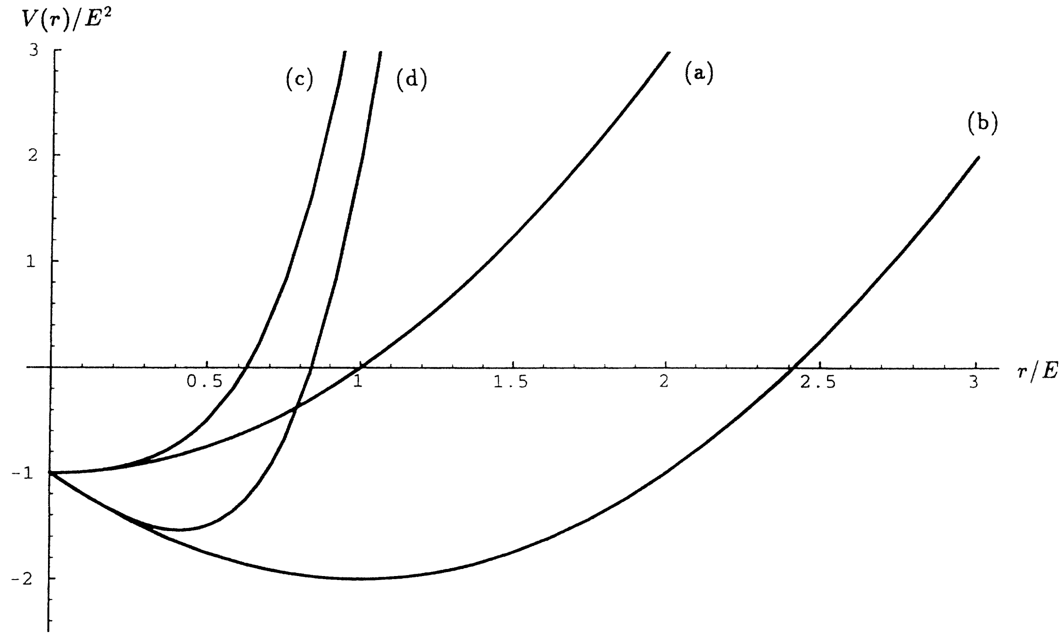

FIG. 2. The potential $V(r)$, Eq. (4.46), for a circular string in the equatorial plane of the four (3+1)-dimensional spacetimes: (a) Minkowski (Min) space, (b) Schwarzschild (S) black hole, (c) anti–de Sitter (AdS) space, and (d) Schwarzschild–anti–de Sitter (SAdS) space. The potentials are plotted for fixed E and we notice the following general relations between the maximal string radii r_m: $r_m^{\rm S} > r_m^{\rm Min} > r_m^{\rm AdS}$, $r_m^{\rm S} > r_m^{\rm SAdS} > r_m^{\rm AdS}$, and $r_m^{\rm Min} > r_m^{\rm SAdS} \Leftrightarrow H^2E^3 > 2M$.

The mathematical solution oscillates between $r = M + \sqrt{M^2+E^2}$ and $r = M - \sqrt{M^2+E^2} < 0$, but because of the causal structure and the curvature singularity the motion stops at $r = 0$. The solution of Eqs. (4.45) is remarkably simple (compare with the point particle case, see, for instance, Ref. [28]),

$$r(\tau) = M + \sqrt{M^2+E^2}\cos\tau ,$$

$$t(\tau) = E\tau + 2M\ln\left|\frac{\tan\tau/2 + (\sqrt{M^2+E^2}-M)/E}{\tan\tau/2 - (\sqrt{M^2+E^2}-M)/E}\right| . \tag{4.53}$$

3. Anti–de Sitter space

Here we take $a = 0 = M$ and $\Lambda \equiv -3H^2 < 0$ in Eqs. (4.45) and (4.46) and find

$$\dot{t} = \frac{E}{1+H^2r^2},\quad V(r) = r^2(1+H^2r^2) - E^2 , \tag{4.54}$$

i.e. [see Fig. 2(c)],

$$V(0) = -E^2,\quad r_m = \frac{1}{\sqrt{2}H}(-1+\sqrt{1+4H^2E^2})^{1/2} < E ,$$
$$V(r) \propto r^4 \quad \text{for } r \gg (1/H, E) . \tag{4.55}$$

The string is oscillating between $r = \frac{1}{\sqrt{2}H}(-1 + \sqrt{1+4H^2E^2})^{1/2}$ and $r = 0$. The solution of Eqs. (4.45) for $r(\tau)$ is

$$r(\tau) = r_m|\mathrm{cn}[(1+4H^2E^2)^{1/4}\tau, k]| , \tag{4.56}$$

which is periodic with period 2ω,

$$\omega = \frac{K(k)}{(1+4H^2E^2)^{1/4}};\quad k = \left(\frac{\sqrt{1+4H^2E^2}-1}{2\sqrt{1+4H^2E^2}}\right)^{1/2} . \tag{4.57}$$

Equation (4.54) can then be integrated, and we obtain, for $t(\tau)$,

$$t(\tau) = \frac{E}{(1+4H^2E^2)^{1/4}(1+H^2r_m^2)} \times\Pi\left(\frac{H^2r_m^2}{1+H^2r_m^2}, (1+4H^2E^2)^{1/4}\tau, k\right) , \tag{4.58}$$

where Π is the incomplete elliptic integral of the third kind.

