

String field theory in the temporal gauge

M. Ikehara,^{1,2} N. Ishibashi,¹ H. Kawai,¹ T. Mogami,^{1,3} R. Nakayama,⁴ and N. Sasakura¹

¹*KEK Theory Group, Tsukuba, Ibaraki 305, Japan*

²*Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan*

³*Department of Physics, Kyoto University, Kitashirakawa, Kyoto 606, Japan*

⁴*Department of Physics, Hokkaido University, Sapporo 060, Japan*

(Received 7 July 1994)

We construct the string field Hamiltonian for $c = 1 - 6/m(m+1)$ string theory in the temporal gauge. In order to do so, we first examine the Schwinger-Dyson equations of the matrix chain models and propose the continuum version of them. The results of boundary conformal field theory are useful in making a connection between the discrete and continuum pictures. The W constraints are derived from the continuum Schwinger-Dyson equations. We also check that these equations are consistent with other known results about noncritical string theory. The string field Hamiltonian is easily obtained from the continuum Schwinger-Dyson equations. It looks similar to the Kaku-Kikkawa Hamiltonian and may readily be generalized to $c > 1$ cases.

PACS number(s): 11.25.Sq, 11.25.Pm

I. INTRODUCTION

String theory provides us with the most promising framework for describing physics at the Planck scale. However, a nonperturbative treatment of string theory is indispensable for relating it to the lower energy phenomena we see. String field theory [1] is expected to make such treatment possible. A string field theory corresponds to a rule to cut the string world sheets into vertices and propagators, or in other words, a way to fix the reparametrization invariance.

Recently a new class of string field theory was proposed for $c = 0$ noncritical string [2]. It is based on a gauge fixing [3] of the reparametrization invariance, which can naturally be considered on dynamically triangulated world sheets. The gauge, which can be called the temporal gauge [4] or the proper time gauge [5], is peculiar in many respects. For example, in this gauge, even a disk amplitude is expressed as a sum of infinitely many processes involving innumerable splitting of strings. It forms a striking contrast to the case of the conformal gauge. The amplitudes can be calculated by using the Schwinger-Dyson (SD) equations of the string field. Actually the SD equations are powerful enough to make a nonperturbative treatment of the $c = 0$ string possible. Indeed, the Virasoro constraints [6] can be derived from the SD equation, and all the results of the matrix model are reproduced. Conversely it was pointed out by Jevicki and Rodrigues [7] that the string field Hamiltonian can be derived from the stochastic quantization of the matrix model. Also in [8], the string field Hamiltonian was deduced from the matrix model.

Therefore if the temporal gauge string field theory is constructed for the critical string, it may be a useful tool to study the nonperturbative effects of string theory. In order to go from $c = 0$ to the critical string, one needs to know a way to introduce matter degrees of freedom on the world sheet. In [9], $c \leq 1$ string field Hamiltonian was

constructed by changing the way of gauge fixing a little. However, it was not possible to derive the W constraints from this Hamiltonian and prove that it really describes a $c \leq 1$ string theory.

In the present work, we will propose a string field theory of a $c = 1 - 6/m(m+1)$ string in the temporal gauge such that the W constraints are deduced from the string field SD equation. Actually we start from the matrix model SD equations, from which the W constraints are deduced. We propose the continuum version of these equations. Since the string field SD equations are in close relation with the matrix model ones, it is easy to construct the string field Hamiltonian once we know the continuum version of matrix model SD equations. Thus the Hamiltonian we construct is naturally related to the W constraints.

The organization of this paper is as follows. In Sec. II, we first consider $c = \frac{1}{2}$ case as an example. After briefly explaining the relation between the temporal gauge string field theory and the matrix model SD equations, we examine the SD equations for the two matrix models which were analyzed by Gava and Narain [10]. We propose the continuum version of these equations and show that the W_3 constraints can be deduced from the continuum equations. We also check if our equations are consistent with other known results of $c = \frac{1}{2}$ string theory. In Sec. III, we generalize the discussion of Sec. II to the case of a $c = 1 - 6/m(m+1)$ string. In Sec. IV, we construct the string field Hamiltonian from the SD equations obtained in Secs. II and III.

II. CONTINUUM SD EQUATIONS FOR A $c = \frac{1}{2}$ STRING

Let us recall the definition of the time coordinate in [3]. Suppose a randomly triangulated surface with boundaries. The time coordinate of a point on the surface is

defined to be the length of the shortest curves connecting the point and the boundaries. In [3], this time coordinate was introduced to study the fractal structure [11] of random surfaces. It was shown that a well-defined continuum limit of such a time coordinate exists at least in the case of a $c = 0$ string. If one takes such a time coordinate t in the continuum limit, the metric will look like

$$ds^2 = dt^2 + h(x, t)[dx + N^1(x, t)dt]^2.$$

In [4], two-dimensional (2D) quantum gravity was studied by further fixing the gauge as $\partial_x h = 0$. Such a gauge was called the temporal gauge. In [5], the gauge $N^1 = 0$ was pursued, which was called the proper time gauge.

In this coordinate system, we cut the surface into time slices. Then the surface can be interpreted as describing a history of strings which keep splitting and joining. In [2], a string field Hamiltonian H describing the evolution of the strings in such a coordinate system was constructed. In this paper, we will call this Hamiltonian the string field Hamiltonian in the temporal gauge. (It can also be called the proper time gauge Hamiltonian.) H is expressed in terms of the creation [annihilation] operator $\Psi^\dagger(l)$ [$\Psi(l)$] of the string. Since each string is labelled only by its length, the string field is a function of the length l . An n -string amplitude corresponds to the world sheets with n boundaries, each of which describes an external string state. Therefore such an amplitude is expressed as

$$\lim_{D \rightarrow \infty} \langle 0 | e^{-DH} \Psi^\dagger(l_1) \cdots \Psi^\dagger(l_n) | 0 \rangle. \quad (2.1)$$

The string amplitudes can be obtained by solving the string field SD equation:

$$\lim_{D \rightarrow \infty} \partial_D \langle 0 | e^{-DH} \Psi^\dagger(l_1) \cdots \Psi^\dagger(l_n) | 0 \rangle = 0. \quad (2.2)$$

This equation means that the string amplitudes do not change if one puts the time evolution operator on all the external string states. In the point of view of 2D quantum gravity, this equation corresponds to the Wheeler-DeWitt equation.

Even if there are matter fields on a dynamically triangulated surface, a time coordinate can be defined in the same way. Here we concentrate on a $c = \frac{1}{2}$ string. Such a string can be realized by putting the Ising model on the random surface. Since the length of a curve on the surface is defined irrespective of the matter, the time coordinate can be defined and the surface is cut into time slices. Again, the surface can be regarded as describing a history of strings. Therefore we will be able to construct a string field Hamiltonian describing the time evolution of the strings. However, in this case, the strings have Ising spins on them. Hence the string field depends not only on the length of the string but also on the spin configuration on it.

In the continuum limit, an Ising spin configuration may be represented by a state¹ of $c = \frac{1}{2}$ conformal field the-

ory (CFT). The splitting and joining of the strings should be described by the three-Reggeon-like vertex for $c = \frac{1}{2}$ CFT, and the string field Hamiltonian will be very complicated. This is the reason why an alternative definition of the time coordinate was taken in [9]. Here we would like to stick to this time coordinate and construct the Hamiltonian in the temporal gauge.

One can obtain a hint on the form of such a Hamiltonian by examining the matrix model SD equations. As was discussed in [2,8], the string field SD equations are closely related to the matrix model SD equations. The latter describe the change of the partition functions corresponding to dynamically triangulated surfaces when one takes a triangle away from a boundary. It is obvious from the definition of the time coordinate that at the discrete level the former equations describe the changes which happen when one takes one layer of triangles from all the boundaries. Therefore, in the continuum limit, the former should be expressed as an integration of the latter.

Hence if we know the continuum limit of the matrix model SD equations, we can figure out what the string field SD equations should be. Then we can infer the form of the string field Hamiltonian. Gava and Narain [10] studied the SD equation for the two matrix model and deduced the W_3 constraints. In this section, we will consider the continuum limit of the equations of Gava and Narain.

A. Continuum limit of the Gava-Narain equation

Let us sketch how Gava and Narain obtained the W_3 constraints. The W_3 constraints are expected to come from equations about the loop amplitudes in which the Ising spins on all the boundary loops are, say, up. Suppose the partition function of the dynamically triangulated surfaces with boundaries on which all the Ising spins are up. If one takes one triangle from a boundary, the following three things can happen (Fig. 1): (1) The boundary loop splits into two; (2) the boundary loop absorbs another boundary; (3) the spin configuration on the boundary loop changes.

