Zones of dynamical instability for rotating string loops

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The equations of first-order perturbations are derived directly in a particular gauge for a stationary rotating string ring in a flat background. The perturbations are decoupled into equatorial (i.e., in the plane of the loop) and azimuthal (i.e., perpendicular to the plane of the loop) plane waves with quantified wavelengths. A polynomial eigenvalue equation for the perturbations defining the pulsation of the plane waves is then written and, after simplification, reduces the condition of stability to the reality of the roots of a third degree polynomial with real coefficients. This condition is equivalent to the positivity of a generalized discriminant and relies only on two parameters which are the longitudinal and transverse characteristic speeds and depends on the internal structure of the string. It is found that, although the azimuthal, axisymmetric, and lowest nonaxisymmetric perturbations are stable, there exist configurations of instability in the equatorial perturbations for all the other spatial modes, especially for classical and ultrarelativistic strings. The whole range of parameters of the problem is then explored analytically and numerically, giving a complete solution to the problem. So, the stability of a particular model of strings can be checked easily. A rate of dynamical decay of the equilibrium state is also defined and calculated in some interesting cases to verify the effective cosmological instability of the loops.

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INTRODUCTION

It has recently been shown [1] that elastic circular rotating string loops with any equation of state are always dynamically stable with respect to azimuthal perturbations whereas, with respect to equatorial perturbations, there can appear unstable states when the velocity of transverse perturbations v is lower than the longitudinal group velocity c. The purpose of the present article is to investigate in more detail the range of parameters for which the loop is unstable, especially in the ultrarelativistic limit which is of cosmological interest for cosmic strings with currents. Although the electromagnetic interaction is neglected in the following, this work effectively covers the case of superconducting cosmic strings, at least to lowest order, since the main effect of electromagnetism on cosmic strings applies to their internal structure and is taken into account in their equation of state. The effect of such interactions, although small, could, however, lead to higher order modifications [2] in some limit cases but this is left for a future work. As for gravitational interactions, we suppose that the background metric is flat and neglect the action of gravitational radiation of the loop, which is clearly justifiable in the short run, and could possibly lead in the medium to long run to a slow secular contraction [3] so that the equilibrium state would remain of the kind discussed here.

David and Shellard [4–6] were the first to point out that for closed cosmic string loops there can exist stationary states sustained by a centrifugal force. The earliest studies of circular equilibrium states of cosmic strings supposed that the support mechanism was provided by electromagnetic forces, but Peter [2] has shown that, for realistic parameter values, electromagnetic forces are too weak to play a dominant role and can be allowed merely as small correction effects. In their pioneering work, Davis and Shellard limited themselves to circular rotating rings for a particular class of superconducting strings. In fact, further investigations [1] have shown that such states are more general: they exist for all kinds of elastic strings and for any, not necessarily circular, shapes. Although Davis and Shellard considered the question of stability with respect to the underlying quantum field theory, they did not investigate stability with respect to macroscopic dynamical perturbations. It should be mentioned that stationary loop states are potentially interesting as seeds in scenarios of galaxy formation [7].

As pointed out by Davis and Shellard [3], supraconducting grand unified theory (GUT) cosmic string loops of the kind considered here will not decay and may therefore create a mass excess in the Universe similar to the one introduced by monopoles, ruling out galaxy formation scenarios using such strings. However, decay by quantum tunneling, whose efficiency is uncertain, may still save these scenarios. Another way to avoid the mass excess problem is to consider strings generated at energies close to the electroweak scale such as the lightweight strings considered by Peter [8]. In this case, the small loops obtained from interaction of infinite strings in galaxy formation scenarios would be equivalent to very massive charged nonbaryonic particles or charged ultramassive particles which could account for dark matter [9]. Another possibility for getting rid of excess loops is the secular instability due to gravitational radiation reaction [10,11] but, because of the weakness of the gravitational coupling constant, this seems very unlikely to be sufficiently rapid to be significant.

The purpose of this work is to consider the possibilities of dynamical instability which, if it occurs, will be very much more rapid than the gravitational back reaction mechanism alluded to above. Section I recalls the basic of elastic string dynamics that will be used throughout this paper. Only the simplest kind of equilibrium state, that of a circular rotating ring, is considered. This enables the problem to be solved algebraically. It may be hoped that other kinds of equilibrium states will have the same general behavior. The equations and the condition of equilibrium for such equilibrium states are derived in Sec. II. In Sec. III the first-order perturbations for the circular loop are rewritten in a new gauge and related to the system of equations obtained in the preceding work [1] with a different gauge. The polynomial characteristic eigenvalue equation for the plane wave perturbations linking their pulsation to their discrete wavelengths is calculated in Sec. IV. It is the same result as in the preceding paper [1] confirming that both gauges lead, as expected, to the same result. In Sec. V, the requirement for stability of the rotating ring is shown to be the reality of all its eigenmode pulsations. This condition can be reduced to a simple criterion, the positivity of a polynomial depending only on two parameters: namely, the two speeds of transverse and longitudinal perturbation. This is worked out numerically over the whole range of parameters so as to get the ranges of stability and instability which are plotted in Fig. 5 below.

The main conclusion is that there is no simple general result: there are values of the two parameters that lead to both stable and unstable rings. In the case of instability, a typical lifetime for the loop to leave the vicinity of its equilibrium state is defined and is estimated in some interesting cases in subsequent sections. Sections VI–XI are devoted to the analysis of cases of particular interest. It is confirmed in Sec. VI that in particular the axisymmetric mode [12], the first nonaxisymmetric mode, and the very short wavelength perturbations are always stable. In Sec. VII some particular equations of state are studied such as the warm cosmic string model [13,14], the constant trace equation of state, and the more familiar Hookean equation of state, and it is shown that these are all stable. Section VIII is devoted to the study of limit values of the two parameters, including the so called "cosmic spring" limit. In Sec. IX, it is shown algebraically that whenever the speed of longitudinal perturbation is greater than that of transverse perturbations, the rotating circular string loop is stable. In the classical and in the ultrarelativistic zones treated, respectively, in Secs. X and XI stability and instability are both possible and their respective ranges of parameters are evaluated. In the ultrarelativistic case, which is of particular interest for cosmic strings, an estimation of the rate of instability of unstable states is evaluated using a set of equations of state covering roughly the unstable zones. Finally, in the Conclusion, the main results found in this paper are discussed. The main issue remaining for future work to investigate is that it is still unclear what happens to unstable rings in the long run: whether the cosmic string loop would ultimately break down and transform its energy completely into radiation (which would resolve the cosmological excess problem mentioned above) or whether, perhaps after self-intersection and production of multiple daughter loops, new stable equilibrium states would be attained. In particular, the qualitative validity of the constant trace and constant determinant equation of state for representing more specific models such as Witten superconducting rings is discussed. An order of magnitude of the dynamical instability of unstable loops is also estimated for the grand unification and electroweak phase transitions.

I. BASICS OF STRING DYNAMICS

We restrict ourselves to a fixed Minkowskian background metric of signature $(-1,1,1,1)$. As in the preceding work [1], we define an orthonormal base (u^μ, v^μ) at each point of the two-dimensional world sheet of the string. This enables us to write the fundamental tensor of the world sheet (which is a projector on the world sheet and its induced metric) as

$$\eta^{\mu\nu} = -u^\mu u^\nu + v^\mu v^\nu .$$

We also define the complementary orthogonal projector

$$\perp^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu} ,$$

which is equivalent in terms of information to the fundamental tensor $\eta^{\mu\nu}$. The two vectors of the base can be taken to be respectively the timelike and spacelike eigenvectors of the stress-energy tensor $T^{\mu\nu}$, which therefore becomes

$$T^{\mu\nu} = U u^\mu u^\nu - T v^\mu v^\nu , \tag{1}$$

where the first eigenvalue U is the energy density and the second eigenvalue T is the tension. These are functionally related by the equation of state, which can be used to define a number density variable ν and an effective mass variable μ given by [1,2]

$$\ln(\nu) = \int \frac{dU}{U-T} , \qquad \ln(\mu) = \int \frac{dT}{T-U} , \tag{2}$$

together with the relation

$$\mu\nu = U - T ,$$

which restricts the number of independent constants of integration from 2 to 1, corresponding just to the freedom to adjust the overall normalization of the number density. These were the primary variables used in the preceding work [1] to derive the equation of motion of a loop. A covariant derivation along the world sheet is also needed: $\bar\nabla_\rho = \eta^\nu{}_\rho \nabla_\nu$, in terms of which the second fundamental tensor of the string two-surface is given by [15]

$$K_{\mu\nu}{}^{\rho} = \eta^{\sigma}{}_{\mu}\bar{\nabla}_{\nu}\eta^{\rho}{}_{\sigma} \ . \tag{3}$$