4. Schwarzschild–anti–de Sitter space

By taking $a = 0$ and $\Lambda = -3H^2 < 0$ in Eqs. (4.45) and (4.46) we are finally in the case of Schwarzschild–anti–de

Sitter spacetime:

$$\dot{t} = \frac{E}{1 + H^2r^2 - 2M/r},$$
$$V(r) = H^2r^4 + r^2 - 2Mr - E^2 \ , \tag{4.59}$$

i.e. [see Fig. 2(d)],

$$V(0) = -E^2, \ \ H^2r_m^4 + r_m^2 - 2Mr_m - E^2 = 0 \ ,$$
$$V(r) \propto r^4 \ \text{ for } r \gg (M, H^{-1}, E) \ . \tag{4.60}$$

Notice that $V(r) \leq -E^2$ inside the horizon, and that dV/dr has only one real (positive) zero. It follows that the r_m equation (4.60) has exactly one positive solution which is then by definition r_m. The explicit (but not very enlightening) expression for r_m as a function of M, H, and E can of course be written down by solving the quartic equation, but we shall not give the result here. The solution of Eqs. (4.45) for $r(\tau)$ can be written in terms of the Weierstrass elliptic $\wp$ function:

$$r(\tau) = r_m - \frac{1}{d_1\wp(\tau - \tau_0) + d_2} \ , \tag{4.61}$$

where the two constants d_1 and d_2 are given by

$$d_1 = 2(r_m - M + 2H^2r_m^3)^{-1} \ ,$$
$$d_2 = \tfrac{1}{6}(1 + 6H^2r_m^2)(r_m - M + 2H^2r_m^3)^{-1} \ . \tag{4.62}$$

The invariants of the Weierstrass function are given by

$$g_2 = \tfrac{1}{12} + 2Mr_mH^2 - H^2r_m^2(1 + H^2r_m^2) \ , \tag{4.63}$$

$$g_3 = \tfrac{1}{216} + \frac{M^2H^2}{4} - \frac{Mr_mH^2}{3} + \frac{H^2r_m^2}{6}(1 + H^2r_m^2) \ , \tag{4.64}$$

from which one can calculate the discriminant, the roots, etc. From Eq. (4.61) it is, however, already clear that the string starts with maximal size r_m for τ=0, it then contracts and eventually collapses into the singularity $r = 0$ (taking for convenience τ_0=0).

From Fig. 2 and the above analysis we conclude that the circular string motion is in fact very similar in the equatorial plane of the four backgrounds of Minkowski space, anti–de Sitter space, Schwarzschild black hole, and Schwarzschild–anti–de Sitter space. In all these cases the string has a maximal size, and then contracts towards r=0. Quantitatively there are of course differences but qualitatively the motion from $r = r_m$ to $r = 0$ is the same. This also includes the (2+1)-dimensional BHAdS spacetime when $J = 0$: according to Fig. 1(b) and Eq. (4.35), the $J = 0$ circular string motion is qualitatively similar to the four cases described above.

This similarity can actually be pushed one step further by considering small perturbations propagating around the circular strings, using the covariant approach of Larsen and Frolov [4]. For a line element in the form (2.1) the circular string is determined by Eqs. (4.10) and (4.11):

$$\dot{t} = E/a(r) \ , \tag{4.65}$$

$$\dot{r}^2 + r^2a(r) = E^2 \ . \tag{4.66}$$

We can introduce a normal vector n^μ perpendicular to the string world sheet ($x^\mu = t, r, \phi$):

$$g_{\mu\nu}\dot{x}^\mu n^\nu = g_{\mu\nu}x'^\mu n^\nu = 0, \ \ g_{\mu\nu}n^\mu n^\nu = 1 \ , \tag{4.67}$$

explicitly given by

$$n^\mu = \left(\frac{\dot{r}}{ra(r)}, \frac{E}{r}, 0\right) \ , \tag{4.68}$$

and satisfying the completeness relation

$$g^{\mu\nu} = \frac{1}{r^2}(x'^\mu x'^\nu - \dot{x}^\mu\dot{x}^\nu) + n^\mu n^\nu \ . \tag{4.69}$$

For circular strings in the equatorial plane of a higher-dimensional spacetime there will also be normal vectors in the directions perpendicular to the plane of the string but they will not concern us here. By defining the comoving physical perturbations δx to be the perturbations in the n^μ direction,

$$\delta x^\mu = \delta x n^\mu \ , \tag{4.70}$$

it can be shown that [38]

$$(\partial_\sigma^2 - \partial_\tau^2)\delta x - \left(\frac{r}{2}\frac{da(r)}{dr} + \frac{r^2}{2}\frac{d^2a(r)}{dr^2} - \frac{2E^2}{r^2}\right)\delta x = 0 \ , \tag{4.71}$$

to first order in the perturbation. To solve this equation one has to first solve Eq. (4.66) for $r(\tau)$. After Fourier expanding δx we get the Schrödinger equation [38]

$$\ddot{C}_n + \left(n^2 + \frac{r}{2}\frac{da(r)}{dr} + \frac{r^2}{2}\frac{d^2a(r)}{dr^2} - \frac{2E^2}{r^2}\right)C_n = 0 \ , \tag{4.72}$$

determining the Fourier components of the comoving perturbations.