The matrix model SD equation is a sum of three kinds of terms corresponding to the above processes. In the first and the second process, only boundaries with all the spins up can appear. The third process is due to the matrix model action. A boundary loop on which one spin is down and all the others are up can appear in this process. In order to derive the W_3 constraints, one should somehow cope with this mixed spin configuration. Gava and Narain then considered the loop amplitudes with one loop having such a spin configuration and all the other loops having all the spins up. They obtained two SD equations corresponding to the processes of taking away the triangle attached to the link on which the Ising spin is down and the one attached to the next link. Those equations also consist of the terms corresponding to the above three processes. With these two equations, one can express the loop amplitude with one mixed spin loop insertion by loop amplitudes with all the spins up. Thus

¹Here the states we mean do not necessarily satisfy the condition $(L_0 - \bar{L}_0)|v\rangle = 0$.

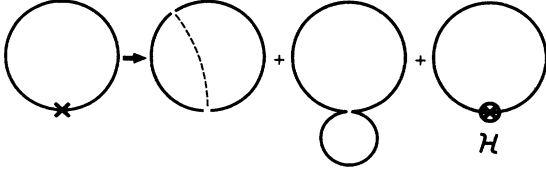


FIG. 1. Processes involved in SD equations. If one deforms the loop on the left hand side at the cross, it either splits into two (the first term on the right hand side), absorbs another loop (the second term), or changes in its spin configuration (the third term). The change in the spin configuration is expressed by an operator \mathcal{H} .

they can obtain closed equations for loop amplitudes with all the spins up, and the W_3 constraints were derived from them.

We would like to rewrite the above procedure in terms of the continuum language. Let us define the continuum loop operator $w(l; |v\rangle)$ as representing a loop with length l and the spin configuration corresponding to $|v\rangle$ which is a state of $c = \frac{1}{2}$ CFT. We take the loop to have one marked point on it. The loop amplitude will be denoted by

$$\langle w(l_1; |v_1\rangle) w(l_2; |v_2\rangle) \cdots w(l_n; |v_n\rangle) \rangle. \quad (2.3)$$

The matrix model SD equation describes the change of the amplitude Eq. (2.3), when one takes a triangle away from a boundary. Now we will construct the continuum version of it, which describes what happens when one deforms the amplitude Eq. (2.3) at a point on a boundary. In principle, by closely looking at the discrete SD equations and taking the continuum limit, one should be able to figure out what the continuum SD equations will be. However, in actuality, it is not an easy task, because of the existence of the nonuniversal parts in the loop operators and the operator mixing between various loop operators. Therefore, here we will construct the contin-

uum SD equations by assuming the following properties of them and check the validity of our assumption later by deriving the W constraints from them.

(1) We will assume that the continuum SD equation also consists of the three terms representing the three processes in the above (Fig. 1): i.e., a loop splitting into two, a loop absorbing another one, and changes in the spin configuration of the loop. Let us call the first two the vertex terms and the last one the kinetic term.

(2) We will assume that when a loop splits into two, the descendant loops should inherit the spin configuration of the original loop. Such a three-string vertex will be expressed by a δ functional of the spin configurations, i.e., the three-Reggeon-like vertex of $c = \frac{1}{2}$ CFT in the continuum limit. The process where a loop absorbs another loop will also be expressed by the three-Reggeon-like vertex.

(3) In the matrix model SD equations, the kinetic terms come from the matrix model action. In the two matrix model, they include terms which change the length of the loop as well as a term which flips the spin. We will assume that in the continuum limit, only the spin flipping term survives.

With all these assumptions, we are able to write down the continuum SD equations. We will present the most general continuum SD equation using such vertices in Sec. IV. Here let us concentrate on a simpler situation, which Gava and Narain considered. In the derivation of the W_3 constraints, they started from loops with all the spins up. Such a spin configuration was represented as a state of $c = \frac{1}{2}$ CFT in [12,13]. Let us denote such a state by $|+\rangle$. It is clear that if such a loop splits into two, it results in two loops with all the spins up. Also if a loop with all the spins up absorbs another one, we obtain another loop with all the spins up. Therefore the process of splitting and merging is particularly simple for such kind of loops. The first SD equation Gava and Narain considered corresponds to the deformation of the loop amplitude Eq. (2.3) with $|v_1\rangle = |v_2\rangle = \cdots = |v_n\rangle = |+\rangle$. The equation in the continuum limit should be

$$\begin{aligned} & \int_0^l dl' \langle w(l'; |+\rangle) w(l-l'; |+\rangle) w(l_1; |+\rangle) \cdots w(l_n; |+\rangle) \rangle \\ & + g \sum_k l_k \langle w(l+l_k; |+\rangle) w(l_1; |+\rangle) \cdots w(l_{k-1}; |+\rangle) w(l_{k+1}; |+\rangle) \cdots w(l_n; |+\rangle) \rangle \\ & + \langle w(l; \mathcal{H}(\sigma)|+\rangle) w(l_1; |+\rangle) \cdots w(l_n; |+\rangle) \rangle \approx 0. \end{aligned} \quad (2.4)$$

Here the first term corresponds to the process 1 in the above, and the second term is for the process 2. The string coupling constant g comes in front of the second term as in the case of a $c = 0$ string [2]. The last term describes the process 3, where the operator $\mathcal{H}(\sigma)$ expresses the local change of the spin configuration. $0 \leq \sigma < 2\pi$ is the coordinate of the point where the local change occurs. The coordinate σ on the loop is taken so that the induced metric on the loop becomes independent of σ . $\sigma = 0$ is taken to be the marked point of the loop. ≈ 0

here means that as a function of l , the quantity has its support at $l = 0$. Therefore the left hand side of Eq. (2.4) is equal to a sum of derivatives of $\delta(l)$. These δ functions correspond to processes in which a string with vanishing length disappears. In the point of view of string field theory, such processes are expressed by the tadpole terms.

Therefore $w(l; \mathcal{H}(\sigma)|+\rangle)$ is supposed to correspond to a loop with one spin flipped to be down because of $\mathcal{H}(\sigma)$, and the rest of the spins up (Fig. 2). In the continuum limit, the operator which is on the domain wall of up and

down spins is identified to be $\phi_{2,1}$ [13] (Fig. 2). Therefore $\mathcal{H}(\sigma)$ may be written as $\lim_{\sigma' \rightarrow \sigma} \phi_{2,1}(\sigma') \phi_{2,1}(\sigma)$. With this operator, we can express everything about the SD equations in terms of the continuum language.

The two other equations which Gava and Narain used were obtained by taking a triangle away from $w(l; \mathcal{H}(\sigma)|+)$. The triangles to be considered were the one attached to the link where $\mathcal{H}(\sigma)$ is inserted and the one next to it. In the continuum limit, these equations will correspond to the following two equations. In one of the equations, we consider a loop $w(l; \mathcal{H}(\sigma)|+)$ and deform at a point near σ and take the limit in which the point tends to σ (Fig. 3). The SD equation becomes

$$\begin{aligned} & \int_0^l dl' \langle w(l'; |+)) w(l-l'; \mathcal{H}(\sigma)|+) w(l_1; |+)) \cdots w(l_n; |+)) \rangle \\ & + g \sum_k l_k \langle w(l+l_k; \mathcal{H}(\sigma)|+) w(l_1; |+)) \cdots w(l_{k-1}; |+)) w(l_{k+1}; |+)) \cdots w(l_n; |+)) \rangle \\ & + \langle w(l; [\mathcal{H}(\sigma)]^2|+) w(l_1; |+)) \cdots w(l_n; |+)) \rangle \approx 0. \end{aligned} \quad (2.5)$$

Here $w(l; [\mathcal{H}(\sigma)]^2|+)$ denotes the limit $\lim_{\sigma' \rightarrow \sigma} w(l; \mathcal{H}(\sigma') \mathcal{H}(\sigma)|+)$. In the other equation, we consider a loop $w(l; \phi_{2,1}(\sigma') \phi_{2,1}(\sigma)|+)$, deform at a point between the two $\phi_{2,1}$ insertions and then take the limit $\sigma' \rightarrow \sigma$ (Fig. 4). The insertion of \mathcal{H} yields $w(l; [\mathcal{H}(\sigma)]^2|+)$ again. However, this time, the loop cannot split or absorb a loop $w(l; |+)$. When a loop splits, two points on the loop should merge. The spin configurations at the two points should coincide in order for this to happen. Now we deform the loop at a point in the down spin region, and the point it merges with should be in the down spin region. Therefore, in the limit $\sigma' \rightarrow \sigma$, no splitting can occur. The loop cannot absorb another loop for the same reason. Hence we obtain

$$\langle w(l; [\mathcal{H}(\sigma)]^2|+) w(l_1; |+)) \cdots w(l_n; |+)) \rangle \approx 0. \quad (2.6)$$

This equation means that the loop $w(l; [\mathcal{H}(\sigma)]^2|+)$ is in a sense “null.” Similar arguments as above show that correlation functions involving such a loop vanish unless there exist any finite regions of down spins on the boundaries.