The internal structure of the particular elastic string model studied is what determines the equation of state relating the energy per unit length U to the tension T of the string. By allowing for arbitrary equations of state, the range of our study extends from classical strings to cosmic ones. But in the following, only two parameters will actually be necessary to explore the stability of a ring: the velocity of transverse perturbations

$$c_T = \sqrt{T/U} \ , \tag{4}$$

which describes the string at equilibrium, and the longitudinal group velocity

$$c_L = \sqrt{-dT/dU} \ , \tag{5}$$

which comes into play in the first-order perturbation equations. A particular equation of state corresponds just to a curve in the square parametrized by these two parameters. The derivatives of the energy per unit length U and of the tension T can be expressed in terms of these two parameters as

$$\bar{\nabla}_{\rho}(\ln U) = A\bar{\nabla}_{\rho}(c_T^2) \ , \tag{6}$$

$$(\bar{\nabla}_{\rho}T)U^{-1} = -Ac_L^2\bar{\nabla}_{\rho}(c_T^2) \ , \tag{7}$$

where we have defined $A = -1/(c_L^2 + c_T^2)$. The dynamical equations of the free moving string are just given by the conservation of the surface stress-energy tensor:

$$\bar{\nabla}_{\mu}T^{\mu\nu} = 0 \ .$$

Since the method used [1,12,15] takes the basic unknowns to be not the fundamental string imbedding coordinates but the tangent vectors u^{μ} and v^{μ}, it will also be necessary to include the Weingarten identity

$$K_{[\mu\nu]}{}^{\rho} = 0 \ ,$$

which is a nontrivial integrability condition to be solved in conjunction with the equations of motion. The full set of equations of motion can thus be written explicitly as the system

$$K_{[\mu\nu]}{}^{\rho} = 0 \Leftrightarrow \perp_{\mu\nu} [u^{\rho}\nabla_{\rho}(v^{\nu}) - v^{\rho}\nabla_{\rho}(u^{\nu})] = 0 \ , \tag{8}$$

$$\perp_{\mu\nu} \bar{\nabla}_{\rho}T^{\rho\nu} = \perp_{\mu\nu} [u^{\rho}\nabla_{\rho}(u^{\nu}) - c_T^2 v^{\rho}\nabla_{\rho}(v^{\nu})] = 0 \ , \tag{9}$$

$$u_{\nu}\bar{\nabla}_{\rho}T^{\rho\nu} = Au^{\rho}\nabla_{\rho}(c_T^2) - (1-c_T^2)u_{\nu}v^{\rho}\nabla_{\rho}(v^{\nu}) = 0 \ , \tag{10}$$

$$v_{\nu}\bar{\nabla}_{\rho}T^{\rho\nu} = -Ac_L^2 v^{\rho}\nabla_{\rho}(c_T^2) + (1-c_T^2)u_{\nu}u^{\rho}\nabla_{\rho}(v^{\nu}) = 0 \ . \tag{11}$$

II. EQUILIBRIUM OF THE ROTATING RING

As was shown in the previous work [1], the stationary equilibrium state of a string loop (i.e., a string loop with a static timelike Killing vector k^{μ}) in a flat background can have an arbitrary geometrical configuration but must have a running velocity v (the speed of the intrinsic rest frame of the string as characterized by the timelike eigenvalue u^{μ} relative to the frame of the static Killing vector k^{μ}) equal to the transverse perturbation speed c_T. Moreover, when the longitudinal group velocity and the velocity of transverse perturbations are not equal (which excludes the case for a warm cosmic string equation of state [13,14] $UT = m^4$), T and U (and so also v) must be constant along the loop.

In this paper, we restrict ourselves to the case of a circular loop, so that, because of the symmetries, every physical scalar U, T, and v must be constant along the loop. Let the ring have a radius r_0, an angular speed Ω, and a Lorentz factor $\gamma = 1/\sqrt{1 - r_0^2\Omega^2}$. The running velocity of the ring is then $v = r_0\Omega$. Spacetime will be described with cylindrical coordinates (t, r, θ, z) so that the equations of the ring will be just $r = r_0$ and $z = 0$, and [12]

$$u^{\mu} = (\gamma, 0, \gamma v, 0) \ , \ v^{\mu} = (\gamma v, 0, \gamma, 0) \ . \tag{12}$$

The integrability condition (8) is then automatically satisfied since both u^{μ} and v^{μ} derive direction from a world sheet. Because of the symmetries, the longitudinal conservation laws (10) and (11) are also automatically satisfied. This leaves (9) as the only nontrivial equation, which gives the expected condition

$$c_T = r_0\Omega = v \ . \tag{13}$$

It will be convenient when perturbing these equations to note that Eqs. (8) and (9), when the equilibrium condition (13) is satisfied, are still satisfied without the extrinsic projector $\perp_{\mu\nu}$.

III. EQUATIONS FOR THE PERTURBATIONS OF THE RING

As in previous work [1], it will be convenient to work not just with the single ring state in which we are ultimately interested, but with a space-filling congruence of such states. Whereas the previous analysis started from a family of equilibrium states characterized by a fixed uniform running velocity v, or, equivalently, a fixed velocity of transverse perturbations c_T after Eq. (13), in the present work we shall instead postulate that the angular velocity Ω of the equilibrium states has a fixed uniform value so that the running velocity v will be a linear function of r. Thus each ring state will have its own $c_T = v = r\Omega$ whose variations are equivalent to those of r. These are just two possible gauge-fixing conditions for the space-filling congruence of equilibrium loops which do not change the characteristic eigenvalue equation and thus the stability results, as will be shown in Sec. IV.

As this paper considers the linear perturbations of a stationary, axially symmetric state, any perturbed quantity x of the problem can be expanded in plane waves of the form $x = x_0 + \delta x e^{i(\omega t - n\theta)}$ where x_0 is the value of x at equilibrium, δx is a first-order constant, n is an integer akin to an angular quantum number, and ω is the pulsation of the perturbation. In particular, the geometry of the perturbed ring can be described in terms of two kinds of modes which are mutually decoupled [1]. In the cylindrical coordinates (r, θ, z) defined in the previous section, where the unperturbed ring is given by $r = r_0$ and $z = 0$, the first kind consists of *azimuthal* perturbations characterized by

$$r = r_0 \quad \text{and} \quad z = \delta z e^{i(\omega t - n\theta)} , \tag{14}$$

while the second kind consists of *equatorial* perturbations characterized by

$$r = r_0 + \delta R e^{i(\omega t - n\theta)} \quad \text{and} \quad z = 0 . \tag{15}$$

Figures 1 and 2 illustrate the configurations (15) and (14), respectively, for some values of n at a given time. The time evolution of these loops is just a uniform rotation around the z axis.

There are five equations in this problem: the four dynamical equations (9)–(11) obtained by conservation of the stress-energy tensor and the equation of state. On the other side, the displacement of the world sheet of the string gives two unknowns (only the transverse displacement of the world sheet is physically significant), and the stress-energy tensor which is a symmetric tensor of rank 2 gives three unknowns. The physical problem is therefore well defined. As the system (9)–(11) depends

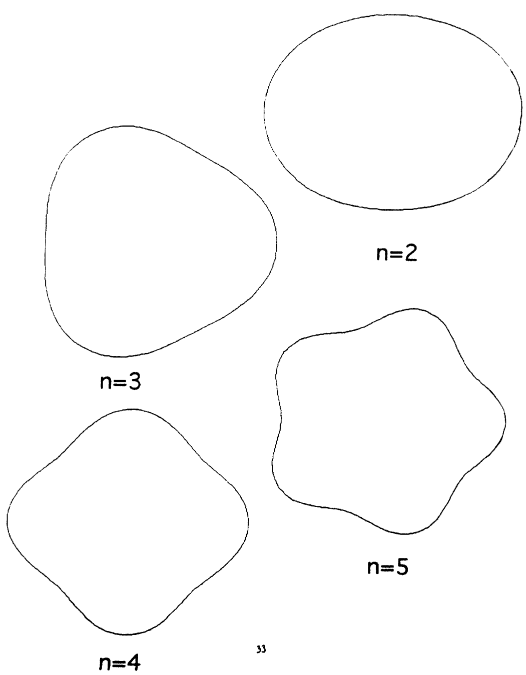


FIG. 1. Illustration of equatorial perturbations for various modes n as expressed in (15).