The four nonrotating spacetimes considered in this subsection are special cases of $a(r) = 1 - 2M/r + H^2r^2$, while the J=0 case of the 2+1 BHAdS spacetime corresponds to $a(r) = \frac{r^2}{l^2} - M$. However, in all these cases it is clear from Eq. (4.72) that the comoving perturbations are regular except near $r = 0$, where the dominant term in the "potential" (4.72) is the E^2 term. The E^2 term depends on $a(r)$ through the denominator r^2 that is obtained by solving Eq. (4.66). In all the nonrotating cases considered in this section it is easy, however, to see that for $r \to 0$ we have $r \approx -E(\tau - \tau_0)$. It follows that not only the unperturbed circular strings but also the comoving perturbations around them behave in a qualitatively similar way in all the nonrotating backgrounds considered in this section. This should be contrasted with the analysis of the string perturbations around the string center of mass (Sec. III), where we found qualitative differences between strings in ordinary (3+1)-dimensional black hole spacetimes and strings in the (2+1)-dimensional BHAdS spacetime. Notice, however, that in both approaches we have only made the comparison for nonrotating spacetimes.

V. CIRCULAR STRINGS WITH UNBOUNDED RADIUS

In the previous section we concluded that the circular string motion is qualitatively similar in the equatorial plane of spacetimes with the line element

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 , \quad (5.1)$$

with

$$a(r) = H^2 r^2 + 1 - \frac{2M}{r} , \quad (5.2)$$

and in the (2+1)-dimensional spacetime

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2 d\phi^2 , \quad (5.3)$$

with

$$a(r) = \frac{r^2}{l^2} - M . \quad (5.4)$$

This similarity can be physically understood in the following way: The dynamics of a circular string in a curved spacetime is determined by the string tension and by the local gravity. The string tension will always try to contract the circular string, while the local gravity can be either attractive or repulsive. For the spacetimes (5.1) and (5.3) the local gravity is proportional to the derivative of $a(r)$. It follows that in the cases represented by Eqs. (5.2) and (5.4) the local gravity is always positive [considering only positive M in Eq. (5.2)], corresponding to attraction. So in these cases both the string tension and the local gravity work in the direction of contraction of the circular string, and therefore all strings collapse to $r = 0$.

The above argument also suggests that we can find qualitatively different circular string motions by considering spacetimes with regions of negative local gravity (repulsion). In such spacetimes we can expect to find regions where the string tension is dominating, regions where the negative local gravity is dominating, and regions where the two opposite effects are of the same order, being a natural balance ensuring the existence of stationary circular strings (such a solution actually exists in de Sitter space [4,6,8]).

We have already seen that in the rotating ($J \neq 0$) (2+1)-dimensional BHAdS spacetime we can have string solutions qualitatively different from the J=0 solutions, namely noncollapsing circular strings (provided $J^2 > 4E^2$). The same happens in the equatorial plane of rotating ($a \neq 0$) (3+1)-dimensional spacetimes in the form (4.43). If $a^2 > E^2\Delta_0^2$ we see from Eq. (4.46) that the potential is positive infinite for $r \to 0$, so that no collapse into $r = 0$ is possible.

A somewhat simpler example is provided by the Reissner-Nordström black hole, which has a region of negative local gravity inside the (outer) horizon. Circular strings in Reissner-Nordström background were investigated in Ref. [38] and indeed noncollapsing solutions were found.

It is also easy to find spacetimes with negative local gravity in the asymptotic region $r \to \infty$. The simplest example is ordinary de Sitter space, which in the static parametrization takes the form (5.1) with $a(r) = 1 - H^2 r^2$. In that case the potential (4.46) goes to minus infinity [$V(r) \propto -r^4$] for $r \to \infty$ and unbounded expanding circular strings are found [6,8]. These solutions and the other types of circular string solutions in de Sitter space were discussed in great detail in Ref. [8], so we shall not say too much about it here. One of the most important results in de Sitter space is the existence of multistring solutions [6–8]. It turns out that for certain ranges of the integration constant E (which is called $-\sqrt{b}/H$ in Ref. [8]) the internal world-sheet time τ is a multivalued (finite or infinite) function of the cosmic time. This means that one single world sheet, where τ runs from $-\infty$ to $+\infty$, can describe finitely or infinitely many different and independent strings in de Sitter space. It is an interesting question whether this feature is also present in other curved backgrounds. Let us now consider briefly two other curved spacetimes in which we find circular string solutions with unbounded r, and multistring solutions.