We propose Eqs. (2.4), (2.5), and (2.6) as the continuum limit of the Gava-Narain equations. As a check of the validity of our equations, let us first see if the disk amplitude of $c = \frac{1}{2}$ string theory satisfies these equations. Let us denote the disk amplitude with a loop $w(l; [\mathcal{H}(\sigma)]^n|+)$ as the boundary by $w_n(l)$. The Laplace transform $\tilde{w}_0(\varsigma) = \int_0^\infty dl e^{-\varsigma l} w_0(l)$ is known as [14]

$$\tilde{w}_0(\varsigma) = (\varsigma + \sqrt{\varsigma^2 - t})^{4/3} + (\varsigma - \sqrt{\varsigma^2 - t})^{4/3}, \quad (2.7)$$

²Because of the reparametrization invariance, correlation functions involving $w(l; [\mathcal{H}(\sigma)]^n|+)$'s do not depend on σ .

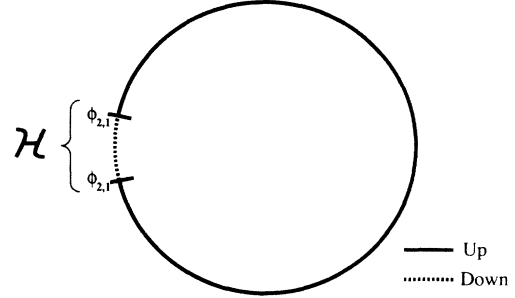


FIG. 2. The action of the operator \mathcal{H} .

where t is the cosmological constant. If our equations really correspond to $c = \frac{1}{2}$ string theory, this disk amplitude should satisfy these equations at the lowest order in the expansion in terms of g . In the Laplace transformed form, the equations to be satisfied are

$$\begin{aligned} & (\tilde{w}_0(\varsigma))^2 + \tilde{w}_1(\varsigma) \approx 0, \\ & \tilde{w}_0(\varsigma) \tilde{w}_1(\varsigma) + \tilde{w}_2(\varsigma) \approx 0, \\ & \tilde{w}_2(\varsigma) \approx 0. \end{aligned} \quad (2.8)$$

Here $\tilde{w}_n(\varsigma)$ denotes $\int_0^\infty dl e^{-\varsigma l} w(l; \mathcal{H}^n(\sigma)|+)$. ≈ 0 here means that the quantity is a polynomial of ς . It is easy to see that the disk amplitude (2.7) and

$$\tilde{w}_1(\varsigma) = -(\varsigma + \sqrt{\varsigma^2 - t})^{8/3} - (\varsigma - \sqrt{\varsigma^2 - t})^{8/3} + t^{4/3}, \quad (2.9)$$

is a solution of (2.8).

This w_1 in Eq. (2.9) is a new kind of amplitude which has never appeared in the literature. It indeed emerges in

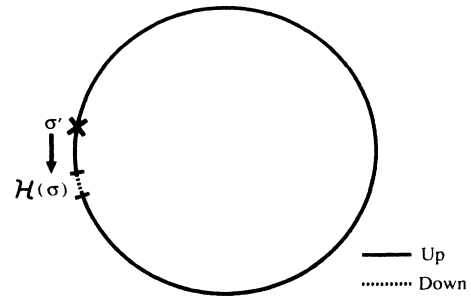


FIG. 3. The SD equation (2.5).

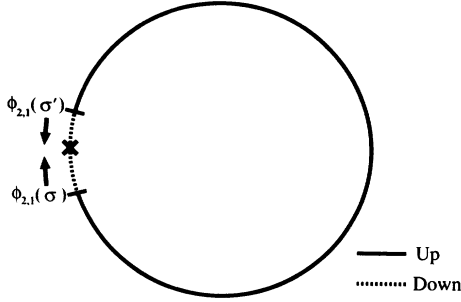


FIG. 4. The SD equation (2.6).

the continuum limit of the matrix model disk amplitude³ $W(P) = \langle \frac{1}{N} \text{tr}(P - A)^{-1} \rangle$. $W(P)$ is a solution to the matrix model SD equations given in [10,15,19]. In the continuum limit, one should take P and the matrix model coupling constant⁴ g to approach the critical value P_* and g_* as $P = P_* + a\varsigma$, $g = g_* + \text{const} a^2 t$, where a is the lattice cutoff. By expanding $W(P)$ in powers of a , one obtains

$$W(P) = b_0 + b_3 \varsigma a + b_4 \tilde{w}_0(\varsigma) a^{\frac{4}{3}} + b_5 \partial_\varsigma \tilde{w}_1(\varsigma) a^{\frac{5}{3}} + O(a^2),$$

where b_i 's are nonuniversal constants. Thus we can see that not only $w_0(l)$ but also $lw_1(l)$ are included in the continuum limit of the disk amplitude $W(P)$. Here

$lw_1(l) \sim \langle w(l; l \int d\sigma \mathcal{H}(\sigma) | +) \rangle_0$ rather than $w_1(l)$ appears because $W(P)$ corresponds to a loop which is invariant under rotation.

We conclude this subsection with a comment on the scaling dimensions. The scaling dimension of the disk amplitude $\langle w(l; | +) \rangle_0$ can be estimated [14] by the Knizhnik-Polyakov-Zamolodchikov-David-Distler-Kawai (KPZDDK) argument [16] to be $L^{-7/3}$, where L denotes the dimension of the boundary length. From the above result, the dimension of $\langle w(l; \mathcal{H}(\sigma) | +) \rangle_0$ is $L^{-11/3}$. The difference $L^{-4/3}$ of the dimensions is attributed to the insertion of the operator $\mathcal{H}(\sigma)$. Notice that Eqs. (2.8) make sense as a continuum SD equation only when $\mathcal{H}(\sigma)$ has such a dimension. It is quite consistent with the identification $\mathcal{H}(\sigma) = \lim_{\sigma' \rightarrow \sigma} \phi_{2,1}(\sigma') \phi_{2,1}(\sigma)$, because the gravitational scaling dimension of $\phi_{2,1}$ on the boundary is [16,17] estimated to be $L^{-2/3}$.

B. Derivation of the W_3 constraints

If our continuum limit SD equation is correct, Eqs. (2.4), (2.5), and (2.6) should yield the W_3 constraints. In this subsection we will show that this is indeed the case. In order to do so, let us define the generating functional of loop amplitudes as

$$Z(J_0(l), J_1(l), J_2(l)) = \left\langle \exp \left(\int_0^\infty dl J_0(l) w(l; | +) + \int_0^\infty dl J_1(l) w(l; \mathcal{H}(\sigma) | +) + \int_0^\infty dl J_2(l) w(l; [\mathcal{H}(\sigma)]^2 | +) \right) \right\rangle. \quad (2.10)$$

Using this generating functional, the SD equations (2.4), (2.5), and (2.6) can be rewritten as

$$\left(\frac{\delta}{\delta J_{n+1}(l)} + \int_0^l dl' \frac{\delta^2}{\delta J_0(l') \delta J_n(l-l')} + \int_0^\infty dl' l' J_0(l') \frac{\delta}{\delta J_n(l+l')} \right) Z|_{J_i(l)=0 \ (i=1,2)} \approx 0 \quad (n=0,1),$$

$$\frac{\delta}{\delta J_2(l)} Z|_{J_i(l)=0 \ (i=1,2)} \approx 0. \quad (2.11)$$

Here we have set the string coupling $g = 1$ for notational simplicity. The fact that the left hand sides of the three equations above do not vanish unless $l \neq 0$ makes further analysis cumbersome. We can see from the analysis of the disk amplitudes in the above that the tadpole terms should exist. However it is possible to show that we can cancel such tadpole term contributions by shifting $J_0(l)$ as $J_0(l) \rightarrow c_1 l^{1/3} + c_2 l^{7/3} + J_0(l)$, and we obtain Eqs. (2.11) with \approx replaced by $=$. Indeed the W

constraints are usually written in terms of such shifted variables [6]. For notational simplicity, we will deal with equations which are obtained after such a shift is done.

It is convenient to use the notation

$$(f * g)(l) = \int_0^l dl' f(l-l') g(l'), \quad (2.12)$$

$$(f \triangleleft g)(l) = \int_0^\infty dl' f(l') g(l+l').$$

³Here A denotes one of the matrices in the two matrix model. Here we follow the notation of Ref. [19].

⁴Do not confuse it with the string coupling constant g .