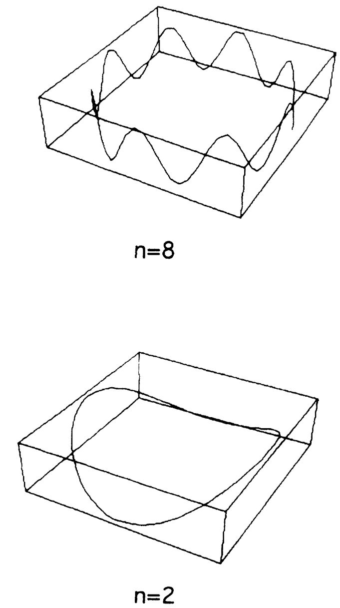


FIG. 2. Perspective views of rings with azimuthal perturbations for some modes n as expressed in (14).

on the spacetime coordinates of the string only through the eigenvectors u^μ and v^μ, we will take the variations of these two vectors as unknowns in

$$\delta u^\mu = \varepsilon^\mu \ , \quad \delta v^\mu = \eta^\mu \ .$$

The orthonormality of these two vectors immediately gives three relations between the perturbed variables:

$$\begin{aligned} u^\mu u_\mu &= -1 \Rightarrow \varepsilon^0 = c_T \varepsilon^\theta \ , \\ v^\mu v_\mu &= -1 \Rightarrow \eta^\theta = c_T \eta^0 \ , \\ u^\mu v_\mu &= 0 \Rightarrow \eta^0 = \varepsilon^\theta \ . \end{aligned}$$

However, the seven remaining variables are still not independent since there is also a condition of integrability that needs to be obeyed by u^μ and v^μ to be able to go back to the world sheet itself: it is the Weingarten identity (8) that must therefore be added to the set of equations of motion. Furthermore, the two thermodynamical unknowns appearing in the equations of motion (9)–(11) are the two speeds c_L^2 and c_T^2 which can be taken as unknowns instead of U and T. We consider in the following the unperturbed values of these two speeds to be given as input parameters (related *a priori* by the equation of state). The perturbation of the speed of transverse perturbations $\delta c_T^2 = \delta v^2$ is denoted α in the following. For convenience, we also define

$$\begin{aligned} V &= c_T^2 \ , \\ C &= c_L^2 \ , \\ X &= u^\rho (\bar{\nabla}_\rho)/i\gamma = \omega - n\Omega \ , \\ Y &= -r v^\rho (\bar{\nabla}_\rho)/i\gamma = n - r^2 \Omega \omega \ . \end{aligned}$$

As remarked at the end of Sec. II, when varying Eqs. (8) and (9), we need not vary $\perp_{\mu\nu}$ and thus we get four transverse equations, two by projecting on the r coordinate and two by projecting on the z coordinate:

$$Y\varepsilon^r + rX\eta^r = 0 \ , \tag{16}$$

$$rX\varepsilon^r + VY\eta^r + 2ir\Omega/\gamma^2\varepsilon^\theta - i\gamma\alpha = 0 \ , \tag{17}$$

$$Y\varepsilon^z + rX\eta^z = 0 \ , \tag{18}$$

$$rX\varepsilon^z + VY\eta^z = 0 \ . \tag{19}$$

The two longitudinal equations derived from (10) and (11) are

$$(2AV+1)\varepsilon^r - iY/\gamma^2\varepsilon^\theta + iArX\gamma\alpha = 0 \ , \tag{20}$$

$$2v\varepsilon^r + (2AC-1)V\eta^r + irX/\gamma^2\varepsilon^\theta - iACY\gamma\alpha = 0 \ . \tag{21}$$

The system (16)–(21) completely describes the dynamics of the perturbations.

If instead of the gauge condition Ω constant adopted at the beginning of this section one uses the condition v constant adopted in the preceding paper [1], then Eqs. (16)–(19) remain unchanged and the two other equations reduce just to

$$\varepsilon^r - v\eta^r - iY/\gamma^2\varepsilon^\theta + iArX\gamma\alpha = 0 \ ,$$

$$v\varepsilon^r - V\eta^r + irX/\gamma^2\varepsilon^\theta - iACY\gamma\alpha = 0 \ .$$

By switching to the variables used in the other paper [1], the perturbation equations are recovered.

IV. THE CHARACTERISTIC EIGENVALUE EQUATIONS

The perturbation equations form a homogeneous linear system of six equations (16)–(21) with six variables which must have nonzero solutions. This condition is equivalent to the annulation of the determinant of the system which gives the eigenvalue equation: a sixth degree polynomial equation giving ω as a function of n and of the two only parameters C and V. In fact, we have a first obvious simplification: as stated in Sec. III, the azimuthal perturbations (ε^z and η^z) of the system decouple from the equatorial perturbations (the rest of the unknowns: ε^r, ε^θ, η^z, and α).

The eigenvalue equation of the azimuthal part of the system, including (18) and (19), can be easily solved:

$$\begin{vmatrix} Y & rX \\ rX & VY \end{vmatrix} = 0 \Leftrightarrow \omega \in \left\{0, \frac{2nv}{r(1+v^2)}\right\} \ . \tag{22}$$

Thus there are two azimuthal modes: one is static and represents the backward moving extrinsic perturbations (moving at speed c_T against the speed v of the loop) and the other is $r\omega = nv_+$ where $v_+ = 2v/(1+V)$ is twice the velocity v (using the Lorentzian sum) and represents the forward moving extrinsic perturbations [1] (moving at speed c_T with the speed v of the loop). The straight lines of solutions for $\omega = 0$ and $\omega = nv_+/r$ are, respectively, $\varepsilon^z = c_T\eta^z$ and $\varepsilon^z = -c_T\eta^z$. It only remains now to express the amplitude of the oscillations of the loop δz, as defined in (14), in terms of the basic unknowns of the problem. This is done by equating the plane tangent to the world sheet defined by (14) with the plane defined by the two perturbed eigenvectors u^μ and v^μ. This gives, when $n \neq 0$,

$$\delta z = \frac{ir\gamma}{n}(\eta^z - v\varepsilon^z) \ .$$

When $n = 0$, the system (22) becomes trivial and the world sheet works out to be

$$z = (\varepsilon^z - v\eta^z)t + (\eta^z - v\varepsilon^z)R\theta \ , \tag{23}$$

which corresponds to two geometries: a circular loop slowly boosted in the z direction or a helix with an almost flat pitch (static or slowly rotating). The helix, although mathematically an approximation of a rotating circle, must be rejected here as an unphysical solution: there is an implicit boundary condition saying that the loop should remain closed during its evolution which in general is automatically satisfied because of the periodicity of the plane wave perturbations.

The equatorial part of the eigenvalue equation obtained from the system (16), (17), (20), and (21) can be computed from the determinant

$$\begin{vmatrix} Y & rX & 0 & 0 \\ rX & VY & 2iv/\gamma^2 & -i\gamma \\ 1+2AV & 0 & -iY/\gamma^2 & iAr\gamma X \\ 2v & (2AC-1)V & irX/\gamma^2 & -i\gamma ACY \end{vmatrix} = 0$$

to get one static mode $\omega = 0$ and a third degree polynomial

$$v^2(1+v^2)(1-c^2v^2)\sigma^3 + 2v^2[c^2 - v^2 - 2(1-c^2v^2)]n\sigma^2 + [4v^2(1-c^2)(n^2-1) - (1+v^2)(c^2-v^2)(n^2+1)]\sigma + 2(c^2-v^2)(n^2-1)n = 0 \ , \tag{24}$$

where $\sigma = r\omega/v = \omega/\Omega$. The amplitude of the oscillations of the loop δR, as defined in (15), is given as in the azimuthal case above when $n \neq 0$ by

$$\delta R = \frac{ir\gamma}{n}(\eta^r - v\varepsilon^r) \ ,$$

or when $\omega \neq 0$ by the similar formula

$$\delta R = \frac{i\gamma}{\omega}(v\eta^r - \varepsilon^r) \ .$$

The case $n = \omega = 0$ will be treated specifically in Sec. VI with the $n = 0$ mode.

The two above results (22) and (24) agree with those already obtained in the preceding paper [1] in a different gauge, which was expected as the eigenvalue pulsations for the perturbations are scalars independent of any gauge-fixing condition.

V. DYNAMIC STABILITY CONDITIONS FOR THE ROTATING LOOP

The eigenvalue equation found above generically defines six distinct modes for each n. The corresponding eigenvectors will then generate a base of the set of solutions. It is clear in this case that if for each n all the pulsations are real, the corresponding perturbed solutions will always remain bounded (they are plane waves with constant amplitude) and so the loop will be stable. On the other hand, if for any one n there is at least one complex pulsation, there will necessarily be a pulsation with negative imaginary part which leads to an exponentially growing perturbation, so that the loop will not be stable. If for a given n there is a (real) multiple solution to the eigenvalue equation and that the dimension of the corresponding subspace of eigenvectors is less than the multiplicity of the eigenvalue, then this mode will be marginally unstable, growing polynomially instead of exponentially.

As was pointed out in the preceding work [1], the above criterion gives immediately that the azimuthal perturbations are all stable, as when $n \neq 0$ there are two real and distinct solutions given by (22), while in the case $n = 0$ it has been found in (23) that there is only one physical solution, that of a loop slowly boosted in the z direction, which is also stable.