A. Circular strings in Schwarzschild–de Sitter space

Schwarzschild–de Sitter space is in the form of Eq. (5.1) with

$$a(r) = 1 - \frac{2M}{r} - H^2 r^2 . \quad (5.5)$$

It is a very interesting spacetime for circular strings for several reasons. First of all it has regions of both positive [$da(r)/dr > 0$] local gravity and regions of negative [$da(r)/dr < 0$] local gravity. Second, it is asymptotically de Sitter, so we expect to find features similar to the "pure" de Sitter case, for instance, the existence of multistrings, as discussed above.

The mathematics of the circular strings is unfortunately going to be more complicated here, as compared to the cases discussed in Sec. IV and in Ref. [8]. From Eq. (5.5) it follows that Schwarzschild–de Sitter spacetime has two horizons (one horizon when equality) provided that $\sqrt{27}HM \leq 1$. Explicitly they are given by ($r_+ \geq r_-$),

$$r_- = M\left(-\frac{1-i\sqrt{3}}{2^{2/3}Z(HM)} - \frac{(1+i\sqrt{3})Z(HM)}{6H^2M^2 2^{1/3}}\right) , \quad (5.6)$$

$$r+ = M\left(\frac{2^{1/3}}{Z(HM)} + \frac{Z(HM)}{3H^2M^2 2^{1/3}}\right) , \tag{5.7}$$

where we introduced the notation

$$Z(HM) \equiv [-54H^4M^4 + \sqrt{-108H^6M^6(1-27H^2M^2)}]^{1/3} , \tag{5.8}$$

and we only consider the region $HM \in\,]0,\, 1/\sqrt{27}]$. The circular string potential (4.46) for $a = 0$ and $\Lambda \equiv 3H^2 > 0$ takes the form

$$V(r) = -H^2r^4 + r^2 - 2Mr - E^2 , \tag{5.9}$$

so that

$$V(0) = V(r_+) = V(r_-) = -E^2 ,$$
$$V(r) \propto -r^4 \quad \text{for } r \gg (M, H^{-1}, E) . \tag{5.10}$$

The potential has a local minimum between $r = 0$ and the inner horizon, and a local maximum at $r = r_0$ between the two horizons,

$$\left.\frac{dV(r)}{dr}\right|_{r=r_0} = 0, \quad \left.\frac{d^2V(r)}{dr^2}\right|_{r=r_0} < 0 , \tag{5.11}$$

where

$$r_0 = M\left(\frac{2^{1/3}}{W(HM)} + \frac{W(HM)}{6H^2M^2 2^{1/3}}\right) , \tag{5.12}$$

and

$$W(HM) \equiv [-108H^4M^4 + \sqrt{-432H^6M^6(2-27H^2M^2)}]^{1/3} . \tag{5.13}$$

Since the string dynamics takes place at the r axis in a $(r, V(r))$ diagram, it is important to know exactly the shape of the potential. If $V(r_0) > 0$ the potential has two zeros between the two horizons, and it will act effectively as a barrier, see Fig. 3. On the other hand, if $V(r_0) < 0$ there is no barrier and nothing can prevent a contracting string from collapsing into the singularity $r = 0$. One finds

$$V(r_0) > 0 \Leftrightarrow \frac{E^2}{M^2} < -H^2M^2\left(\frac{r_0}{M}\right)^4 + \left(\frac{r_0}{M}\right)^2 - 2\left(\frac{r_0}{M}\right) , \tag{5.14}$$

where the right-hand side of the (second) inequality depends on HM only. The inequality (5.14) also provides the critical value of (E/M) in terms of HM, when the potential equals zero for $r = r_0$.

The mathematical solution of Eqs. (4.45) for $r(\tau)$, determining the invariant string size as a function of τ, can be formally obtained from the Schwarzschild–anti–de Sitter case, discussed in subsection IV B, by simply changing the sign of H^2 in Eqs. (4.59)–(4.64). The parameter r_m defined in Eq. (4.60), however, can no longer be interpreted as the maximal string size. In the present case we can just take r_m to be any of the complex roots of the quartic equation (4.60) (with H^2 replaced by $-H^2$), and then also the (complex) constant τ_0 introduced in Eq. (4.61) must be carefully chosen to obtain a real $r(\tau)$ for real τ.

In the present paper we shall not go into a complete analysis of the various types of solutions. We will restrict ourselves by considering only the "degenerate" case $V(r_0)$=0, where the Weierstrass function reduces to an elementary function. In that case it is tempting to take $r_0 = r_m$, but then the constants d_1 and d_2, defined in Eq. (4.62), diverge, so we have to take one of the other roots of Eq. (4.60). After a little algebra we find

$$r(\tau) = r_0 + \frac{4(6H^2r_0^2-1)\exp[-\sqrt{6H^2r_0^2-1}(\tau-\tau_0)]}{H\{\exp[-\sqrt{6H^2r_0^2-1}(\tau-\tau_0)] - 4H^2r_0^2\}^2 - 4H(6H^2r_0^2-1)} . \tag{5.15}$$

It is clear from the potential (Fig. 3) that there are two qualitatively different types of solutions, which we shall call $r_+(\tau)$ and $r_-(\tau)$.