Then the first line of (2.11) can be rewritten in a simpler form:

$$\left[\frac{\delta}{\delta J_{n+1}(l)} + \left(\frac{\delta}{\delta J_0} * \frac{\delta}{\delta J_n} \right) (l) + \left((lJ_0) \triangleleft \frac{\delta}{\delta J_n} \right) (l) \right] Z|_{J_i(l)=0 \ (i=1,2)} = 0 \quad (n=0,1). \quad (2.13)$$

By solving $\delta/\delta J_2(l)$ in terms of $\delta/\delta J_0(l)$ and $lJ_0(l)$, and substituting it into the second line of (2.11), we obtain

$$\left[\left(\frac{\delta}{\delta J_0} * (lJ_0) \right) \triangleleft \right]^2 \frac{\delta}{\delta J_0} (l) Z = 0. \quad (2.14)$$

Here $J_i(l) = 0$ ($i=1,2$) is implicitly understood. Also we always understand \triangleleft as an operation to the right: $A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_n = A_1 \triangleleft [A_2 \triangleleft (\dots \triangleleft A_n) \dots]$. To deduce the W_3 constraints from (2.14), one should subtract the nonuniversal part of Z . In usual cases, the nonuniversal parts exist in the disk and the cylinder amplitudes. However, after the shift of $J_0(l)$ discussed above, the disk amplitude vanishes. Hence only the subtraction of the nonuniversal part of the cylinder amplitude is needed:

$$Z = Z_{\text{non}} Z_{\text{univ}}, \quad (2.15)$$

$$Z_{\text{non}} = \exp \left(\frac{1}{2} \int_0^\infty dl dl' J_0(l) C_{\text{non}}(l, l') J_0(l') \right).$$

Then, substituting (2.15) into (2.14), we obtain the SD equation for the universal part of the partition function merely by shifting the derivative:

$$\frac{\delta}{\delta J_0(l)} = D(l) + \int_0^\infty dl' C_{\text{non}}(l, l') J_0(l'), \quad (2.16)$$

where $D(l)$ denotes the derivative for the universal part.

Next we will specify the $C_{\text{non}}(l, l')$ and $D(l)$, and deduce the W_3 constraints explicitly. It is more convenient to work in the Laplace transformed variables:

$$\tilde{f}(\varsigma) = \int_0^\infty dl \exp(-l\varsigma) f(l). \quad (2.17)$$

In such variables, the operations $*$ and \triangleleft defined in (2.12) are expressed as

$$(\tilde{f} * \tilde{g})(\varsigma) = \tilde{f}(\varsigma) \tilde{g}(\varsigma), \quad (2.18)$$

$$(\tilde{f} \triangleleft \tilde{g})(\varsigma) = - \int \frac{d\varsigma'}{2\pi i} \tilde{f}(-\varsigma') \frac{\tilde{g}(\varsigma) - \tilde{g}(\varsigma')}{\varsigma - \varsigma'}.$$

The nonuniversal part can be obtained by the orthogonal polynomial technique [18,14]. Substituting it into (2.16), we obtain

$$\frac{\delta}{\delta \tilde{J}_0(\varsigma)} = \tilde{D}(\varsigma) + \frac{1}{3} \int \frac{d\varsigma'}{2\pi i} \tilde{K}(-\varsigma') \frac{2 - (\frac{\varsigma'}{\varsigma})^{1/3} - (\frac{\varsigma'}{\varsigma})^{2/3}}{\varsigma - \varsigma'}, \quad (2.19)$$

where $\tilde{K}(\varsigma) = -\tilde{J}'_0(\varsigma) = \int dl \exp(-l\varsigma) l J_0(l)$. The universal part of the partition function depends only on some fractional moments of the currents $\tilde{J}_r = \int d\varsigma \tilde{J}_0(-\varsigma) \varsigma^{-r-1}$ ($r = n + \frac{1}{3}, n + \frac{2}{3}$) with n non-negative integers. So the $D(\varsigma)$ will be expanded in the form

$$\tilde{D}(\varsigma) = \sum_{r>0} \varsigma^{-r-1} \frac{\partial}{\partial \tilde{J}_r}, \quad (2.20)$$

$$r = n + \frac{1}{3}, n + \frac{2}{3},$$

with n non-negative integers.

Substituting (2.19) and (2.20) into (2.14), we have obtained the following result after a long calculation:

$$\left\{ : \left[\left(\tilde{D} + \frac{\tilde{K}}{3} \right)^3 \right]_{\leq -1} : + \frac{3}{2} \left(\tilde{D} + \frac{1}{3} \hat{K}_3 \right) \left(: \left[\left(\tilde{D} + \frac{\tilde{K}}{3} \right)^2 \right]_{\leq -1} : + \frac{2}{27\varsigma^2} \right) \right\} Z_{\text{univ}} = 0, \quad (2.21)$$

where

$$\hat{K} = \sum_{4>0} \varsigma^{r-1} r \tilde{J}_r, \quad (2.22)$$

$$\hat{K}_3 \tilde{f} = \frac{1}{2\pi i} \int d\varsigma' \frac{\tilde{K}(-\varsigma')}{\varsigma - \varsigma'} \left\{ 2\tilde{f}(\varsigma') - \left[\left(\frac{\varsigma'}{\varsigma} \right)^{1/3} + \left(\frac{\varsigma'}{\varsigma} \right)^{2/3} \right] \tilde{f}(\varsigma) \right\},$$

and $[\cdot]_{\leq -1}$ means taking all the terms with negative integral powers of ς , and $::$ denotes the normal ordering such that $\partial/\partial \tilde{J}_r$'s are put on the right of \tilde{J}_r 's.

Expanding Eq. (2.21) asymptotically in powers of ς^{-1} , we obtain the following constraints for the partition functions,

$$W_n^3 Z_{\text{univ}} = 0 \quad (n = -2, -1, \dots), \quad (2.23)$$

$$L_n Z_{\text{univ}} = 0 \quad (n = -1, 0, \dots),$$

where L_n and W_n^3 are defined through expanding in ς the operators appearing in (2.21):

$$\sum_{n=-2}^{\infty} W_n^3 \varsigma^{-n-3} = : \left[\left(\tilde{D} + \frac{\tilde{K}}{3} \right)^3 \right]_{\leq -1} :, \quad (2.24)$$

$$\frac{2}{3} \sum_{n=-1}^{\infty} L_n \varsigma^{-n-2} = : \left[\left(\tilde{D} + \frac{\tilde{K}}{3} \right)^2 \right]_{\leq -1} : + \frac{2}{27\varsigma^2}.$$

(2.23) coincides with the W_3 constraints [6] for the partition function.

C. Loop with mixed spin configurations

So far in this section we have mainly dealt with only loops with all the spins up. As a check of the validity of our continuum SD equations, we will show that they can be applied to the loops with mixed spin configurations which were considered in [19].

Let us consider a loop which is divided into two connected regions of up and down spins. We denote such a loop by $w(l_1, l_2)$ where l_1 and l_2 are the length of the up and down regions, respectively (Fig. 5). We will discuss the disk amplitude $\langle w(l_1, l_2) \rangle_0$ with such a boundary in this subsection. In [19], the discrete counterpart $W^{(2)}(P, Q)$ of

$$\tilde{w}(\varsigma_1, \varsigma_2) = \int_0^\infty dl_1 \int_0^\infty dl_2 e^{-\varsigma_1 l_1 - \varsigma_2 l_2} \langle w(l_1, l_2) \rangle_0$$

was given by solving the matrix model SD equations. By taking the continuum limit of $W^{(2)}(P, Q)$ one can obtain $\tilde{w}(\varsigma_1, \varsigma_2)$. It turns out that $\tilde{w}(\varsigma_1, \varsigma_2)$ mixes with $\tilde{w}_0(\varsigma_1)$ and $\tilde{w}_0(\varsigma_2)$. Therefore we should subtract a multiple of $W(P) + W(Q)$ from $W^{(2)}(P, Q)$ in taking the continuum limit. In the continuum limit, $a \rightarrow 0$, $P = P_* + a\varsigma_1$, $Q = P_* + a\varsigma_2$, one obtains the expansion

$$\begin{aligned} W^{(2)}(P, Q) - \alpha[W(P) + W(Q)] \\ = d_0 + d_3(\varsigma_1 + \varsigma_2)a + d_5 a^{5/3} \tilde{w}(\varsigma_1, \varsigma_2) + O(a^2), \end{aligned}$$