In the equatorial case, the problem is to find whether the third degree equation (24) has three real roots without multiple solutions, three real roots with multiple solutions, or only one real root. Using the Cardan formulas, it is possible for a general third degree polynomial

$$aY^3 + bY^2 + cY + d = 0 \tag{25}$$

to find a generalized discriminant given by

$$b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2 \tag{26}$$

whose sign provides the required information: if it is strictly positive then there are three distinct real roots, if it is zero then there are three real roots with multiple solutions, and if it is strictly negative then there is only one real solution. Before applying this criterion, Eq. (24) can be simplified by noting that only the squares of v and c appear in it. The range of variation of the last parameter n can also be compactified by letting $\varsigma = \sigma/n = r\omega/nv$ and

$$K = 1/n^2$$
$$\text{with } K \in F \equiv \{1/m^2, m \text{ a nonzero integer}\} \cup \{\infty\} . \tag{27}$$

Equation (24) then becomes

$$V(1+V)(1-CV)\varsigma^3 + 2V[C - V - 2(1-CV)]\varsigma^2$$
$$+[4V(1-C)(1-K) - (1+V)(C-V)(1+K)]\varsigma$$
$$+2(C-V)(1-K) = 0 . \tag{28}$$

The condition of stability of the ring is thus that the modified eigenvalue equation (28), which depends only on the two parameters (V, C), have three real roots for every $K \in F$ [as defined in (27)]. The condition obtained by replacing the eigenvalue equation (28) in the pseudodiscriminant (26), expanding, and simplifying is

$$P(K, V, C) > 0 \quad \text{for all } K \in F , \tag{29}$$

where

$$P(K, V, C) = a_3K^3 + a_2K^2 + a_1K + a_0 \tag{30}$$

and

$$\begin{aligned}
a_0 &= (1-V)^6(C-V)^2C ,\\
a_1 &= (1-V)^4(-V^3 + 21V^2C + 28V^3C + 3V^4C + 41VC^2\\
&\quad +72V^2C^2 + 41V^3C^2 + 3C^3 + 28VC^3\\
&\quad +21V^2C^3 - VC^4) ,\\
a_2 &= (1-V)^2(-18V^3 - 24V^4 - 2V^5 - 57V^2C + 4V^3C\\
&\quad +78V^4C + 20V^5C + 3V^6C - 56VC^2 + 104V^3C^2\\
&\quad -56V^5C^2 + 3C^3 + 20VC^3 + 78V^2C^3 + 4V^3C^3\\
&\quad -57V^4C^3 - 2VC^4 - 24V^2C^4 - 18V^3C^4) ,\\
a_3 &= (1+V)(1-CV)(3V - V^2 + C - 3CV)^3 .
\end{aligned}$$

In the limiting case when the polynomial $P(K, C, V) = 0$, there is a (real) multiple solution. The loop is still stable if the dimension of the associated subspace of eigenvalue solutions is equal to the multiplicity of the eigenvalue, while if it is not there are polynomially growing solutions and the loop is marginally unstable. So, in this special case, additional care must be taken and the dimension of the eigenvalue subspace must be worked out.

As stated above, the criterion of stability (29) depends only on the two speeds (c_T, c_L) which can vary between 0 and the speed of light, set here to 1. It is therefore possible to plot numerically the regions in the (C, V) plane where the equilibrium states are unstable over the whole range of parameters. In this plane, after the definition (5) of $C = c_L^2$ and (4) of $V = c_T^2$, an equation of state is given by a curve in the plane. To get a numerical plot of the regions of instability, I have evaluated numerically for a given mode of perturbations n the polynomial $P(1/n^2, C, V)$ on a 500×500 regular grid covering all the possible values of C and V. The points where this value is negative are equilibrium states for which the corresponding mode is unstable. These points have been plotted on Figs. 3 and 4 for, respectively, the $n = 2$ and $n = 3$ modes. The regions of instability of an equilibrium state are obtained when the regions of instability for all modes are superimposed. I have done this in Fig. 5 by superimposing the regions of instability of all the modes $n < 400$.

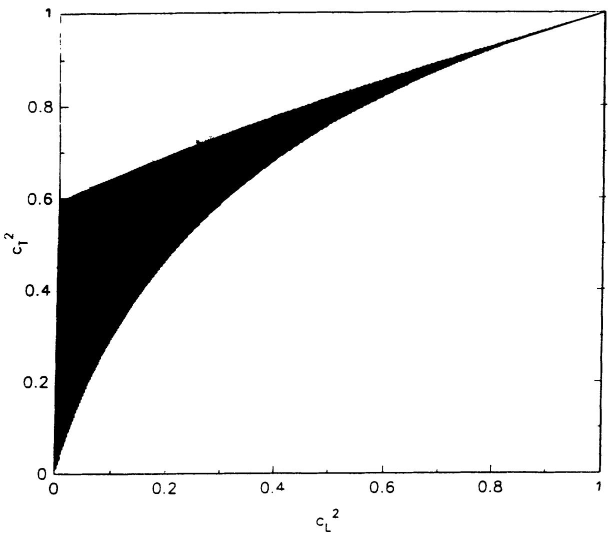


FIG. 3. The region of instability is plotted (in black) on the plane parametrized by the squared longitudinal and transverse perturbations, c_L^2 and c_T^2, on the basis of numerical evaluation of the polynomial $P(1/n^2, C, V)$ given by (38) on a regular grid of 500×500 points for the particular case of the mode $n = 2$.

When one mode is unstable, then the rate of decay of the equilibrium state can be estimated from C, V, and n only as

$$|\mathrm{Im}(\omega/\Omega)| = |\mathrm{Im}(\sigma)| = n|\mathrm{Im}(\varsigma)| \ .$$

The effective lifetime τ of the equilibrium state can also be defined as

$$\tau = |\mathrm{Im}(\omega)|^{-1} \ . \tag{31}$$

This is proportional to an unknown parameter Ω, the angular speed of the circular loop. An estimation of the

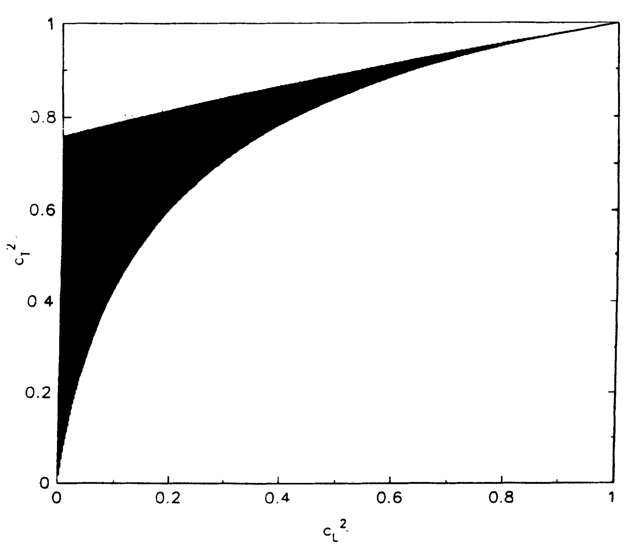


FIG. 4. Same as Fig. 3 for the mode $n = 3$.

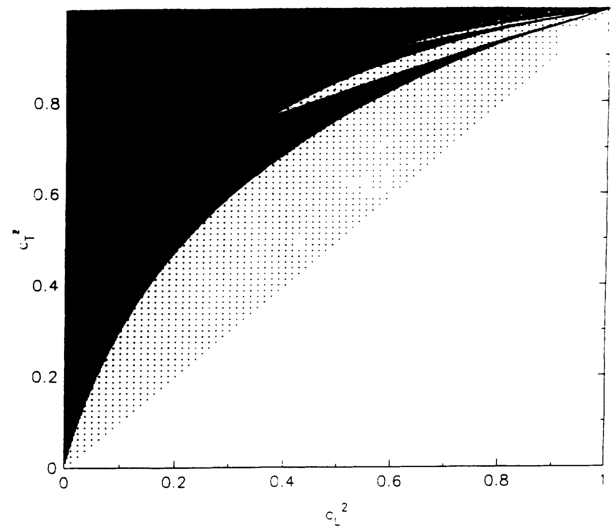


FIG. 5. Same as Figs. 3 and 4 for a superposition of all modes up to $n = 400$ (the more extended region of potential secular instability due to gravitational radiation reaction is indicated by dots).

order of magnitude of this parameter for cosmic strings has been given in the Conclusion. Anyway, the rate of decay (31) can be deduced algebraically from the eigenvalue equations (24) or (28) with the Cardan formulas, although as a very unwieldly formula. This formula can, however, be useful to compute numerically this rate of decay in some interesting cases.

In the following Secs. VI–XI, the criterion (29) is investigated algebraically in some interesting cases to clarify and complete the understanding of Figs. 3–5.

VI. STABILITY FOR PARTICULAR SIMPLE MODES

We first consider the axisymmetric mode $n = 0$, which corresponds to $K = +\infty$ [where K was defined in (27)]. This mode corresponds to a perturbed loop which, after Eq. (15), remains circular. As

$$a_3 = (1+V)(1-CV)\{V[(1-V)+2(1-C)]+C(1-V)\}$$

is positive, we see that the criterion (29) is satisfied, in agreement with previous results obtained through a different method [12] and with our previous conclusion [1] obtained directly from (24):

$$\sigma(V(1+V)(1-CV)\sigma^2 - \{V[(1-V)+2(1-C)] + C(1-V)\}) = 0 \ .$$

It is clear that this equation always has three real solutions. But with the static mode which always exists with (24), 0 is a multiple solution. Thus the dimension of the corresponding eigenvector subspace must be found by solving (16), (17), (20), and (21) with $X = Y = 0$. There are two cases. If $c = v$, which is always the case for the warm cosmic string model [13,14], the subspace of eigenvectors has two dimensions, so that the ring is stable. If $c \neq v$, the subspace of eigenvectors has only one dimension: $\varepsilon^r = \eta^r = 0$ and $\alpha = 2v\varepsilon^\theta/\gamma^3$, so linearly growing solutions must be allowed for. The above solution simply shows a loop at equilibrium with a slightly enlarged radius and a correspondingly enlarged speed of transverse perturbation c_T to keep the condition of equilibrium (13) satisfied. The second solution for this mode must be a linearly growing radius and nonzero radial components of the eigenvectors u^μ and v^μ but, when integrated to recover the string world sheet, the solution appears to be that of an infinite spiraling string, which must be dropped as unphysical as the loop should remain closed during its evolution. This problem does not arise for other modes as the periodicity of the nonconstant plane wave solutions found for the perturbation ensure the same periodicity for the geometry of the loop, and thus its closure. This is the same problem as for the helix solution found for the $n = 0$ azimuthal perturbations in (23).