For

$$\tau_0 = \frac{1}{\sqrt{6H^2r_0^2-1}} \ln \frac{2 - 8H^2r_0^2 + 2\sqrt{6H^2r_0^2-1}\sqrt{3H^2r_0^2-1}}{Hr_0} , \tag{5.16}$$

we have the solution $r_+(\tau)$ with the properties

$$r_+(-\infty) = r_0, \quad r_+(0) = 0 . \tag{5.17}$$

This circular string starts with its maximal size r_0 at $\tau = -\infty$, passes the inner horizon and falls into the singularity $r = 0$ at τ=0.

The choice

$$\tau_0 = \frac{1}{\sqrt{6H^2r_0^2-1}} \ln(4Hr_0 + 2\sqrt{6H^2r_0^2-1}) \tag{5.18}$$

leads to a different type of solution, $r_-(\tau)$, with

$$r_-(-\infty) = r_0, \quad r_-(0) = \infty, \quad r_-(\infty) = r_0 . \tag{5.19}$$

This solution is very similar to the multistring solution

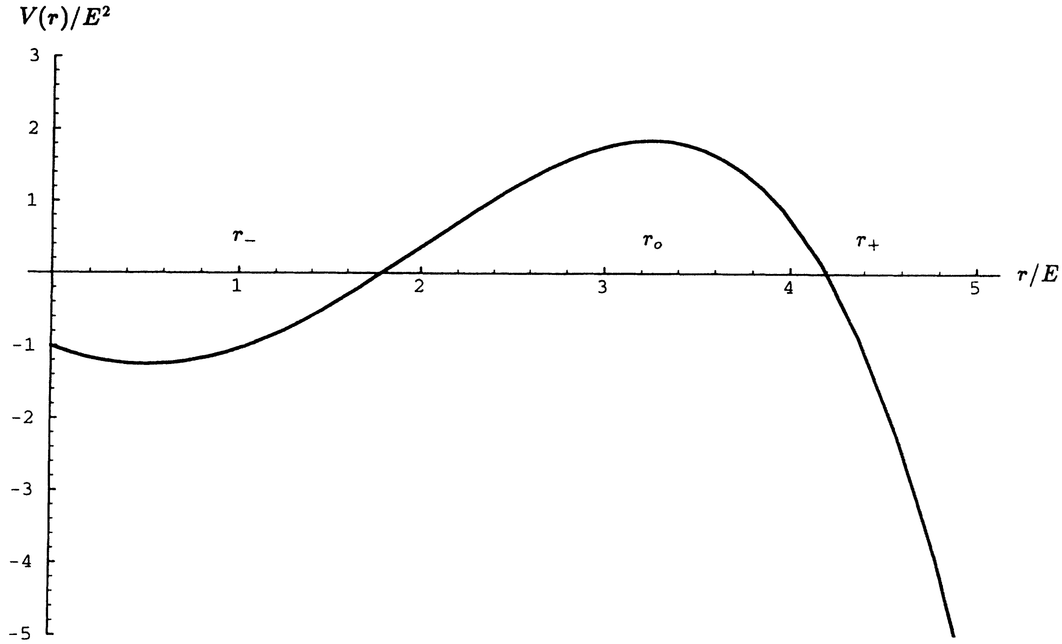

FIG. 3. The potential $V(r)$, Eq. (5.9), for a circular string in the ordinary Schwarzschild–de Sitter spacetime (SdS). For $V(r_0) \geq 0$ there are qualitatively different types of solutions, since $V(r)$ acts as a barrier.

discussed in Refs. [6,8] for strings in de Sitter space. Each of the two world-sheet time intervals $\tau \in]-\infty, 0]$ and $\tau \in [0, \infty[$ corresponds to the physical time interval $]-\infty, \infty[$, that is, the world-sheet time τ is a two-valued function of the physical time, and Eqs. (5.15), (5.18), and (5.19) describe a multistring (two) solution.

More generally, from the potential (Fig. 3) and the similarity between the Schwarzschild–de Sitter and the "pure" de Sitter spacetimes outside the horizon, we can expect multistrings for any values of the parameters (M, H, E).

B. Circular strings in the black string background

As our final example of circular strings in curved spacetimes, we consider the (2+1)-dimensional black string