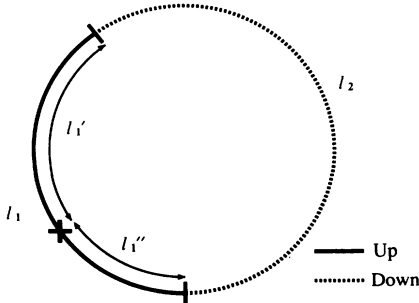


FIG. 5. The mixed spin configuration.

where α and d_i 's are nonuniversal constants. The coefficient of $a^{5/3}$ may be identified as $\tilde{w}(\varsigma_1, \varsigma_2)$ which is given as

$$-\frac{\tilde{w}_0(\varsigma_1)^2 + \tilde{w}_0(\varsigma_1)\tilde{w}_0(\varsigma_2) + \tilde{w}_0(\varsigma_2)^2 - 3t^{4/3}}{\varsigma_1 + \varsigma_2}. \quad (2.25)$$

On the other hand, we can construct the continuum SD equation for $\langle w(l_1, l_2) \rangle_0$ as in the previous section. If we deform the loop at a point in the up spin region, we obtain

$$\begin{aligned} \int_0^{l_1'} dl w_0(l) \langle w(l_1 - l, l_2) \rangle_0 \\ + \int_0^{l_1''} dl w_0(l) \langle w(l_1 - l, l_2) \rangle_0 + \langle w_1(l_1, l_2) \rangle_0 \approx 0. \end{aligned}$$

Here l_1' and $l_1'' = l_1 - l_1'$ are the distances from the point of deformation to the two domain walls (Fig. 5). $w_1(l_1, l_2)$ denotes the loop with one \mathcal{H} insertion at the point. If we deform the loop $w_1(l_1, l_2)$, we obtain the following two equations as in the previous section:

$$\begin{aligned} \int_0^{l_1'} dl w_1(l) \langle w(l_1 - l, l_2) \rangle_0 + \int_0^{l_1''} dl w_0(l) \\ \times \langle w_1(l_1 - l, l_2) \rangle_0 + \langle w_2(l_1, l_2) \rangle_0 \approx 0, \end{aligned}$$

$$\int_0^{l_2} dl \langle w(l_1', l) \rangle_0 \langle w(l_1'', l_2 - l) \rangle_0 + \langle w_2(l_1, l_2) \rangle_0 \approx 0,$$

where $w_2(l_1, l_2)$ denotes the loop with two \mathcal{H} insertions at the point. Since the loop $\langle w(l_1, l_2) \rangle_0$ now has the down spin region, the second equation does not imply $\langle w_2(l_1, l_2) \rangle_0 \approx 0$ contrary to the previous case. By eliminating $w_1(l_1, l_2)$ and $w_2(l_1, l_2)$ from the above equations, one obtains a closed equation for $\langle w(l_1, l_2) \rangle_0$. It is easy to check that the disk amplitude Eq. (2.25) satisfies this equation. Although we have not tried yet, it is in principle possible to do the same thing for loops with more complicated spin configurations.

III. CONTINUUM SD EQUATIONS FOR $c = 1 - 6/m(m+1)$ STRING

It is straightforward to construct continuum SD equations for $c = 1 - 6/m(m+1)$ string in the same way as in the previous section. In this section, we will elucidate $m = 4$ case as an example. We will show that we can derive the W constraints from the SD equation.

A. SD equations

As a generalization of the two matrix model, $c = 1 - \frac{6}{m(m+1)}$ string can be realized by the $(m-1)$ -matrix chain model. The matrices M_i are labeled by an integer i ($i = 1, \dots, m-1$). The matter degrees of freedom

are represented by this “spin” variable i . Each spin can be considered to correspond to a vertex of the Dynkin diagram of A_{m-1} so that the matrix chain potential $\sum_i \text{tr}(M_i M_{i+1})$ is written as $\sum_{i,j} C_{ij} \text{tr}(M_i M_j)$ by the connectivity matrix C_{ij} of the Dynkin diagram.

In this case, a string is labeled by its length and the spin configuration. In the continuum limit, the matrix configuration can be expressed by a state in $c = 1 - 6/m(m+1)$ CFT. The W constraints can be obtained by considering SD equations involving strings on which all the spins are 1. In [13], various boundary configurations in the A_m restricted solid on solid (RSOS) models [20] are identified with a state in $c = 1 - 6/m(m+1)$ CFT. The RSOS realization of $c = 1 - 6/m(m+1)$ CFT is a bit different from the matter realization in the matrix chain model, in which the A_{m-1} Dynkin diagram is related to $c = 1 - 6/m(m+1)$. However, as in the Ising case, the fixed boundary conditions in the matrix chain may be identified with a boundary condition in which the spins on the boundary and those of the neighbors of the boundary are fixed in the RSOS model. Such a boundary condition is labeled by an integer r ($r = 1, \dots, m-1$) [13] and we will identify it with the spin configuration where all the spins are r in the matrix chain. We will denote

the loop on which all the spins are 1 by $w(l; |1\rangle)$.

The SD equations are constructed as in the Ising case. We will illustrate $m = 4$ case as an example. Let us consider the SD equation corresponding to the deformation of a loop amplitude:

$$\langle w(l; |1\rangle) w(l_1; |1\rangle) \cdots w(l_n; |1\rangle) \rangle. \quad (3.1)$$

The continuum SD equations are constructed assuming the following.

(1) The SD equations consist of three kinds of terms illustrated in Fig. 1.

(2) The splitting and merging process is written by using the three-Reggeon-like vertex which represents a δ functional of the spin configurations.

(3) For the kinetic terms, only the terms in which spins are flipped survive in the continuum limit of the matrix model SD equation. In the matrix chain model, such terms come from the matrix chain potential $\sum_i \text{tr}(M_i M_{i+1})$. Therefore a spin i , $1 < i < m-1$ is flipped to $i-1$ and $i+1$, and 1 and $m-1$ are flipped to 2 and $m-2$, respectively.

The equation corresponding to the deformation of Eq. (3.1) at a point on a boundary becomes

$$\begin{aligned} & \int_0^l dl' \langle w(l'; |1\rangle) w(l-l'; |1\rangle) w(l_1; |1\rangle) \cdots w(l_n; |1\rangle) \rangle \\ & + g \sum_k l_k \langle w(l+l_k; |1\rangle) w(l_1; |1\rangle) \cdots w(l_{k-1}; |1\rangle) w(l_{k+1}; |1\rangle) \cdots w(l_n; |1\rangle) \rangle \\ & + \langle w(l; \mathcal{H}(\sigma)|1\rangle) w(l_1; |1\rangle) \cdots w(l_n; |1\rangle) \rangle \approx 0. \end{aligned} \quad (3.2)$$

$\mathcal{H}(\sigma)$ here represents an insertion of a tiny region on which the spins take the value 2. This insertion comes from the matrix chain potential $\sum_i \text{tr}(M_i M_{i+1})$. In the continuum, the operator which is at the domain wall between the regions of spin 1 and 2 is again identified to be $\phi_{2,1}$ [13]. Therefore $\mathcal{H}(\sigma)$ insertion here can be replaced by $\lim_{\sigma' \rightarrow \sigma} \phi_{2,1}(\sigma') \phi_{2,1}(\sigma)$.

We can go on to obtain equations involving $w(l; [\mathcal{H}(\sigma)]^2|1\rangle)$. If one deforms $w(l; \mathcal{H}(\sigma)|1\rangle)$ at a point near σ and take the limit in which the point tends to σ , one obtains

$$\begin{aligned} & \int_0^l dl' \langle w(l'; |1\rangle) w(l-l'; \mathcal{H}(\sigma)|1\rangle) w(l_1; |1\rangle) \cdots w(l_n; |1\rangle) \rangle \\ & + g \sum_k l_k \langle w(l+l_k; \mathcal{H}(\sigma)|1\rangle) w(l_1; |1\rangle) \cdots w(l_{k-1}; |1\rangle) w(l_{k+1}; |1\rangle) \cdots w(l_n; |1\rangle) \rangle \\ & + \langle w(l; [\mathcal{H}(\sigma)]^2|1\rangle) w(l_1; |1\rangle) \cdots w(l_n; |1\rangle) \rangle \approx 0. \end{aligned} \quad (3.3)$$

So far the Eqs. (3.2) and (3.3) have the same form as the Ising case Eqs. (2.4) and (2.5). A difference comes in when one tries to obtain Eq. (2.6). If one deforms $w(l; \phi_{2,1}(\sigma') \phi_{2,1}(\sigma)|1\rangle)$ at a point between the two $\phi_{2,1}$ insertions and then takes the limit $\sigma' \rightarrow \sigma$, one obtains not only the loop $w(l; \mathcal{H}^2(\sigma)|1\rangle)$ but also a loop with an insertion of a tiny region on which the spins are 3 (Fig. 6). The boundary operator which is at the domain wall between 1 and 3 regions is identified with $\phi_{3,1}$ [13]. Therefore we obtain an equation

$$\langle w(l; [\mathcal{H}(\sigma)]^2|1\rangle) w(l_1; |1\rangle) \cdots w(l_n; |1\rangle) \rangle + \langle w(l; [\phi_{3,1}(\sigma)]^2|1\rangle) w(l_1; |1\rangle) \cdots w(l_n; |1\rangle) \rangle \approx 0. \quad (3.4)$$

This equation reflects the fusion rule $\phi_{2,1} \phi_{2,1} \sim \phi_{1,1} + \phi_{3,1}$. \mathcal{H} should be identified with the $\phi_{1,1}$ part of the product $\phi_{2,1} \phi_{2,1}$.