The first nonaxisymmetric case $n = 1$ whose geometry is that of an ellipse is easy to solve, as $\omega = 0$ is a solution of (28). This also makes 0 a multiple solution that must be investigated in more detail. Letting $A = V - C$ and $B = 1 - CV$, after extraction of the second static mode (28) reduces to a quadratic equation whose discriminant is positive for $A \leq 0$, while for $A \geq 0$, it is also positive:

$$\Delta = (A+2B)^2 - 2AB(1+V)^2V \geq (A+2B)^2 - 8AB \geq (A-2B)^2 \geq 0 \; .$$

Exactly the same result would have been obtained from (29) as the pseudodiscriminant (30) reduces exactly to the discriminant computed above. To establish the stability of these $n = 1$ modes, the eigenstate subspace of static perturbations remains to be computed. Once more, there are two cases: either $c = v$ and the subspace of eigenvectors has two dimensions while 0 has a multiplicity of 3, or $c \neq v$ and the subspace has one dimension while 0 has a multiplicity of 2, so in any case one dimension is missing from the eigenstate subspace and there exists in both cases a linearly growing solution. In fact, after integration of the world sheet, it can be seen that this linearly growing solution corresponds to a slowly Lorentz boosted nonperturbed solution. So this mode also is stable.

At the opposite limit when $n \to \infty$ are the very wiggly rings, to which corresponds the case $K = 0$. In fact, this limit is outside the range of physical validity of the model, as when the wavelength of the perturbations decreases, it will eventually reach the order of magnitude of the thickness of the string, at which time the string approximation ceases to be valid. For instance, for electroweak cosmic strings, the radius of a typical equilibrium ring is about a hundred times the thickness of the string, which means that beyond the $n = 10$ mode this formalism would not be valid anymore. In the $K = 0$ limit, the criterion (29) is obviously always satisfied whenever $a_0 \neq 0$ (i.e., $c \neq v$, $c \neq 0$, and $v \neq 1$) and by continuity of the polynomial $P(K,V,C)$, the loop is stable for all large $n = 1/\sqrt{K}$. In the cases when $a_0 = 0$, the loop is only marginally stable for $n \to \infty$ and therefore nothing can be deduced by continuity on the stability of the loop for n large. The corresponding cases $c = v$, $c = 0$, and $v = 1$ will be studied specifically in Secs. VII and VIII.

In the following, when the stability of some sets of the parameters c_T and c_L is investigated, we need only check it for $n \geq 2$ or equivalently for $K \leq \frac{1}{4}$, as we have shown that the other first two modes are stable.

VII. STABILITY FOR PARTICULAR SIMPLE EQUATIONS OF STATE

Among the specially simple equations of state that can be envisaged for a cosmic string, the one most frequently used in earlier discussions [16,17] is that characterized by the constant trace equation $U + T = 2T_0$ which corresponds to the case $c = 1$, subject to the supposition that $V \neq 1$ (leaving the special case $V = 1$ to be considered later in the ultrarelativistic limit in Sec. XI). The polynomial $P(K,V,1)$ can be simplified to

$$\begin{aligned}(1-V)^4[(1+V)^4K^3 &+ 3(1+V^4)K(1+K) \\ &+(32V^2+70V+32)VK(1-K) \\ &+(36V^2+44V+36)VK + (1-V)^4] \\ &\geq (1-V)^8 > 0 \; ,\end{aligned}$$

which shows that the ring is stable in the neighborhood of the segment $v \in [0;1[$ and $c = 1$. This agrees with the numerical results shown in Fig. 5 at the right border of the square corresponding to $c = 1$.

The "warm" cosmic string model [13,14] or constant determinant equation of state $UT = m^4$ gives the case $c = v$ (and $v \neq 0$). Zero is obviously a solution of the characteristic eigenvalue equation (28), which reduces to a quadratic equation and can be exactly solved as

$$V(1-V)(1+V)\sigma[v\sigma - (n+1)v_+][v\sigma - (n-1)v_+] = 0 \; ,$$

where $v_+ = 2v/(1+V)$ was defined before as the Lorentzian double of v. It is easy to verify that for all modes n the eigenvalue subspace of static solutions is of dimension 2 and that the loop is therefore stable when $c = v$. This result has already been observed [1] and was to be expected as the warm cosmic string model has been shown [13] to be completely integrable in flat space. The criterion (29) gives

$$P(K,V,V) = 64V^3(1+V)^2(1-V)^4K(K-1)^2 \; . \quad (32)$$

The stability of loops for which $C - V$ is small cannot be deduced because the polynomial (30) tends to zero for n large ($K = 0$). In this limit, the eigenvalue equation (24) has two equal solutions $\sigma = nv_+/v$. As the first derivatives of $P(0,V,C)$ are all zero when $C = V$, to check the stability of the ring at $K = 0$ around the segment $C = V$, we must consider the Hessian of $P(0,C,V)$ when $C = V$

which gives

$$P(0,V+X,V+Y) = 2V^3(1-V)^6(X^2 - XY + Y^2) .$$

This is definite positive when $v \notin \{0,1\}$ which ensures that in the neighborhood of $(0,V,V)$ the stability criterion (29) is satisfied everywhere. Thus we can conclude that the ring is stable in the neighborhood of the diagonal $c = v$ and $v \notin \{0,1\}$. This result agrees with the numerical results shown in Fig. 5 where the diagonal corresponds to the case $c = v$.

We consider now the constant tension equation of state $T = T_0 \neq 0$ which corresponds to $c = 0$ (and $v \neq 0$). This is a simple equation of state which, however, does not correspond to any physical model. $P(K,V,0)$ reduces to

$$V^3K[(1+V)(3-V)^3K^2 \\ -2(1-V)^2(V^2-12V+9)K-(1-V)^4]$$

which is necessarily negative near $K = 0$. This can also be checked directly on Eq. (28) by making an expansion for $K = 0$ around the multiple solution $\zeta = 1$. The instability of the loop when $c \approx 0$ is verified by the numerical results shown in Fig. 5 as the left border of the square corresponding to $c = 0$ is plotted in black. To get the rate of decay of these unstable modes, the imaginary part of σ must be calculated for each n using the general Cardan formulas and the largest must be retained, leading to the dominant rate of decay. This imaginary part has been plotted numerically as a function of $1/[n(1-V)]$ in Fig. 6 and shows that the maximum imaginary part (leading to the dominant instability) is reached when, for a given V, n is much larger than $1/(1-V)$. In this case, as expressed in (31), the rate of decay is simply

$$\tau = 1/\Omega .$$

If instead of cosmic strings we are concerned with strings of the kind familiar under terrestrial laboratory conditions, the simplest kind of equation of state that is relevant will have the form traditionally named after Hooke, with a tension proportional to the elongation. I

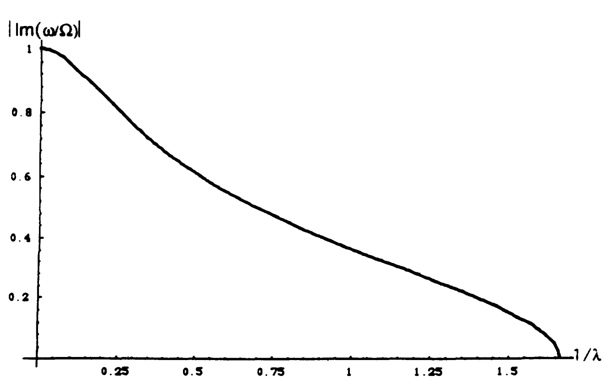


FIG. 6. Plot of the module of the imaginary part of the complex reduced pulsations σ when $c = 0$.

show here that for a Hookean equation of state the rotating rings will always be stable. Using the number density ν as defined in (2), the Hookean equation of state is expressible as

$$T = Y\left(\frac{1}{\nu} - \frac{1}{\nu_0}\right) , \qquad (33)$$

and the energy per unit length U can be derived from the relations (2) as

$$U = Y\left(m\nu + \frac{1}{2\nu} - \frac{1}{\nu_0}\right) , \qquad (34)$$

where m is an integration constant. The speeds c_T and c_L can then be deduced from their expressions in (4) and (5), and their ratio is

$$\frac{c_T^2}{c_L^2} = \left(1 - \frac{\nu}{\nu_0}\right)\left(1 - \frac{1-\frac{\nu}{\nu_0}}{m\nu^2 + \frac{1}{2} - \frac{\nu}{\nu_0}}\right) , \qquad (35)$$

which is always smaller than 1, using the positivity of c_T^2 and of the elongation $1/\nu - 1/\nu_0$ of the string. By the theorem [1] proved in Sec. IX that the relation $c_T < c_L$ guarantees the stability of a circular string loop, it follows that circular string loops of this Hookean kind are never unstable.