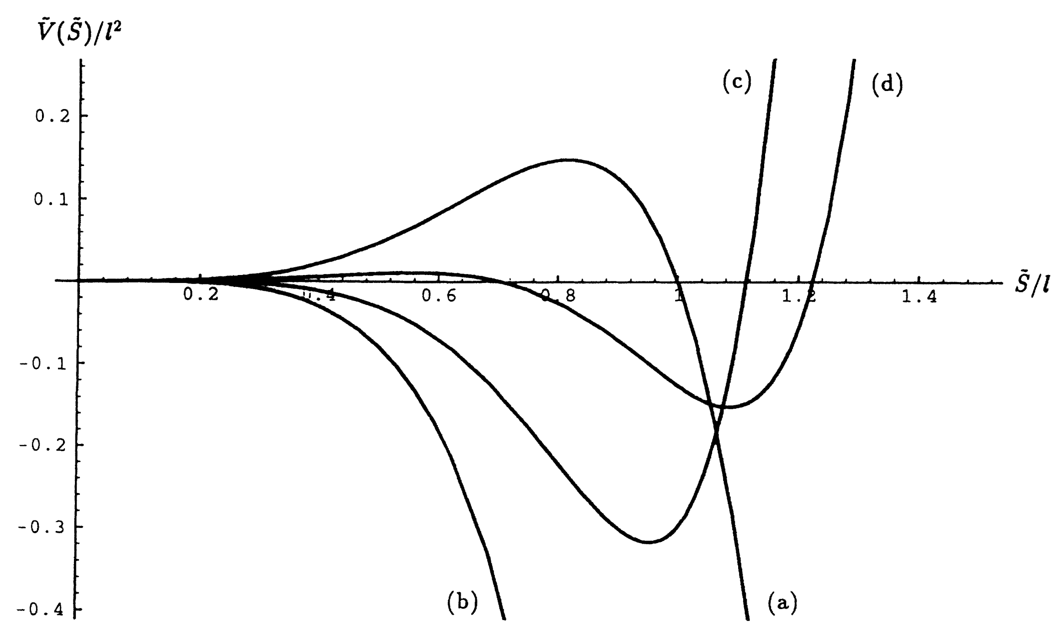

FIG. 4. The potential $\tilde{V}(\tilde{S})$, Eq. (5.27), for a circular string in the black string background. For $M > 0$ and $M^2l^2 - J^2 \geq 0$ (the latter is the condition for the existence of horizons) there are four qualitatively different cases: (a) $J = 0$, $E^2l^2 < 1$; (b) $J = 0$, $E^2l^2 \geq 1$; (c) $J \neq 0$, $E^2l^2 \geq 1$; (d) $J \neq 0$, $E^2l^2 < 1$.

of Horne and Horowitz [26]. For our purposes it is most convenient to use the stationary (but nonstatic) parametrization (2.13), which in the general form (4.1) is

$$\tilde{g}_{tt} = M - \frac{J^2}{4r^2}, \;\; \tilde{g}_{rr} = \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1} ,$$
$$\tilde{g}_{\phi\phi} = \frac{1}{r^2}, \;\; \tilde{g}_{t\phi} = \frac{1}{l} , \tag{5.20}$$

where the tilde reminds us that the black string is dual to the (2+1)-dimensional BHAdS spacetime (2.6). From the analysis of Sec. IV we can then write down the Eqs. (4.9)–(4.11), determining the circular string motion in the background (5.20):

$$\dot{r}^2 + \tilde{V}(r) = 0, \;\; \dot{t} = \frac{-E}{M - J^2/4r^2 - r^2/l^2} ,$$
$$\dot{f} = \frac{E}{(l/r^2)(M - J^2/4r^2) - 1/l} . \tag{5.21}$$

For the r equation we find a potential which is dual to the potential for the (2+1)-dimensional BHAdS spacetime, in the sense that

$$\tilde{V}(R) = \tilde{g}^{rr}(E^2\tilde{g}^{tt} + \tilde{g}_{\phi\phi}) , \tag{5.22}$$

[compare with Eqs. (4.11) and (4.15)], and given explicitly by

$$\tilde{V}(r) = \frac{J^2}{4r^4} - \frac{M}{r^2} + \frac{1}{l^2} - E^2 . \tag{5.23}$$

It follows that

$$\tilde{V}(r_\pm^2) = -E^2, \; \tilde{V}(\infty) = \frac{1}{l^2} - E^2 , \tag{5.24}$$

where $r_\pm$ are the two horizons (2.7), unchanged by the duality transformation. For $r \to 0$ the potential is positive infinite for $J \neq 0$, and negative infinite for J=0 (taking M positive).

One effect of the duality transformation has been to change the asymptotic behavior of the circular string potential, compare with Eq. (4.15). In the case that $1/(l^2) - E^2 < 0$ the potential approaches a negative constant value, which gives rise to string solutions of unbounded r. In this sense we will find solutions of com-

TABLE I. String motion described by the string perturbation series approach in the 2+1 BHAdS, ordinary black hole AdS, de Sitter (dS), and black string backgrounds. Notice the difference between the string motion in the 2+1 BHAdS and in the other spacetimes, while the strings in the black string background behave similarly as in the ordinary black hole backgrounds.