Thus $w(l; [\mathcal{H}(\sigma)]^2|1\rangle)$ is not null in this case. Rather we can prove $w(l; [\mathcal{H}(\sigma)]^3|1\rangle)$, which is defined as a limit

$$\lim_{\sigma_3 \rightarrow \sigma_1} w(l; \mathcal{H}(\sigma_3) \mathcal{H}(\sigma_2) \mathcal{H}(\sigma_1)|1\rangle), \quad \sigma_3 > \sigma_2 > \sigma_1,$$

is null by the following sequence of SD equations:

$$\begin{aligned}
& \lim_{\sigma_3 \rightarrow \sigma_1} [\langle w(l; \mathcal{H}(\sigma_3) \mathcal{H}(\sigma_2) \mathcal{H}(\sigma_1) | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle \\
& \quad + \langle w(l; \phi_{3,1}(\sigma_3) \phi_{3,1}(\sigma_2) \mathcal{H}(\sigma_1) | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle] \approx 0, \\
& \lim_{\sigma_3 \rightarrow \sigma_1} [\langle w(l; \phi_{3,1}(\sigma_3) \mathcal{H}(\sigma_2) \phi_{3,1}(\sigma_1) | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle \\
& \quad + \langle w(l; \phi_{3,1}(\sigma_3) \phi_{3,1}(\sigma_2) \mathcal{H}(\sigma_1) | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle] \approx 0, \\
& \lim_{\sigma_3 \rightarrow \sigma_1} [\langle w(l; \phi_{3,1}(\sigma_3) \mathcal{H}(\sigma_2) \phi_{3,1}(\sigma_1) | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle] \approx 0.
\end{aligned} \tag{3.5}$$

Here $\sigma_3 > \sigma_2 > \sigma_1$ in all the equations. For example, the first equation corresponds to the deformation of the amplitude $\langle w(l; \phi_{2,1}(\sigma_3) \phi_{2,1}(\sigma_2) \mathcal{H}(\sigma_1) | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle$ at a point between the two $\phi_{2,1}$ insertions [Fig. 7(a)]. In the limit $\sigma_3 \rightarrow \sigma_1$, splitting and absorbing of loops does not contribute to the equation and we obtain the first equation in the above. The derivations of the other two equations are also illustrated in Fig. 7. Thus we can prove

$$\lim_{\sigma_3 \rightarrow \sigma_1} [\langle w(l; \mathcal{H}(\sigma_3) \mathcal{H}(\sigma_2) \mathcal{H}(\sigma_1) | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle] \approx 0.$$

For general m , we can again identify \mathcal{H} with the $\phi_{1,1}$ part of the product $\phi_{2,1} \phi_{2,1}$. We can prove, by similar manipulations,

$$\lim_{\sigma_{m-1} \rightarrow \sigma_1} [\langle w(l; \mathcal{H}(\sigma_{m-1}) \cdots \mathcal{H}(\sigma_1) | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle] \approx 0 \quad (\sigma_{m-1} > \cdots > \sigma_1). \tag{3.6}$$

Therefore $w(l; [\mathcal{H}(\sigma)]^{m-1} | 1) \rangle$ becomes null for $c = 1 - 6/m(m+1)$ string theory. As a generalization of Eq. (3.3), we have

$$\begin{aligned}
& \int_0^l dl' \langle w(l'; | 1) \rangle w(l-l'; [\mathcal{H}(\sigma)]^j | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle \\
& \quad + g \sum_k l_k \langle w(l+l_k; [\mathcal{H}(\sigma)]^j | 1) \rangle w(l_1; | 1) \cdots w(l_{k-1}; | 1) \rangle w(l_{k+1}; | 1) \cdots w(l_n; | 1) \rangle \\
& \quad + \langle w(l; [\mathcal{H}(\sigma)]^{j+1} | 1) \rangle w(l_1; | 1) \cdots w(l_n; | 1) \rangle \approx 0,
\end{aligned} \tag{3.7}$$

for $j = 0, \dots, m-2$. With Eqs. (3.6) and (3.7), the W constraints will be derived in the next subsection.

We will conclude this subsection with a comment on the scaling dimensions again. For general m , the scaling dimension of the disk amplitude $\langle w(l; | 1) \rangle_0$ is $L^{-(2m+1)/m}$. The gravitational scaling dimension of $\phi_{r,1}$ on the boundary is $L^{-(m+1)(r-1)/2m}$ and again has the right dimension for the continuum SD equations to make sense.

B. Derivation of the W constraints

Let us rewrite Eqs. (3.6) and (3.7) into equations for the generating functional of the loop amplitudes:

$$Z^{(m)}(J_i(l)) = \left\langle \exp \left[\sum_{i=0}^{m-1} \int_0^\infty dl J_i(l) w(l; [\mathcal{H}(\sigma)]^i | 1) \right] \right\rangle.$$

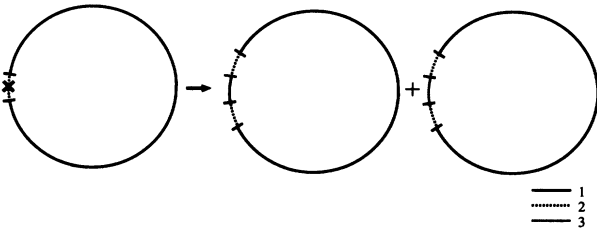


FIG. 6. The SD equation (3.4).

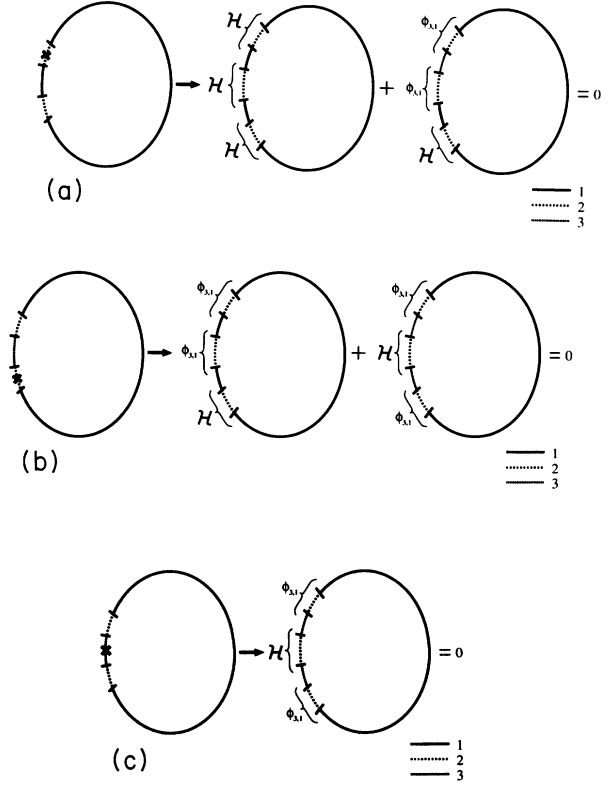


FIG. 7. The SD equation (3.5).