VIII. STABILITY AT THE LIMIT VALUES OF THE PARAMETERS

The case $v = 0$ (and $c \neq 0$) corresponds to the limit of an equation of state where T vanishes. Unlike the above cases, it cannot be associated with an equation of state because $T = 0$ would also entail $c = 0$. However, strings with vanishing tension have been studied in the so called "cosmic spring" limit [18–20]. The criterion (29) reduces just to

$$C^3(K+1)^3 > 0 .$$

Thus, by continuity of $P(K,V,C)$, the ring is stable in the neighborhood of the segment $c \in]0;1]$ and $v = 0$. This is in agreement with the numerical results shown in Fig. 5.

The case $v = 1$ (and $c \neq 1$) is also a limiting case of other models as it corresponds to $U = T$. This relation cannot be extrapolated as a valid equation of state as it would give $c^2 = -1$. But it is an interesting limit, as the equation of state for cosmic strings of the kind studied by Witten [21] is in this zone $c_T \approx 1$. In this case, Eq. (24) can easily be solved:

$$2(1-C)(\sigma-1)(\sigma-1/n)(\sigma+1/n) = 0 .$$

This evidently has three real roots so that this case is clearly stable. However, no simple conclusion can be deduced for $v \lesssim 1$ because, when n tends to infinity, there are three equal solutions and so complex solutions and instability may arise in the vicinity. These results can be verified using the criterion (29): $P(K,1,C) = [2(1-C)]^4K^3 \geq 0$ and goes to zero for $K = 0$. To study the stability around $V = 1$, we let $V = 1 - \zeta$ and expand $P(K,V,C)$ at first order for small ζ and K, which gives

$$T(\kappa,C)=\frac{1}{\zeta^6}P(K,1-\zeta,C)$$
$$\sim 16(1-C)^4\kappa^3-(1-C)^2(44C^2+40C^2+40C+44)\kappa^2$$
$$+(-C^4+52C^3+154C^2+52C-1)\kappa+C(1-C)^2\ , \qquad (36)$$

where $\kappa = K/\zeta^2$ in the set $\{1/(n\zeta)^2\}$ (n is an integer greater than 2). The only influence of $\zeta = 1 - V$ on the sign of $T(\kappa, C)$ is to enlarge the interval of variation of κ arbitrarily, so that the sign is now to be studied for κ varying over the whole range of real positive numbers. The derivative of $T(\kappa, C)$ with respect to κ is quadratic and always has a positive discriminant. Its larger solution is

$$\kappa_0=\frac{11+10C+11C^2+2(1+C)\sqrt{31-46C+31C^2}}{12(1-C)^2}\ . \qquad (37)$$

If $T(\kappa, C)$ as defined by (36) reaches a negative value, it must be at $\kappa = \kappa_0$. Figure 7 gives a numerical plot of $(1-C)^2T(\kappa_0, C)$ as a function of C, and this shows that the zone around $V = 0$ is always unstable except for $C \simeq 1$, where some more caution is required, and which will be treated fully in Sec. XI in the ultrarelativistic limit. So when $v \lesssim 1$ the loop is always unstable. This agrees with the numerical simulation shown in Fig. 5 at the upper border of the square corresponding to $v = 1$.

IX. THE STABILITY OF THE ZONE $c_L > c_T$

It has been shown in the preceding work [1] that when $c > v$ the ring is always stable. This is in agreement with the numerical results shown in Fig. 5. In other words, all the regions of short time scale dynamic instability are confined within the region $c > v$ (marked by dots in Fig. 5) that is expected to be subject to the very weak gravitational reaction instability effect that was mentioned in the Introduction [10,11].

The proof of stability for $c > v$ is recalled and extended to show a particular set of curves in the (C, V) plane which always lead to unstable loops. It will prove interesting to study a particular set of unstable points in

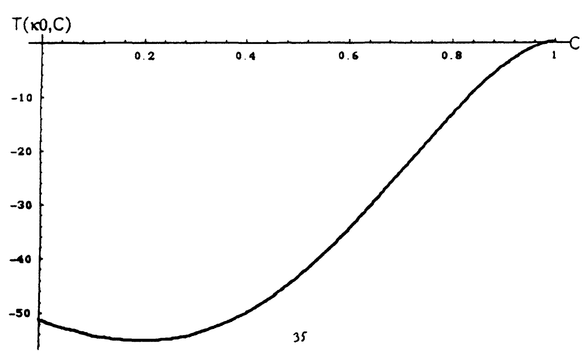


FIG. 7. Plot of $(1-C)^2T(\kappa_0, C)$ as given by the expressions (36) and (37).

the ultrarelativistic limit in Sec. XI and estimate their rate of instability. For $c > v$, Eq. (28) is first reduced to the intersection of a parabola depending on the parameters only through the rescaled variable

$$x=\frac{1+V}{2}\varsigma$$

with a hyperbola whose vertical scale is proportional to the dimensionless "distention" parameter

$$\Gamma=\frac{C-V}{v_+^2(1-CV)}\ .$$

In terms of this parameter, Eq. (28) can be written in the convenient form

$$(x-1)^2-\frac{1}{n^2}=\Gamma(1-v_+^2)\left(x-1+\frac{1}{n^2}\right)\frac{x}{x+\Gamma}\ . \qquad (38)$$

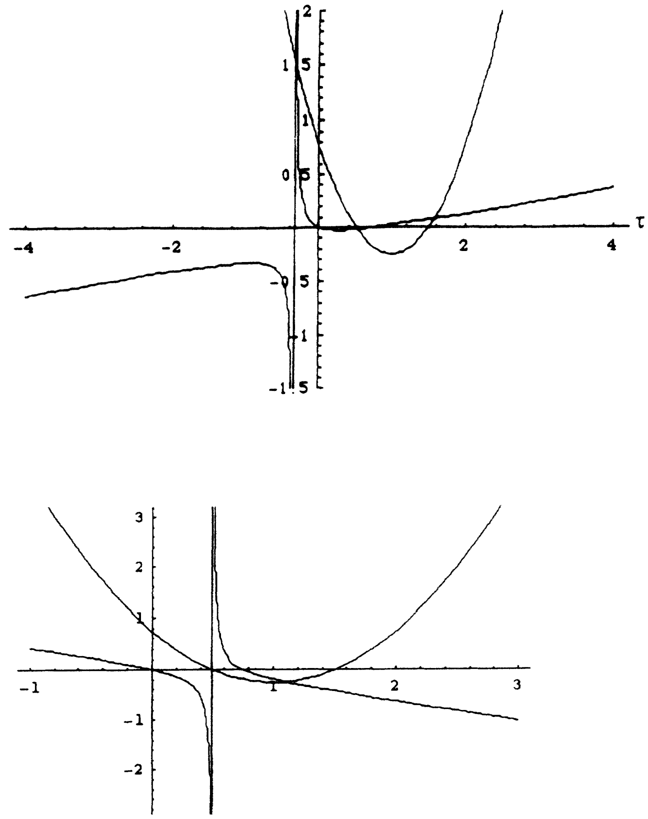


FIG. 8. (a) Plot of the parabola and the hyperbola expressed in (38) for the particular set of parameters $c = \frac{2}{3}$, $v = \frac{1}{2}$, and $n = 2$ as expressed in (39). (b) Plot of the parabola and the hyperbola expressed in (38) for the particular set of unstable parameters corresponding to $v = 0.5$, $n = 3$, and $\Gamma = -\frac{2}{3}$ as given by (40).

The parabola and the hyperbola have been plotted numerically in Fig. 8(a) for the particular set of values of parameters

$$c = \tfrac{2}{3}\ , \quad v = \tfrac{1}{2}\ , \quad n = 2\ , \qquad (39)$$

which are generic for the case $c > v$ and correspond to $\Gamma = \frac{175}{512}$. If the difference between the two sides of Eq. (38) is called $R(x)$, then its limits are

$$\lim_{-\infty} R = +\infty, \quad \lim_{+\infty} R = +\infty\ ,$$

$$\lim_{-\Gamma^-} R = \mathrm{sgn}\left(\Gamma + 1 - \frac{1}{n^2}\right)\infty\ ,$$

$$\lim_{-\Gamma^+} R = \mathrm{sgn}\left(-\Gamma - 1 + \frac{1}{n^2}\right)\infty\ ,$$

$$R(0) = 1 - \frac{1}{n^2} \geq 0\ , \quad R(1) = -\frac{1}{n^2}\left(1 + \Gamma\frac{(1 - v_+^2)}{1 + \Gamma}\right)\ .$$

Thus in the case when $\Gamma > 0$ the polynomial $R(x)$ goes to zero three times: once for $-\Gamma < x < 0$, once for $0 < x < 1$, and one last time for $x > 1$. It can be verified in Fig. 8(a) that the parabola and the hyperbola expressed in (38) intersect each other three times in the intervals where the polynomial $R(x)$ goes to zero. The other interesting result worth mentioning is that for each positive integer n there is a corresponding value

$$\Gamma = -1 + 1/n \qquad (40)$$

of the distention parameter for which the system is always unstable. Figure 8(b) shows the parabola and hyperbola expressed in (38) for a generic value $v_+ = \frac{1}{2}$, $n = 2$, and the corresponding value $\Gamma = -1 + 1/n = -\frac{1}{2}$. The equation of the unstable lines defined above in (40) can be expanded as

$$C = V\frac{-(3 + V)(1 - V) + 4/n}{(1 + 3V)(1 - V) + 4V^2/n}\ . \qquad (41)$$

These lines of unstable loops will prove interesting for studying the rate of decay of the rings for a particular set of curves and especially around $(c, v) = (1, 1)$ where the unstable zones shrink to lines. This case will be further studied in Sec. XI in the ultrarelativistic case.