String perturbation series approach $x^\mu(\tau,\sigma) = q^\mu(\tau) + \eta^\mu(\tau,\sigma) + \xi^\mu(\tau,\sigma) + \cdots$ $\eta^\mu = n^\mu_R \delta x^R, \; \delta x^R(\tau,\sigma) = \sum_n C^R_n(\tau) e^{-in\sigma}; \; R = 1,\ldots,(D-1)$ $\ddot{C}_{nR} + (n^2\delta_{RS} - R_{\mu\rho\sigma\nu} n^\mu_R n^\nu_S \dot{q}^\rho \dot{q}^\sigma) C^S_n = 0$	
2+1 black hole AdS $\ddot{C}_{n\perp} + \left(n^2 + \frac{m^2}{l^2}\right) C_{n\perp} = 0$ $\ddot{C}_{n\parallel} + \left(n^2 + \frac{m^2}{l^2}\right) C_{n\parallel} = 0$ No instability, $\omega_n = \sqrt{n^2 + m^2/l^2}$ $\delta x^R, \xi^\mu$ bounded everywhere. $\delta x_R = \sum_n [A_{nR} e^{-i(n\sigma+\omega_n\tau)} + \tilde{A}_{nR} e^{-i(n\sigma-\omega_n\tau)}]$ $m^2 = 2\sum_n \left(2n^2 + \frac{m^2}{l^2}\right) \sum_R A_{nR}\tilde{A}_{-nR}$	Ordinary ($D \geq 4$) black hole AdS $\ddot{C}_{n\perp} + \left(n^2 + m^2H^2 + \frac{Mm^2}{r^3}\right) C_{n\perp} = 0$ $\ddot{C}_{n\parallel} + \left(n^2 + m^2H^2 - \frac{2Mm^2}{r^3}\right) C_{n\parallel} = 0$ Instability, $r_{\mathrm{inst}} = \left(\frac{2Mm^2}{1+m^2H^2}\right)^{1/3}$ Unbounded perturbations $C_{n\parallel}$ for $r(\tau) \approx (3m\sqrt{M/2})^{2/3}(\tau_0 - \tau)^{2/3} \to 0$: $\ddot{C}_{n\parallel} - \frac{4}{9(\tau_0-\tau)^2} C_{n\parallel} = 0$ $m^2(H=0) = 4\sum_n n^2 \sum_R A_{nR}\tilde{A}_{-nR}$
Ordinary ($D \geq 4$) de Sitter $\ddot{C}_{nR} + (n^2 - m^2H^2) C_{nR} = 0$ Unbounded modes for $\lvert n\rvert < mH$ Instability, $\omega_n = \sqrt{n^2 - m^2H^2}$ $\delta x_R = \sum_n [A_{nR} e^{-i(n\sigma+\omega_n\tau)} + \tilde{A}_{nR} e^{-i(n\sigma-\omega_n\tau)}]$ $m^2 = 2\sum_n (2n^2 - m^2H^2) \sum_R A_{nR}\tilde{A}_{-nR}$	Black string $\ddot{C}_{n\perp} + n^2 C_{n\perp} = 0$ $\ddot{C}_{n\parallel} + \left(n^2 - \frac{2Mm^2}{lr}\right) C_{n\parallel} = 0$ Instability, $r_{\mathrm{inst}} = \frac{2Mm^2}{l}$ Unbounded perturbations $C_{n\parallel}$ for $r(\tau) \approx \frac{Mm^2}{l}(\tau_0 - \tau)^2 \to 0$: $\ddot{C}_{n\parallel} - \frac{2}{(\tau_0-\tau)^2} C_{n\parallel} = 0$ String mass formula not yet known.

pletely different type than the solutions discussed in subsection IV A, that were always bounded. Another effect of the duality transformation however, is, to change the expression for the invariant string size, defined in Eq. (4.14),

$$\tilde{S}(\tau) = \sqrt{\tilde{g}_{\phi\phi}} = \frac{1}{r(\tau)} \; . \tag{5.25}$$

Writing Eqs. (5.21) in terms of the invariant string size yields

$$\dot{\tilde{S}}^2 + \tilde{V}(\tilde{S}) = 0, \;\; \dot{t} = \frac{-E}{M - (J^2/4)\tilde{S}^2 - (l\tilde{S})^{-2}} \; , \tag{5.26}$$

$$\dot{f} = \frac{E}{l[M - (J^2/4)\tilde{S}^2]\tilde{S}^2 - 1/l} \; ,$$

where

$$\tilde{V}(\tilde{S}) = \tilde{S}^4 \left(\frac{J^2}{4}\tilde{S}^4 - M\tilde{S}^2 + \frac{1}{l^2} - E^2 \right) \; . \tag{5.27}$$

For nonzero J, which means nonzero charge for the black string [26], we have only bounded configurations (finite $\tilde{S}$), while for J=0 and $M > 0$, unbounded configurations will exist as well. The potential (5.27) is shown in Fig. 4 for various values of the parameters. The simplest case ($J = 0$, $E^2 l = 1$) Eqs. (5.26) and (5.27) are solved by

$$\tilde{S}(\tau) = \frac{1}{\sqrt{|2\sqrt{M}\tau|}} \; , \tag{5.28}$$

$$t(\tau) = \frac{1}{2E\sqrt{M}} \ln \left| \frac{2E^2}{\sqrt{M}}\tau - 1 \right| \; , \tag{5.29}$$

$$\phi(\tau,\sigma) = \sigma - \tau - \frac{\sqrt{M}}{2E^2} \ln \left| \frac{2E^2}{\sqrt{M}}\tau - 1 \right| \; , \tag{5.30}$$

so that

TABLE II. Circular exact string solutions in the indicated backgrounds. $S(\tau)$ is the invariant string size. The motion is exactly and completely solved in terms of elliptic functions; [$\wp(\tau - \tau_0)$ stands for the Weierstrass elliptic function which reduces to elementary functions for zero cosmological constant or for particular combinations of the spacetime parameters]. The properties of the potential $V(r)$ [$\tilde{V}(r)$ in the dual black string background] determine general properties of the string dynamics. Unstable strings (expanding with unbounded size) and multistring solutions are present for potentials unbounded from below for $r \to \infty$, while bounded strings (and no multistring solutions) correspond to $V(\infty) = +\infty$. When $V(0) < 0$, strings can collapse into a point.