Eqs. (3.6) and (3.7) become as follows:

$$\left[\frac{\delta}{\delta \tilde{J}_{n+1}} + \left(\frac{\delta}{\delta \tilde{J}_0} * \frac{\delta}{\delta \tilde{J}_n} \right) + \tilde{K} \triangleleft \frac{\delta}{\delta \tilde{J}_n} \right] (\varsigma) \\ \times Z^{(m)}|_{\tilde{J}_i(\varsigma)=0(i>0)} = 0 \quad (n = 0, 1, \dots, m-2), \\ \frac{\delta}{\delta \tilde{J}_{m-1}(\varsigma)} Z^{(m)}|_{\tilde{J}_i(\varsigma)=0(i>0)} = 0. \quad (3.8)$$

We have assumed that the tadpole term is canceled by an appropriate shift of $J_0(l)$. Solving $\delta/\delta \tilde{J}_i (i > 0)$'s recursively and substituting $\delta/\delta \tilde{J}_{m-1}$ into the second line of (3.8), we obtain

$$\left[\left(\frac{\delta}{\delta \tilde{J}_0} * \tilde{K} \triangleleft \right)^{m-1} \frac{\delta}{\delta \tilde{J}_0} \right] (\varsigma) Z^{(m)} = 0. \quad (3.9)$$

Here $\tilde{J}_i(\varsigma) = 0$ ($i > 0$) is implicitly understood. The subtraction of the nonuniversal part will be

$$\frac{\delta}{\delta \tilde{J}_0(\varsigma)} = \tilde{D}(\varsigma) + \int \frac{d\varsigma'}{2\pi i} \tilde{K}(-\varsigma') \tilde{G}^{(m)}(\varsigma, \varsigma'), \quad (3.10)$$

$$\tilde{G}^{(m)}(\varsigma, \varsigma') = \frac{1}{m} \frac{m-1 - \sum_{i=1}^{m-1} \left(\frac{\varsigma'}{\varsigma}\right)^{\frac{i}{m}}}{\varsigma - \varsigma'}.$$

This is a simple generalization of the known cases $m = 2, 3$.

The $\tilde{D}(\varsigma)$ will be generalized to

$$\tilde{D}(\varsigma) = \sum_{r>0} \varsigma^{-r-1} \frac{\partial}{\partial \tilde{J}_r}, \\ r = n + \frac{1}{m}, n + \frac{2}{m}, \dots, n + \frac{m-1}{m} \quad (3.11)$$

with n non-negative integers.

Our expectation is that, substituting (3.10) and (3.11) into (3.9), one will obtain the W_m constraints for the universal part of the partition function. We have performed the calculations explicitly for the cases up to $m = 4$. For $m = 4$, we have obtained, after a long calculation,

$$\left[W_4(\varsigma) - \frac{3}{4}[\tilde{K}W_3]_{\geq 0}(\varsigma) + \frac{3}{8}[\tilde{K}[\tilde{K}L]_{\geq 0}]_{\geq 0}(\varsigma) + \frac{4}{3}(\tilde{D}(\varsigma) + \frac{1}{4}K_q(\varsigma))\left\{\frac{3}{4}W_3(\varsigma) - \frac{3}{8}[\tilde{K}L]_{\geq 0}(\varsigma)\right\} \right. \\ \left. + \frac{1}{2} \left(2 : [(\tilde{D}(\varsigma) + \frac{1}{4}K_q(\varsigma))^2]_n : - [\tilde{K}\tilde{D}]_{\geq 0}(\varsigma) - \frac{1}{4}[\tilde{K}K_q]_{\geq 0}(\varsigma) + \frac{3}{10} \frac{\partial^2}{\partial \varsigma^2} \right) L(\varsigma) \right] Z_{\text{univ}}^{(m=4)} = 0, \quad (3.12)$$

where $[\cdot]_n$ means taking all the terms with nonintegral powers of ς , and

$$[AB]_{\geq 0}(\varsigma) = - \int \frac{d\varsigma_1}{2\pi i} \frac{A(-\varsigma_1)B(\varsigma_1)}{\varsigma - \varsigma_1}, \\ \frac{1}{2}L(\varsigma) = : \left[\left(\tilde{D} + \frac{1}{4}\tilde{K} \right)^2 \right]_{\leq -1} : + \frac{5}{64\varsigma^2}, \\ \frac{3}{4}W_3(\varsigma) = : \left[\left(\tilde{D} + \frac{1}{4}\tilde{K} \right)^3 \right]_{\leq -1} :, \quad (3.13)$$

$$W_4(\varsigma) = : \left[\left(\tilde{D} + \frac{1}{4}\tilde{K} \right)^4 \right]_{\leq -1} : - : \left[\left(\frac{\partial}{\partial \varsigma} \left(\tilde{D} + \frac{1}{4}\tilde{K} \right) \right)^2 \right]_{\leq -1} : \\ + \left(\frac{1}{5} \frac{\partial^2}{\partial \varsigma^2} + \frac{15}{32\varsigma^2} \right) : \left[\left(\tilde{D} + \frac{1}{4}\tilde{K} \right)^2 \right]_{\leq 1} : + \frac{105}{(64)^2\varsigma^4}, \\ K_q(\varsigma) = \int \frac{d\varsigma_1}{2\pi i} \tilde{K}(-\varsigma_1) \frac{(\frac{\varsigma_1}{\varsigma})^{1/4} + (\frac{\varsigma_1}{\varsigma})^{1/2} + (\frac{\varsigma_1}{\varsigma})^{3/4}}{\varsigma - \varsigma_1}.$$

Here the definition of \tilde{K} follows that in (2.22) with the summation over r following (3.11). Expanding Eq. (3.12) asymptotically in ς^{-1} , one obtains the W_4 constraints for the partition functions,

$$L_n Z_{\text{univ}}^{(m=4)} = 0 \quad (n = -1, 0, \dots), \\ W_n^3 Z_{\text{univ}}^{(m=4)} = 0 \quad (n = -2, -1, \dots), \\ W_n^4 Z_{\text{univ}}^{(m=4)} = 0 \quad (n = -3, -2, \dots), \quad (3.14)$$

where L 's and W 's are defined through expanding in ς the operators appearing in (3.13):

$$L(\varsigma) = \sum_{n=-1} L_n \varsigma^{-n-2}, \\ W_3(\varsigma) = \sum_{n=-2} W_n^3 \varsigma^{-n-3}, \\ W_4(\varsigma) = \sum_{n=-3} W_n^4 \varsigma^{-n-4}. \quad (3.15)$$

These coincide with the W_4 constraints [6,21]. We conjecture that W_m constraints can be derived from Eqs. (3.8) also for $m \geq 5$.

IV. STRING FIELD HAMILTONIAN

The discussions in the previous sections imply that the continuum SD equations we proposed really describe $c = 1 - 6/m(m+1)$ string theory. In this section we will infer the form of the string field Hamiltonian from these equations.

In order to do so, we need the SD equation corresponding to the deformation of loops more general than

$w(l; |+) , w(l; \mathcal{H}(\sigma)|+) , w(l_1, l_2)$, etc., which were discussed in the previous sections. For those loops, the vertex terms look particularly simple. In order to write down the continuum SD equations for more general loops, we should introduce three-Reggeon-like vertex for $c = 1 - 6/m(m+1)$ CFT. Here let us express a state of a string (with a marked point) as $|v\rangle_l$ by its length l and the spin configuration $|v\rangle$. We define a product $*$ so that

$$|v_1\rangle_{l_1} * |v_2\rangle_{l_2},$$

represents a loop made by merging the two loops $|v_1\rangle_{l_1}$ and $|v_2\rangle_{l_2}$ at the marked points, with the spin configuration inherited from them (Fig. 8). Then the continuum SD equation for generic loops will be expressed as

$$\begin{aligned} & \int_0^l dl' \sum_{|v'\rangle, |v''\rangle, |v'_i\rangle * |v''_i\rangle_{l-i}, |v\rangle_l} \langle w(l'; |v'\rangle) w(l-l'; |v''\rangle) w(l_1; |v_1\rangle) \cdots w(l_n; |v_n\rangle) \rangle \\ & + g \sum_k l_k \int_0^{2\pi} d\sigma' \langle w(l+l_k; |v\rangle_l * (e^{i\sigma' \mathcal{P}} |v_k\rangle_{l_k})) w(l_1; |v_1\rangle) \cdots w(l_{k-1}; |v_{k-1}\rangle) w(l_{k+1}; |v_{k+1}\rangle) \cdots w(l_n; |v_n\rangle) \rangle \\ & + \langle w(l; \mathcal{H}(\sigma)|v\rangle) w(l_1; |v_1\rangle) \cdots w(l_n; |v_n\rangle) \rangle \approx 0. \end{aligned} \quad (4.1)$$

Here \mathcal{P} is the operator of rotation of a loop. $\mathcal{H}(\sigma)$ is identified with $\lim_{\sigma' \rightarrow \sigma} \phi_{2,1}(\sigma') \phi_{2,1}(\sigma)$.