X. THE CLASSICAL LIMIT

The classical limit corresponds to the limit where $c = 0$ and $v = 0$. In this limit, the eigenvalue equation (24) reduces to nothing and as expected the polynomial (30) goes to zero: $P(K, 0, 0) = 0$. An expansion of $P(K, V, C)$ must be made. The first significant order is the third, and using $\lambda = C/V$ we get

$$P(K, V, \lambda V) \sim [(3 + \lambda)^3K^3 - (18 + 57\lambda + 56\lambda^2 - 3\lambda^3)K^2 + (-1 + 21\lambda + 41\lambda^2 + 3\lambda^3)K + \lambda(1 - \lambda)^2]V^3 = f(\lambda, K)V^3\ .$$

A first analysis of $f(\lambda, K)$ shows that the loop is always unstable around $\lambda = 0$, which is in agreement with the result already obtained in the case $C = 0$ in Sec. VII. At the other extreme, when λ tends to infinity which corresponds to $V = 0$, $f(\lambda, K) \sim \lambda^3(1 + K)^3$, the loop is always stable as already shown at the beginning of Sec. VII. So there exists a transition between a stability and an instability zone around this point. We intend to find the steepness λ of this frontier between stability and instability that appears clearly in Fig. 5. Because this is a transition between stability and instability, the polynomial (30) must go to zero, or equivalently $f(\lambda, K) = 0$. There are an infinity of solutions to this equation in λ corresponding to the infinity of possible values of $K = 1/n^2$ with $n > 1$, but because of the overlapping of instability zones for each n only one appears in Fig. 5. This one can be seen by comparing Figs. 3 and 5 to correspond to the mode $n = 2$ or else $K = \frac{1}{4}$. The equation

$$f(\lambda, \tfrac{1}{4}) = \frac{125\lambda^3 + 313\lambda^2 + 199\lambda - 61}{64} = 0$$

gives only one positive solution $\lambda \simeq 0.222$, which must therefore be the gradient of the tangent to the transition between stability and instability in Fig. 5. The transition from stability to instability around the classical limit $c_T = c_L = 0$ thus happens around

$$\frac{c_L}{c_T} = 0.47\ . \qquad (42)$$

Since the case of classical strings applies to the kinds of strings that are found in an ordinary laboratory context, an experiment can be made to determine the effective stability of rotating rings with this kind of equation of state, and instability should be found to arise in some cases. However, as shown above, the strings with a Hookean equation of state are always stable, so, to find instability, such an experiment would have to be conducted with strings having a stiffer kind of equation of state.

XI. ULTRARELATIVISTIC LIMIT

The ultrarelativistic limit corresponding to strings with $c \simeq 1$ and $v \simeq 1$ is of particular interest as cosmic strings fall in this regime. The limit $c = v = 1$ is very singular as Eq. (24) disappears. This can be checked on the polynomial (30) which goes to zero: $P(K, 1, 1) = 0$. Therefore we work around this limit by letting $V = 1 - \zeta$ and $C = 1 - \xi$ and expanding $P(K, C, V)$ for small ξ and ζ, keeping only the lowest order for each coefficient a_0, a_1, a_2, and a_3 of the polynomial in (30):

$$P(K,1-\zeta,1-\zeta) \approx 16K[(\zeta+\zeta)^2K-4\zeta^2]^2+\zeta^6(\zeta-\xi)^2 .$$

In the ultrarelativistic limit, the loop is thus stable whenever the first significant order does not vanish, i.e., whenever $\zeta \neq \xi$ and $\xi \neq \zeta(2n-1)$. The first case, corresponding to $c = v$, has already been studied with the warm cosmic string equation of state in Sec. VII and is stable. In the latter case $\xi = \zeta(2n-1)$ the calculations must be taken to second order, which gives

$$P(K,1-\zeta,1-\zeta(2n-1+x))$$

$$\approx 256n^2\zeta^4\{[x+(n-1)(2n-1)\zeta]^2-2n(n-1)^3\zeta^2\} .$$

Therefore there are always values of x and ζ for which the loop is unstable except in the cases $n = 0, 1$ which are already known to be stable [for instance, at $x = -(n-1)(2n-1)\zeta$]. The zones of instability are delimited by two parabolas with the same tangent and on the same side of their tangent which is explicitly given by

$$C = (2n-1)V - 2(n-1) . \tag{43}$$

This has been verified using the numerical results shown in Figs. 3–5. In Figs. 3 and 4, the gradients of the shrinking lines of instability are, respectively, about $\frac{1}{3}$ and $\frac{1}{5}$ in agreement with the above result of $1/(2n-1)$. Moreover, it can be seen in Fig. 5 that the lines of instability are, as expected, on the same side of their tangent.

To study more precisely these vanishing lines of instability, I will use the particular set of unstable lines found at the end of Sec. IX, defined by Eqs. (40) or (41). Figure 9(a) shows in the plane (C,V) these curves for various values of n and it is seen that these curves tend to the $V = 1$ line when n becomes large. The expression (40) for the lines can be put in Eq. (38) to get a characteristic equation depending on v_+^2 and $k = 1/n$ only, in the form

$$x^3 - [2+(1-k)v_+^2]x^2$$

$$+(1-k)[2+k+k^2+(1-k^2)v_+^2]x$$

$$+(1-k)^2(1+k) = 0 . \tag{44}$$

There is also a condition of positivity of C in (41) which is

$$V \geq 2\sqrt{1-1/n}-1 .$$

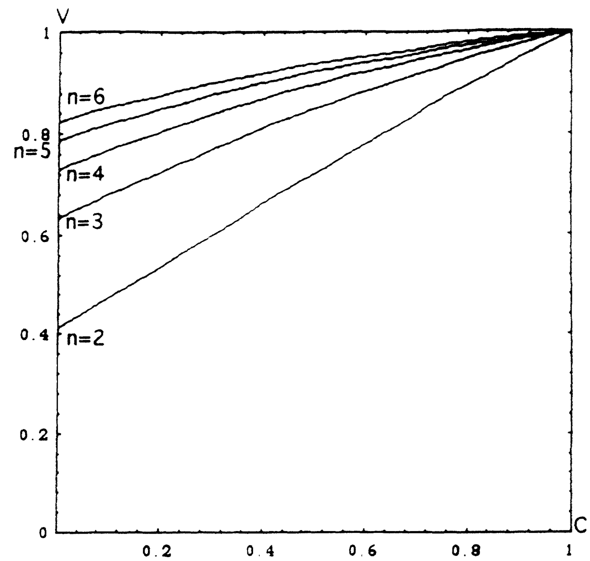
Applying the Cardan formula to the third degree equation (44) above to get the imaginary part of the complex solutions, we get an algebraic expression for

$$|\mathrm{Im}(\omega/\Omega)| = \frac{2n}{1+V}|\mathrm{Im}(x)| ,$$

which has been plotted for $n \in \{2,3,4,5,6\}$ in Fig. 9(b) as a function of C (because V is varying on a vanishing interval when n tends to infinity). The imaginary part of the complex solutions of the characteristic equation (24) are interesting to know as they are closely related to the rate of decay of the unstable stationary equilibrium state through the expression (31). The tangent to the above curves at $C = V = 1$ can be expressed algebraically in the form

$$\frac{d|\mathrm{Im}(\omega/\Omega)|}{dC}\Big|_{C=1^-} = -\frac{1}{8}\sqrt{2\left(1-\frac{1}{n}\right)} . \tag{45}$$

It appears clearly in Fig. 9(b) that the curves seem to



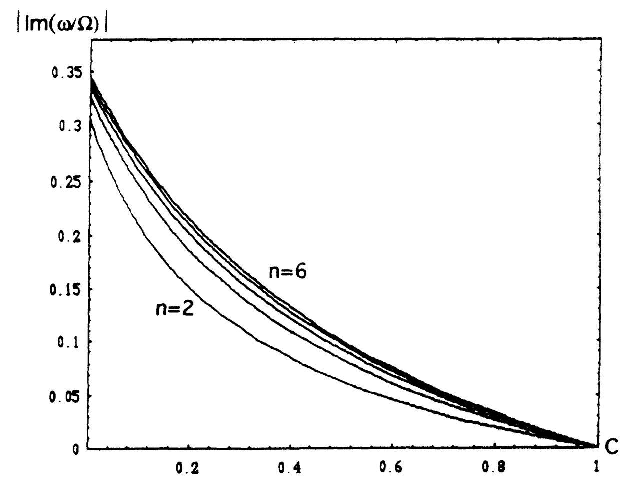


FIG. 9. (a) Plot of the unstable lines as expressed in (40) or (41) for various values of n. (b) Plot of the module of the imaginary part of the complex solutions of (44) or (24) for the unstable lines shown in (a).