Exact circular strings

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + 2g_{t\phi}(r)dtd\phi + g_{\phi\phi}d\phi^2$$
$$t = t(\tau), \;\; r = r(\tau), \;\; \phi = \sigma + f(\tau), \;\; (\theta = \pi/2)$$
$$\dot{r}^2 + V(r) = 0, \;\;\; V(r) = g^{rr}(E^2 g^{tt} + g_{\phi\phi}), \;\;\; \dot{t} = -Eg^{tt}, \;\;\; \dot{f} = -Eg^{t\phi}$$
$$ds^2 = g_{\phi\phi}(-d\tau^2 + d\sigma^2), \text{ i.e., } S(\tau) = \sqrt{g_{\phi\phi}(r(\tau))}$$

2+1 black hole AdS	Ordinary ($D \geq 4$) black hole AdS
$V(r) = r^2(r^2/l^2 - M) + J^2/4 - E^2$	$V(r) = r^2(1 + H^2r^2) - 2Mr - E^2$
$V(0) = J^2/4 - E^2$, $V(r \to \infty) \propto r^4 \to \infty$	$V(0) = -E^2$, $V(r \to \infty) \propto r^4 \to \infty$
$S(\tau) = r(\tau) = r_m - [c_1\wp(\tau - \tau_0) + c_2]^{-1}$	$S(\tau) = r(\tau) = r_m - [c_1\wp(\tau - \tau_0) + c_2]^{-1}$
$S_{\max} = r_m = \sqrt{(Ml^2/2)}\left(1 + \sqrt{1 - 4V(0)/Ml^2}\right)^{1/2}$	$V(r_m) \equiv 0$, $S_{\max} = r_m = r_m(M,H,E)$
For $J^2 > 4E^2$, $S_{\min} > 0$, no collapse.	String contracts from $r(0) = r_m$ until
For $J^2 \leq 4E^2$, $S_{\min} = 0$, collapse.	it collapses into the r=0 singularity.
No unbounded string size.	No unbounded string size.
No multistring solutions.	No multistring solutions.

Ordinary ($D \geq 4$) black hole dS	Black string
$V(r) = r^2(1 - H^2r^2) - 2Mr - E^2$	$\tilde{V}(r) = J^2/(4r^2) - M/r^2 + 1/l^2 - E^2$
$V(0) = -E^2$, $V(r \to \infty) \propto -r^4 \to -\infty$	$\tilde{V}(0) = \infty$ $(J \neq 0)$, $\tilde{V}(0) = -\infty$ $(J = 0)$
	$\tilde{S}(\tau) = 1/r(\tau)$, $\tilde{V}(\infty) = 1/l^2 - E^2$
Contracting, expanding or stationary	
solutions, depending on the balance	For $J \neq 0$, all solutions are bounded
between the string tension and the	(finite $\tilde{S}$). For $J = 0$ unbounded size
local gravity.	solutions exist as well.
$M = 0: \; S^2(\tau) = H^{-2}\wp(\tau - \tau_0) + H^{-2}/3$	$J = 0$, $El = 1$: $\tilde{S}(\tau) = (\lvert 2\sqrt{M}\tau \rvert)^{-1/2}$
Contracting, expanding, or oscillating.	$\tilde{S}(-\infty) = 0$, $\tilde{S}(0) = \infty$, $\tilde{S}(\infty)$=0
Unbounded string size.	Unbounded string size.
Multistring solutions.	Multistring solutions.

$$\tilde{S}(-\infty) = 0, \quad \tilde{S}(0) = \infty, \quad \tilde{S}(+\infty) = 0,$$
$$t(-\infty) = \infty, \quad t(0) = 0, \quad t(\infty) = \infty \; . \tag{5.31}$$

This solution is somewhat similar to the two-string solution found in de Sitter space (Refs. [6,8]) and in Schwarzschild–de Sitter space (subsection V A), and indicates that multistring solutions are a generic feature of the black string spacetime, too.

In the general case the mathematical solution of Eqs. (5.26) and (5.27), or alternatively of Eqs. (5.21) and (5.22), can be obtained in terms of elliptic functions (elementary functions for $J = 0$). This solution and its physical interpretation are to be discussed elsewhere.

VI. CONCLUSION

We have studied the string propagation in the 2+1 BHAdS and black string backgrounds. We found the first- and second-order perturbations around the string center of mass as well as the mass formula, and compared with the ordinary black hole AdS spacetime. We found the exact general evolution of circular strings in all these backgrounds, and in the black hole de Sitter spacetime: in all these cases the solutions were expressed closely and completely in terms of either elementary or elliptic functions. The physical properties of the string motion in these backgrounds have been discussed. A summary of the main features and conclusions of our paper is given in Tables I and II.

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