The SD equation describes a deformation of a loop at a point on it. If we integrate it over the position of the point, we obtain the deformation induced by the string field Hamiltonian in the temporal gauge. Let $\Psi(l; |v\rangle)$ ($\Psi^\dagger(l; |v\rangle)$) denote the annihilation (creation) operator of a string with length l and the spin configuration $|v\rangle$ satisfying

$$[\Psi(l; |v\rangle), \Psi^\dagger(l'; |v'\rangle)] = l \int_0^{2\pi} d\sigma \langle v' | e^{i\sigma \mathcal{P}} | v \rangle \delta(l - l'). \quad (4.2)$$

Namely the commutator of $\Psi(l; |v\rangle)$ and $\Psi^\dagger(l'; |v'\rangle)$ is nonzero only when $l = l'$ and $|v\rangle$ coincides with $|v'\rangle$ up to rotation. The string field Hamiltonian can be obtained from Eq. (4.1) as

$$\begin{aligned} H &= \sum_{|v_i\rangle} \int_0^\infty dl_1 \int_0^\infty dl_2 \Psi^\dagger(l_1; |v_1\rangle) \Psi^\dagger(l_2; |v_2\rangle) \Psi(l_1 + l_2; |v_1\rangle_{l_1} * |v_2\rangle_{l_2}) \\ &+ g \sum_{|v_i\rangle} \int_0^\infty dl_1 \int_0^\infty dl_2 \Psi^\dagger(l_1 + l_2; |v_1\rangle_{l_1} * |v_2\rangle_{l_2}) \Psi(l_1; |v_1\rangle) \Psi(l_2; |v_2\rangle) \\ &+ \sum_{|v\rangle} \int_0^\infty dl \Psi^\dagger(l; \mathcal{H}(0)|v\rangle) \Psi(l; |v\rangle) + \sum_{|v\rangle} \int_0^\infty dl \rho(l; |v\rangle) \Psi(l; |v\rangle). \end{aligned} \quad (4.3)$$

Here $\rho(l; |v\rangle)$ expresses the tadpole term and it has its support at $l = 0$.

The string amplitudes can be expressed by using this Hamiltonian as follows:

$$\begin{aligned} & \langle w(l_1; |v_1\rangle) w(l_2; |v_2\rangle) \cdots w(l_n; |v_n\rangle) \rangle \\ &= \lim_{D \rightarrow \infty} \langle 0 | e^{-DH} \Psi^\dagger(l_1; |v_1\rangle) \cdots \Psi^\dagger(l_n; |v_n\rangle) | 0 \rangle. \end{aligned}$$

The string field SD equation can be obtained as

$$\lim_{D \rightarrow \infty} \partial_D \langle 0 | e^{-DH} \Psi^\dagger(l_1; |v_1\rangle) \cdots \Psi^\dagger(l_n; |v_n\rangle) | 0 \rangle = 0.$$

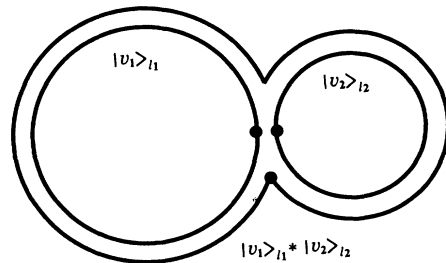


FIG. 8. The product $*$.

It is obvious from the construction of H that this SD equation can be written as an integration of the SD equation in Eq. (4.1).

We can estimate the dimension of the geodesic distance D from the above Hamiltonian. The scaling dimension of various quantities can be estimated most easily by considering terms involving strings on which all the spins are aligned. For example, for a $c = 1 - 6/m(m+1)$ string, the scaling dimension of g is given as $[g] = l^{-2(2m+1)/m}$ which coincides with the matrix model result [18]. The dimension of D becomes $[D] = L^{1/m}$. This fact may be checked by numerical simulations.

Thus we have constructed the string field Hamiltonian using the three-Reggeon-like vertices. We should, however, remark that Eq. (4.3) is a formal expression. As was clear from the discussions in the previous sections, the states such as $|1\rangle$ play important roles in the analysis of the SD equations. However, such states have divergent norms in the usual definition of the norms of states in CFT. Therefore we should adopt a different norm (e.g., one defined by Cardy [13]) in Eqs. (4.2) and (4.3) to make the Hamiltonian applicable to such states. Accordingly the definition of the three-Reggeon-like vertices ought to be changed. We will pursue these problems elsewhere.

V. CONCLUSIONS

In this paper we proposed the continuum SD equations for a $c = 1 - 6/m(m+1)$ string. It was checked

that the SD equations are consistent with all the known results of noncritical string theory. Especially the W constraints were derived from the SD equations. The W constraints essentially come from the fact that the loop operator $w(l; [\mathcal{H}(\sigma)]^{m-1}|1\rangle)$ is null. In the continuum picture, it was proved by using the results of boundary CFT.

We constructed the temporal gauge string field Hamiltonian from the SD equations. The Hamiltonian looks similar to the Hamiltonian of the light-cone gauge string field theory [1], involving only three string interactions. Since the form of the Hamiltonian is almost the same for any c , it might be possible to construct the temporal gauge Hamiltonian in the same way for the $c > 1$ case, especially for the critical string. This will be left to the future investigations.

ACKNOWLEDGMENTS

We would like to thank M. Fukuma, T. Kawai, Y. Kitazawa, Y. Matsuo, M. Ninomiya, J. Nishimura, N. Tsuda, and T. Yukawa for useful discussions and comments. N.S. was supported by the Japanese Society for the Promotion of Science for Japanese Junior Scientists and the Grant-in-Aid for the Scientific Research from the Ministry of Education No. 06-3758.

-
- [1] M. Kaku and K. Kikkawa, *Phys. Rev. D* **10**, 1110 (1974); **10**, 1823 (1974); W. Siegel, *Phys. Lett.* **151B**, 391 (1985); **151B**, 396 (1985); E. Witten, *Nucl. Phys.* **B268**, 253 (1986); H. Hata, K. Itoh, T. Kugo, H. Kunitomo, and K. Ogawa, *Phys. Lett. B* **172**, 186 (1986); **172**, 195 (1986). A. Neveu and P. West, *Phys. Lett.* **168B**, 192 (1986); M. Kaku, *Int. J. Mod. Phys. A* **9**, 3103 (1994).
 - [2] N. Ishibashi and H. Kawai, *Phys. Lett. B* **314**, 190 (1993).
 - [3] H. Kawai, N. Kawamoto, T. Mogami, and Y. Watabiki, *Phys. Lett. B* **306**, 19 (1993).
 - [4] M. Fukuma, N. Ishibashi, H. Kawai, and M. Ninomiya, *Nucl. Phys. B* (to be published).
 - [5] R. Nakayama, *Phys. Lett. B* **325**, 347 (1994).
 - [6] M. Fukuma, H. Kawai, and R. Nakayama, *Int. J. Mod. Phys. A* **6**, 1385 (1991); R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Nucl. Phys.* **B348**, 435 (1991).
 - [7] A. Jevicki and J. Rodrigues, *Nucl. Phys.* **B421**, 278 (1994).
 - [8] Y. Watabiki, Report No. INS-1017, hep-th xxx.lanl.gov - 9401096 [*Nucl. Phys. B* (submitted)].
 - [9] N. Ishibashi and H. Kawai, *Phys. Lett. B* **322**, 67 (1994).
 - [10] E. Gava and K. S. Narain, *Phys. Lett. B* **263**, 213 (1991).
 - [11] M. E. Agishtein and A. A. Migdal, *Int. J. Mod. Phys. C* **1**, 165 (1990); *Nucl. Phys.* **B350**, 690 (1991).
 - [12] N. Ishibashi, *Mod. Phys. Lett. A* **4**, 251 (1989).
 - [13] J. L. Cardy, *Nucl. Phys.* **B324**, 581 (1989).
 - [14] G. Moore, N. Seiberg, and M. Staudacher, *Nucl. Phys.* **B362**, 665 (1991).
 - [15] J. Alfaro, *Phys. Rev. D* **47**, 4714 (1993); D. V. Boulatov, *Mod. Phys. Lett. A* **8**, 557 (1993).
 - [16] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, *Mod. Phys. Lett. A* **3**, 819 (1988); F. David, *ibid.* **3**, 1651 (1988); J. Distler and H. Kawai, *Nucl. Phys.* **B321**, 509 (1989).
 - [17] E. Martinec, G. Moore, and N. Seiberg, *Phys. Lett. B* **263**, 190 (1991).
 - [18] E. Brézin and V. Kazakov, *Phys. Lett. B* **236**, 144 (1990); M. Douglas and S. Shenker, *Nucl. Phys.* **B335**, 635 (1990); D. Gross and A. Migdal, *Phys. Rev. Lett.* **64**, 127 (1990); *Nucl. Phys.* **B340**, 333 (1990).
 - [19] M. Staudacher, *Phys. Lett. B* **305**, 332 (1993).
 - [20] G. E. Andrews, R. J. Baxter, and P. J. Forrester, *J. Stat. Phys.* **35**, 193 (1984); V. Pasquier, *Nucl. Phys.* **B285**, 162 (1987).
 - [21] M. Fukuma, H. Kawai, and R. Nakayama, *Commun. Math. Phys.* **143**, 371 (1992).