tend to a limit when n tends to infinity. This can be confirmed from (44) by letting

$$\alpha = (1 - V)/k \approx (1 - C)/(1 + C) ,$$

$$\lambda = (x - 1)/k = (1 + V)\sigma/2 \approx \sigma ,$$

and working at first significant order in k and $(x - 1)$. Then $|\mathrm{Im}(\sigma)| = |\mathrm{Im}(\lambda)|$ with λ satisfying the equation

$$\lambda^3 + \lambda^2 + (\alpha^2/4 - 1)\lambda - 1 = 0 . \quad (46)$$

In particular, at $C = 0$ for large n, the value of $|\mathrm{Im}(\sigma)|$ tends to the imaginary part of the complex solutions of the equation $\lambda^3 + \lambda^2 - 3\lambda/4 - 1 = 0$, that is, about 0.358; at $C = 1$ for large n, (46) becomes $\lambda^3 + \lambda^2 - \lambda - 1 = (\lambda - 1)(\lambda + 1)^2 = 0$, so a development must be made around $\lambda = -1$ and $C = 1$, giving a complex part $(1 - C)/4\sqrt{2}$, which is of course the limit of (45) for large n. These two results are in good agreement with what can be seen in Fig. 9(b).

CONCLUSION

This investigation has shown the complexity of the situation in the ultrarelativistic regime that is of interest for cosmic string applications. The main remaining problem left for future work is the long term outcome for unstable loops. Further work will also be necessary to establish the precise location of the equilibrium states for particular (more or less realistic) cosmic string models, starting with the Witten type toy model analyzed by Peter [10,22] (which lies in the zone $c_T \simeq 1$ and must therefore have a nontrivial behavior with respect to dynamical stability of loops) in relation to the interpenetrating zones of stability and instability that are illustrated in Fig. 5. This only requires that the two speeds c_T and c_L obtained from the particular model of superconducting strings be put in the criterion (29).

It has been seen that, being always stable, the equations of state for which the speed of transverse perturbation is lower than that for longitudinal perturbations, $c_T < c_L$, have qualitatively a different dynamic behavior than the other equations of state for which $c_T > c_L$, where the situation is less conclusive. As pointed out above, it seems that Witten superconducting strings lie in the latter, less decisive zone where $c_T > c_L$. The constant trace equation of state $U + T = 2T_0$, which corresponds to the simplest correction to the Goto-Nambu model, can be seen to rest in the first zone where $c_T < c_L$. The great stability of these strings does not correctly represent the situation for cosmic strings, in which it has been seen that instability may arise. The constant determinant equation of state $UT = T_0^2$ (which characterizes the "warm string" model for a wiggly Goto-Nambu string), although stable also, has a nontrivial speed of longitudinal perturbations c_L and is more "central" in the ultrarelativistic zone in the sense that it lies between the two zones $c_T < c_L$ and $c_T > c_L$. Thus this equation of state appears to be dynamically a better approximation to Witten models than the constant trace model. This model has also the interesting property [13] of being completely solvable in flat background.

It is to be remarked that the results obtained in this formalism could be tested in the classical zone, as described in Sec. X, by looking for the instability predicted in Eq. (42) when $c_L/c_T < 0.47$. This could probably be reached with material in the nonlinear regime (as the Hookean equation of state is stable) by taking rapidly rotating loops, thereby increasing c_T, which is equal to the rotational speed at equilibrium by Eq. (13).

It is also possible to estimate an order of magnitude for the rate of instability τ of a loop. It appears numerically, from the expression for the imaginary part of a complex solution, that for an unstable loop the imaginary part of the solutions for ω/Ω of the characteristic equation (24) remains bounded by 1, and is generally of order unity except in the transition zones between stability and instability, where it can be lower by some orders of magnitude:

$$|\mathrm{Im}(\omega/\Omega)| \lesssim 1 . \quad (47)$$

The typical radius of a loop at equilibrium has been roughly estimated by Carter [9]. If we call η the mass scale (expressed in Planck mass) at which the strings are formed, the typical radius of a loop (expressed in Planck length unit) was found to be

$$r \approx \eta^{-7/6} . \quad (48)$$

Thus for unstable loops not in the classical zone, the time scale of instability of a circular loop (expressed in Planck time) can be estimated from (47), (13), and (31) as

$$\tau \approx \eta^{-7/6} . \quad (49)$$

For cosmic strings formed at electroweak mass scale, the time scale of instability is extremely short:

$$\tau \approx 10^{-25}\ \mathrm{s} .$$

It can be seen that for heavier cosmic strings, like those arising at grand unification, the time scale of instability will be even shorter. We thus conclude that unstable circular loops will not survive long enough in the Universe to be of any cosmological importance.

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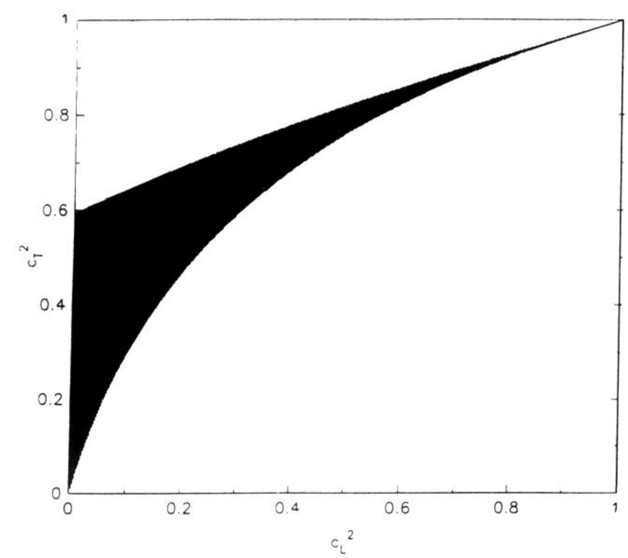

FIG. 3. The region of instability is plotted (in black) on the plane parametrized by the squared longitudinal and transverse perturbations, c_L^2 and c_T^2, on the basis of numerical evaluation of the polynomial $P(1/n^2, C, V)$ given by (38) on a regular grid of 500×500 points for the particular case of the mode $n = 2$.

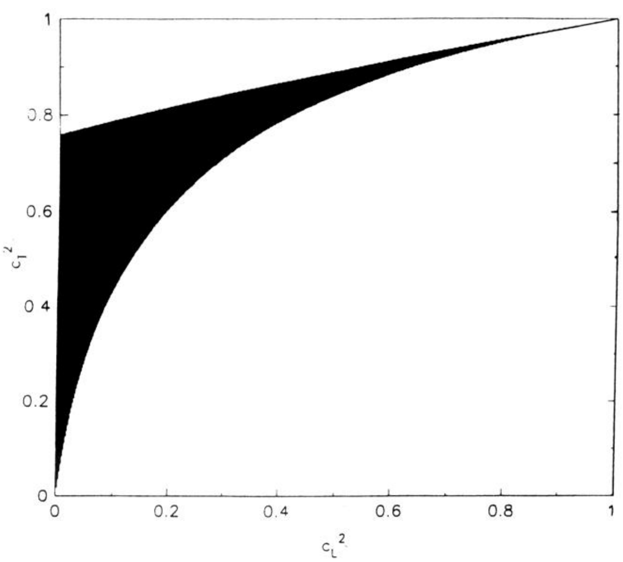

FIG. 4. Same as Fig. 3 for the mode $n = 3$.

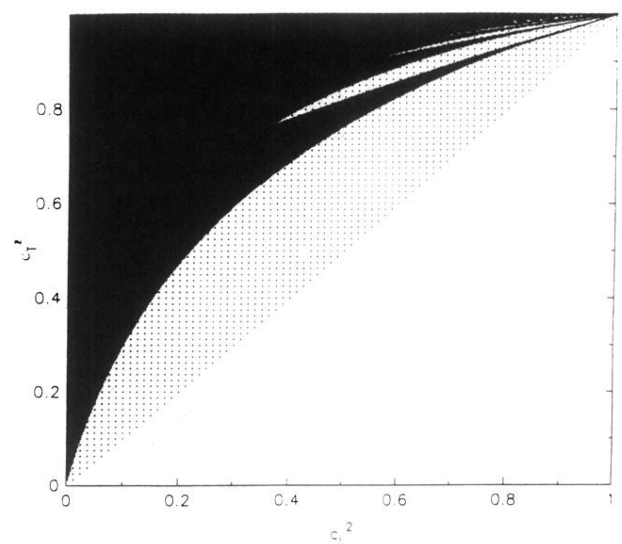

FIG. 5. Same as Figs. 3 and 4 for a superposition of all modes up to $n = 400$ (the more extended region of potential secular instability due to gravitational radiation reaction is indicated by dots).